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Spline Collocation for Fredholm Integral Equations

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Abstract

The collocation methods based on cubic B-spline, are developed to approximate solution of the second and first kind Fredholm integral equations.First we collocate the solution by B-spline and the Newton-Cotes formula is used to approximate integral. Convergence analysis has been investigated and proved that the quadratur rule is fourth order convergent. The presented methods are tested to the problem, and the absolute error in the solution are compared with existing methods [1, 2, 7, 10] to verify the accuracy and convergent nature of proposed methods.

Keywords: Fredholm integral equations, Cubic B-spline, Newton-Cotes, Collocation, Convergence analysis, Regularization.

1 Introduction

We consider the following non homogeneous Fredholm integral equations of the first kind

$$y(x) = \int_a^b k(x,t)f(t)dt, \qquad a \leq x \leq b \tag{1}$$

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and the second kind

$$f(x) = \int_a^b k(x,t)f(t)dt + y(x), \qquad a \leq x \leq b \tag{2}$$

where k and y(x) are given continuous and known functions.

The numerical solution of (1) and (2) by using various methods have been investigated by several authors who rely on the approximation of the integral appearing in (1) and (2) by some quadrature. In [6], the problem is solved by using the rationalized wavelets spline functions. Haar wavelets are also applied for solving these problems in [7]. The hybrid functions are also used in [4].The sinc-Galerlin method is applied for solving problem in [12] and the sinc-collocation method for Fredholm integral equations is used in [11]. Using of a global approximation to the solution of a linear Fredholm integral equation of the second kind is constructed by means of the cubic spline quadrature in [8]. In [13] is used the Taylor series expansions for unknown function and kernel of linear and nonlinear integral equations of the second kind. In this paper, we use cubic B-spline collocation for approximation unknown function and use of the Newton-Cotes rules for approximating integral, moreover for solving Fredholm integral equation first kind, at first, we converted the first kind to the second kind by regularization method by using [3].

2 The method

To develop the collocation methods Based on cubic B-spline for solutions of Fredholm integral equations first and second, Let π be an uniform partition in the interval [a, b] $\pi : \{a = t_0 < t_1 < ... < t_{n+2} = b\}$, where $h = \frac{b-a}{n+2}$, $t_i = a+ih, t_i = x_i, i = 0, ..., n+2$. We introduce the spline space $S_3(\pi) = v \in C^2[a, b]; v|_{[t_i,t_{i+1}]} \in P_3, \qquad i = 0, 1, ..., n + 2$, where P_3 is the class of cubic polynomials. By introducing knots $t_{-2} < t_{-1} < t_0$ and $t_{n+2} < t_{n+3} < t_{n+4}$ and the functions $B_i(t)$ defined by

$$B_i(t) = \frac{1}{h^3}\begin{cases} (t-t_{i-2})^3, \quad if\ t \in [t_{i-2}, t_{i-1}] \\ h^3 + 3h^2(t-t_{i-1}) + 3h(t-t_{i-1})^2 - 3(t-t_{i-1})^3, \quad if\ t \in [t_{i-1}, t_i] \\ h^3 + 3h^2(t_{i+1}-t) + 3h(t_{i+1}-t)^2 - 3(t_{i+1}-t)^3, \quad if\ t \in [t_i, t_{i+1}] \\ (t_{i+2}-t)^3, \quad if\ t \in [t_{i+1}, t_{i+2}] \\ 0, \quad \text{otherwise.} \end{cases} \tag{3}$$

$$S(t) = \sum_{i=-2}^{n+2} c_i B_i(t), \tag{4}$$

Following regularization method in [3], the given integral equation (1) can be converted to

$$\int_a^b \int_a^b \overline{k(x,z)} k(z,t) f(t) dz dt + \alpha f(x) = \int_a^b \overline{k(x,t) y(t)} dt. \tag{5}$$

For some α the solution of problem (5) is identical to that of problem (1).

By using (3) and Newton-Cotes methods we get a system of equation as follows

$$h \sum_{i=-1}^{n+1} [w_i \sum_{j=-2}^{n+2} c_j B_j(t_i) \int_a^b \overline{k(x_l,z)} k(z,t_i) dz] + \alpha \sum_{j=-2}^{n+2} c_j B_j(x_l) = \int_a^b \overline{k(x_l,t) y(t)} dt, \tag{6}$$

where $z_l = a + lh, \quad h = \frac{b-a}{n+2} \quad l = 0, ..., n+2 \quad t_l = x_l = z_{l+1} \quad l = -1, ..., n+1.$

By solving the system (6) we obtain the vector c_j and also we set $c_{-2} = c_{n+2} = 0$ in order to have the cubic B-spline relations,then the approximate solution can be obtain by (4).

In case of the second kind integral equation (2), by using cubic B-spline (3) we can approximated the solution and also we can approximate the integral in (2) by Newton-Cotes type methods, when n is even then we have $h = \frac{b-a}{n+2}$ the Simpson rule can be used and when n is odd we have to use the three-eighth rule.

$$\sum_{i=-2}^{n+2} c_i B_j(x_j) - h \sum_{i=-2}^{n+2} c_i [w_{j,-1} k(x_j, t_{-1}) B_i(t_{-1}) + w_{j,0} k(x_j, t_0) B_i(t_0) + \ldots + w_{j,n+1} k(x_j, t_{n+1}) \times B_i(t_{n+1})] = y(x_j) \tag{7}$$

where $z_j = a + jh$, $h = \frac{b-a}{n+2}$ $j = 0, ..., n+2$ $t_j = x_j = z_{j+1}$ $j = -1, ..., n+1$.
By solving the system (7) we obtain the vector c_i and also we set $c_{-2} = c_{n+2} = 0$ in order to have the cubic B-spline relations,then with substituting c_i in (4) we obtain the approximate solution of (2).

3 Error analysis: convergence of the approximate solution

To study the convergence analysis, first we need to recall the following basic theorem in [3].

Remark 3.1 *The most immediate error analysis for spline approximates S to a given function f defined on an interval [a, b] follows from the first and second integral relations. Throughout our discussion $\pi : \{a = t_0 < t_1 < ... < t_{n+2} = b\}$ is partition in [a, b] and $h = \frac{b-a}{n+2}$ is the mesh of our partition.*
If $f \in C^4[a,b]$, then $\|D^j(f - S)\|_\infty \leq \gamma_j h^{4-j}$, $j = 0, 1, 2, 3, 4$

Where $\|f\|_\infty = \max_{0 \leq i \leq n+2} \sup_{t_{i-1} \leq t \leq t_i} |f(t)|$ and D^j the $j - th$ derivative (see [9], P.112).
The numerical method is said to be convergent if the solution of the approximating set of equations converges to the solution of the exact problem as the step length h tends to zero; that is, if $\lim_{h \to 0} |x_{i+1} - x_i| = 0$. Consider the equation

$$f(x) = \int_a^b k(x,t) f(t) dt + y(x), \; a \leq x \leq b \tag{8}$$

and suppose that at $x = x_i$, where $z_i = a + ih, i = 0, ..., n+2, x_i = t_i = z_{i+1}, i = -1, ..., n+1$ the quadrature formula

$$\int_a^{a+ih} k(x_i, t) f(t) dt = h \sum_{j=-1}^{n+1} w_{ij} k(x_i, t_j) f(t_j) + E_{i,t}(k(x_i, t) f(t)). \tag{9}$$

At $x = x_i, i = -1, ..., n+1$, by substituting (9)in(8) we have

$$f(x_i) = y(x_i) + h\sum_{j=-1}^{n+1} w_{ij}k(x_i,t_j)f(t_j) + E_{i,t}(k(x_i,t)f(t)) \quad ,i = -1, ..., n+1 \quad (10)$$

and corresponding approximating function equations are

$$S(x_i) = y(x_i) + h\sum_{j=-1}^{n+1} w_{ij}k(x_i,t_j)S(t_j) \quad ,i = -1, ..., n+1. \quad (11)$$

Thus we have

$$f(x_i) - S(x_i) = h\sum_{j=-1}^{n+1} w_{ij}k(x_i,t_j)(f(t_j) - S(t_j)) + E_{i,t}(k(x_i,t)f(t)), i = -1, ..., n+1. \quad (12)$$

We set $e_i = f(x_i) - S(x_i)$, it follows that

$$|e_i| \leq h\sum_{j=-1}^{n+1} |w_{ij}||k(x_i,t_j)|\|e_j\| + E_{i,t}(k(x_i,t)f(t)), \qquad i = -1, ..., n+1. \quad (13)$$

Let $w = \max_{i,j}|w_{i,j}|$ and $e = \max_{-1\leq i\leq n+1}|e_i|$, $|e_i| \leq \{|E_{i,t}(k(x_i,t)f(t))| + hkw(n+3)e\}$, hence $|e_i| \to 0$ as $h \to 0$, we may write equivalently $\|e_i\| = \mathcal{O}(h^{p+1}) + \mathcal{O}(h^{q+1})$, where the error in the quadrature rule is $\mathcal{O}(h^5)$ and the error in the function approximate is $\mathcal{O}(h^5)$. If we set $r = min(p,q) = min(4,4)$ then we say the quadrature rule is convergent of order 4.

4 Numerical examples

To compare our computed results and justify the accuracy and efficiency of our presented methods we consider the following examples which are considered by [1, 2, 7, 10] . The solution of the given examples is obtained for different values of n. The RMS errors solutions $E = (\frac{1}{n}\sum_{i=0}^{n}[f(x_i) - S(x_i)]^2)^{\frac{1}{2}}$ where $f(x)$ is the exact solution and $S(x)$ is the approximated solution of integral equation which are given by the suggested methods.

Example 4.1 *Consider the following Fredholm integral equation of the first kind with exact solution* $f(x) = e^x$

$$\int_0^1 e^{x^\beta * t} f(t)dt = \frac{e^{x^\beta+1} - 1}{x^\beta + 1}, \qquad \beta = 1, \quad 0 \leq x \leq 1.$$

Example 4.2 *Consider the problem*

$$\int_0^1 (x^2 + t^2)^{\frac{1}{2}} f(t)dt = \frac{1}{3}[(1 + x^2)^{\frac{3}{2}} - x^3], \qquad 0 \leq x \leq 1$$

with the exact solution $f(x) = x$.

We solved these examples by our presented method (5) and (6), we solved these problems with different values of $h = \frac{1}{3}, \frac{1}{4}, ..., \frac{1}{10}$ to compare our results with results in [1, 10] the RMS errors in the solutions are tabulated in table 1, 2 which shows that the error in the solutions for our methods decreases by reducing the values of h, but in [10] when $h < \frac{1}{6}$ suddenly the error increased more over in [1] by reducing h error will reduced and in comparison with [10] it is better but our is more accurate too.

Table 1. The RMS errors for example 4.1 for different value of h.

h	$Methodof[10]$	$Methodof[1]$	$OurMethod$	α
$\frac{1}{3}$	2.6(-04)^2	1.3(-01)	2.8(-03)	10^{-4}
$\frac{1}{4}$	1.5(-05)	6.5(-02)	5.6(-04)	10^{-4}
$\frac{1}{5}$	7.1(-07)	2.2(-02)	3.9(-04)	10^{-6}
$\frac{1}{6}$	7.9(-07)	2.6(-03)	6.6(-05)	10^{-6}
$\frac{1}{7}$	7.0(-04)	7.7(-05)	2.8(-05)	10^{-7}
$\frac{1}{8}$	8.7(-02)	7.5(-05)	8.2(-06)	10^{-7}
$\frac{1}{9}$	2.8(00)	6.4(-05)	4.5(-06)	10^{-7}
$\frac{1}{10}$	2.5(00)	6.0(-05)	2.0(-06)	10^{-7}

Table 2. The RMS errors for example 4.2 for different value of h.

h	$Method of [10]$	$Method of [1]$	$Our Method$	α
$\frac{1}{3}$	3.8(-04)	1.1(-01)	3.3(-03)	10^{-4}
$\frac{1}{4}$	1.5(-04)	6.2(-02)	6.8(-04)	10^{-5}
$\frac{1}{5}$	8.4(-05)	3.3(-02)	3.6(-04)	10^{-5}
$\frac{1}{6}$	5.8(-05)	2.3(-02)	3.9(-04)	10^{-5}
$\frac{1}{7}$	4.7(-05)	2.8(-02)	4.0(-04)	10^{-5}
$\frac{1}{8}$	4.3(-05)	8.9(-03)	1.5(-04)	10^{-6}
$\frac{1}{9}$	3.1(-05)	1.5(-02)	1.9(-04)	10^{-6}
$\frac{1}{10}$	6.4(-06)	1.4(-02)	8.5(-05)	10^{-7}

Example 4.3

$$f(x) = e^{2x+\frac{1}{3}} - \frac{1}{3}\int_0^1 e^{2x-\frac{5t}{3}} f(t)dt, \qquad 0 \leq x \leq 1.$$

with the exact solution $f(x) = e^{2x}$.

Example 4.4

$$f(x) = -\frac{e^x(N\pi - eN\pi\cos(N\pi) + e\sin(N\pi))}{1 + N^2\pi^2} + \sin(N\pi x) + \int_0^1 e^{x+t} f(t)dt, \qquad 0 \leq x \leq 1.$$

with the exact solution $f(x) = \sin(N\pi x)$.

Now we consider the following second kind Fredholm integral equations.
We solved these examples by our presented method on (7), to compare our results with results in [2, 7], we used $n = 30$ and $h = \frac{1}{32}$ for example 4.3. The computed solutions are compared with exact solution and the absolute error in particular points are tabulated in table 3 which show that our results are more accurate in comparison with [2, 7]. And we solved problem 4.4 with different values of $h = \frac{1}{3}, ..., \frac{1}{10}$ the RMS errors in the solutions is tabulated in table 4 which shows that our method is comparison with method in [7] is more accurate and also when h is decreasing the error in [7] increased but our method in comparison with method in [2] are accurate.

Table 3. The errors $\|E\|$ in solution of example 4.3 at particular points for $h = \frac{1}{32}$.

x	$Method of [2]$	$Method of [7]$	$Our Method$
0	1.65(-04)	3.18(-02)	1.85(-11)
0.1	1.95(-04)	2.32(-02)	2.38(-11)
0.2	4.69(-04)	9.48(-03)	2.87(-11)
0.3	5.65(-04)	1.11(-02)	3.46(-11)
0.4	3.39(-04)	4.11(-02)	4.17(-11)
0.5	4.24(-04)	8.65(-02)	5.04(-11)
0.6	5.31(-04)	6.31(-02)	6.47(-11)
0.7	1.27(-03)	2.57(-02)	7.80(-11)
0.8	1.53(-03)	3.04(-02)	9.41(-11)
0.9	9.20(-04)	1.11(-01)	1.13(-10)
1	1.10(-03)	2.26(-01)	1.37(-10)

Table 4. The RMS errors in solution of example 4.4 with different values of h.

h	$Methodof[7]$	$Methodof[2]$	$OurMethod$
$\frac{1}{3}$	4.162(-01)	4.508(-03)	1.797(-04)
$\frac{1}{4}$	5.547(-01)	2.806(-03)	1.914(-04)
$\frac{1}{5}$	6.930(-01)	7.627(-03)	1.296(-02)
$\frac{1}{6}$	8.310(-01)	4.242(-03)	1.293(-03)
$\frac{1}{7}$	9.686(-01)	1.073(-02)	1.122(-02)
$\frac{1}{8}$	1.105972	5.686(-03)	2.916(-03)
$\frac{1}{9}$	1.242802	1.383(-02)	8.348(-03)
$\frac{1}{10}$	1.379134	7.114(-03)	4.690(-03)

5 Conclusions

We have shown that approximations to Fredholm integral equations of the first and second kind can be obtained by using certain simple numerical quadrature rules and collocation spline. our computed results by the suggested methods are comparable with the methods in [1, 2, 7, 10] and also we verified that the presented method can be applied with large number of n,our methods are stable and also when h is decreasing the error in the solution for our method is also decreasing.

References

[1] Babolian E., Delves L.M. (1979) "A augmented Galerkin matrix for first kind integral equations," J. Inst. Math. Appl, 157-174.

[2] Babolian E., Marzban H.R., Salmani M. (2008) "Using triangular orthogonal functions for solving Fredholm integral equations of the second kind," Appl. Math. Comput, 201, 452-464.

[3] Delves L.M., Mohamad J.L., Computational methods for integral equations, Cambridge University Press, 1985.

[4] Kajani M.T., Vencheh A.H. (2005) "Solving second kind integral equations with hybrid Chebyshev and block-pulse functions," Appl. Math. Comput, 163, 71-77.

[5] Maleknejad K., Lotfi T. (2007) "Using Wavelet for numerical solution of Fredholm integral," Proceeding of the World Congress on Engineering, Vol II.

[6] Maleknejad K., Mesgarani H., Nizad T. (2002) "Wavelet-Galerkin solution for Fredholm integral equation of the second kind," Int. J. Eng. Sci, 13, 75-80.

[7] Maleknejad K., Mirzaee F. (2005) "Using rationalized Harr wavelet for solving linear integral equations," Appl. Math. Comput, 160, 579-587.

[8] Netravali A.N., Figueiredo R.J.P. (1974) "Spline approximation to the solution of the linear Fredholm integral equation of the second kind," SIAM. J. Numer. Anal, 11, 538-549.

[9] Prenter P.M., Spline and variationl methods, Wiley-Iterscience, New-York, 1975.

[10] Rashed M.T. (2003) "An expansion method to treat integral equations," Appl. Math. Comput, 135, 65-72.

[11] Rashidinia J., Zarebnia M. (2005) "Numerical solution of linear integral equations by using Sinc-collocation method," Appl. Math. Comput, 168, 806- 822.

[12] Stenger F. (1979) "A Sinc-Galerkin method of solution of boundary value problems," Math. Comput, 33, 85-109.

[13] Wazwaz A.M., Khuri S.A. (1996) "Two methods for solving integral equations," Appl. Math. Comput, 55, 79- 89.