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Numerical solution of fifth order KdV equations by homotopy perturbation method

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Abstract

In this paper, an application of homotopy perturbation method is applied to finding the solutions of a generalized fifth order KdV (gfKdV) equation. Then we obtain the exact solitary-wave solutions and numerical solutions of the gfKdV equation for the initial conditions. The numerical solutions are compared with the known analytical solutions. Their remarkable accuracy are finally demonstrated for the gfKdV equation.

Keywords: Homotopy perturbation method, Sawada-Kotera equation, Lax's fifth order KdV equation, Solitary-wave solution.

1 Introduction

In recent years, the application of the homotopy perturbation method (**HPM**) [11, 13] in nonlinear problems has been developed by scientists and engineers, because this method continuously deforms the difficult problem under study into a simple problem which is easy to solve. The homotopy perturbation method [12], proposed first by He in 1998 and was further developed and improved by He [13, 14, 17]. The method

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yields a very rapid convergence of the solution series in the most cases. Usually, one iteration leads to high accuracy of the solution. Although goal of He's homotopy perturbation method was to find a technique to unify linear and nonlinear, ordinary or partial differential equations for solving initial and boundary value problems. Most perturbation methods assume a small parameter exists, but most nonlinear problems have no small parameter at all. A review of recently developed nonlinear analysis methods can be found in [15]. Recently, the applications of homotopy perturbation theory among scientists were appeared [1-5], which has become a powerful mathematical tool, when it is successfully coupled with the perturbation theory [13, 16, 17].
In this work we would like to implement the **HPM** to the gfKdV equation [22] which can be shown in the form

$$u_t + au^2u_x + bu_xu_{xx} + cuu_{xxx} + du_{xxxxx} = 0, \tag{1.1}$$

where a, b, c and d are constants. This equation has been known as the general form of the fifth-order KdV equation. Eq. (1.1) is known as Laxs fifth order KdV equation with $a = 30$, $b = 30$, $c = 10$ and $d = 1$ and the Sawada-Kotera equation with $a = 45$, $b = 15$, $c = 15$ and $d = 1$ [19, 21].

2 Basic idea of homotopy perturbation theory

To illustrate **HPM** consider the following nonlinear differential equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{2.1}$$

with boundary conditions:

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma, \tag{2.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function and Γ is the boundary of the domain Ω.

The operator A can be generally divided into two parts F and N, where F is linear, whereas N is nonlinear. Therefore, Eq. (2.1) can be rewritten as follows:

$$F(u) + N(u) - f(r) = 0. \tag{2.3}$$

He [18] constructed a homotopy $v : \Omega \times [0, 1] \longrightarrow \mathrm{R}$ which satisfies:

$$H(v, p) = (1 - p)[F(v) - F(v_0)] + p[A(v) - f(r)] = 0, \tag{2.4}$$

or

$$H(v, p) = F(v) - F(v_0) + pF(v_0) + p[N(v) - f(r)] = 0, \tag{2.5}$$

where $r \in \Omega$, $p \in [0, 1]$ that is called homotopy parameter, and v_0 is an initial approximation of (2.1). Hence, it is obvious that:

$$H(v, 0) = F(v) - F(v_0) = 0, \qquad H(v, 1) = A(v) - f(r) = 0, \tag{2.6}$$

and the changing process of p from 0 to 1, is just that of $H(v, p)$ from $F(v) - F(v_0)$ to $A(v) - f(r)$. In topology, this is called deformation, $F(v) - F(v_0)$ and $A(v) - f(r)$ are called homotopic. Applying the perturbation technique [20], due to the fact that $0 \le p \le 1$ can be considered as a small parameter, we can assume that the solution of (2.4) or (2.5) can be expressed as a series in p, as follows:

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots, \tag{2.7}$$

when $p \to 1$, (2.4) or (2.5) corresponds to (2.3) and becomes the approximate solution of (2.3), i.e.,

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \ldots. \tag{2.8}$$

The series (2.8) is convergent for most cases, and the rate of convergence depends on $A(v)$, [12].

3 Solution of the gfKdV equation by homotopy perturbation method

Consider the following the standard form of the gfKdV equation (1.1) in an operator form:

$$L_t(u) + a(Ku) + b(Mu) + c(Su) + dL_x(u) = 0, \tag{3.1}$$

where the notations $Ku = u^2u_x$, $Mu = u_xu_{xx}$ and $Su = uu_{xxx}$ symbolize the nonlinear term, respectively. The notation $L_t = \frac{\partial}{\partial t}$ and $L_x = \frac{\partial^5}{\partial x^5}$ symbolize the linear differential operators. Assuming the inverse of the operator L_t^{-1} exists and it can conveniently be taken as the definite integral with respect to t from 0 to t, i.e., $L_t^{-1} = \int_0^t (.)dt$. Thus, applying the inverse operator L_t^{-1} to (3.1) yields:

$$L_t^{-1}L_t(u) = -aL_t^{-1}(Ku) - bL_t^{-1}(Mu) - cL_t^{-1}(Su) - dL_t^{-1}L_x(u). \tag{3.2}$$

Therefore, it follows that:

$$u(x,t) - u(x,0) = -aL_t^{-1}(Ku) - bL_t^{-1}(Mu) - cL_t^{-1}(Su) - dL_t^{-1}L_x(u). \tag{3.3}$$

Since initial value is known and decompose the unknown function $u(x,t)$ as a sum of components defined by the decomposition series $u(x,t) = \sum_0^\infty v_n(x,t)$ with v_0 identified as $u(x,0)$.

For solving this equation by **HPM**, let $F(u) = u(x,t) - h(x,t) = 0$, where $h(x,t) = u(x,0)$. Hence, we may choose a convex homotopy such that:

$$\begin{aligned} H(v,p) &= v(x,t) - h(x,t) \\ &\quad -p\int_0^t [a(Ku(x,t)) + b(Mu(x,t)) + c(Su(x,t)) + dL_x(u(x,t))]dt = 0. \end{aligned} \tag{3.4}$$

Substituting (2.7) into (3.4) and equating the terms with identical powers of p, we have:

$p^0 : v_0(x,t) = h(x,t)$,

$$p^1 : v_1(x,t) = \int_0^t [av_0^2(v_0)_x + b(v_0)_x(v_0)_{xx} + cv_0(v_0)_{xxx} + dL_x(v_0)]dt,$$

$$p^2 : v_2(x,t) = \int_0^t [a(2v_0v_1(v_0)_x + v_0^2(v_1)_x) + b((v_0)_x(v_1)_{xx} + (v_1)_x(v_0)_{xx}) +$$

$$c(v_0(v_1)_{xxx} + v_1(v_0)_{xxx}) + dL_x(v_1)]dt,$$

$$p^3 : v_3(x,t) = \int_0^t [a(2v_0v_2(v_0)_x + v_1^2(v_0)_x + 2v_0v_1(v_1)_x + v_0^2(v_2)_x) + b((v_0)_x(v_2)_{xx} +$$

$$(v_1)_x(v_1)_{xx} + (v_2)_x(v_0)_{xx}) + c(v_0(v_2)_{xxx} + v_1(v_1)_{xxx} + v_2(v_0)_{xxx}) + dL_x(v_2)]dt,$$

$$p^4 : v_4(x,t) = \int_0^t [a(2v_0v_3(v_0)_x + 2v_1v_2(v_0)_x + v_1^2(v_1)_x + 2v_0v_2(v_1)_x + v_0^2(v_3)_x) +$$

$$b((v_0)_x(v_3)_{xx} + (v_1)_x(v_2)_{xx} + (v_2)_x(v_1)_{xx} + (v_3)_x(v_0)_{xx}) + c(v_0(v_3)_{xxx} +$$

$$v_1(v_2)_{xxx} + v_2(v_1)_{xxx} + v_3(v_0)_{xxx}) + dL_x(v_3)]dt,$$

$$\vdots$$

So we can calculate the terms of $u = \sum_{n=0}^{\infty} v_n$, term by term, otherwise by computing some terms say k, $u \approx \varphi_k = \sum_{n=0}^{k} v_n$, where $u = \lim_{k\to\infty} \varphi_k$ an approximation to the solution would be achieved.

4 Test examples

Example 1. We first consider Lax's fifth-order KdV equation [19, 21] is given with the initial condition by:

$$\begin{cases} u_t + 30u^2u_x + 30u_xu_{xx} + 10uu_{xxx} + u_{xxxxx} = 0, \\ u(x,0) = 2k^2(2 - 3\tanh^2(k(x - x_0))). \end{cases} \tag{4.1}$$

A homotopy can be readily constructed as follows:

$$u(x,t) - h(x,t) - p\int_0^t (30u^2u_x + 30u_xu_{xx} + 10uu_{xxx} + u_{xxxxx})dt = 0. \qquad (4.2)$$

Substituting (2.7) into (4.2), and equating the terms with identical powers of p, we have:

$p^0 : v_0(x,t) = h(x,t) \Rightarrow\ v_0(x,t) = 2k^2(2 - 3\tanh^2(k(x - x_0))),$

$$p^1 : v_1(x,t) = \int_0^t [30v_0^2(v_0)_x + 30(v_0)_x(v_0)_{xx} + 10v_0(v_0)_{xxx} + L_x(v_0)]dt$$

$$\Rightarrow\quad v_1(x,t) = 12k^7t\text{sech}^7(k(x - x_0))[302\sinh(k(x - x_0))$$

$$- 57\sinh(3k(x - x_0)) + \sinh(5k(x - x_0))],$$

$$p^2 : v_2(x,t) = \int_0^t [30(2v_0v_1(v_0)_x + v_0^2(v_1)_x) + 30((v_0)_x(v_1)_{xx} + (v_1)_x(v_0)_{xx})$$

$$+ 10(v_0(v_1)_{xxx} + v_1(v_0)_{xxx}) + L_x(v_1)]dt$$

$$\Rightarrow\quad v_2(x,t) = 6k^{12}t^2sech^{12}(k(x - x_0))[-7862124 + 9738114\cosh(2k(x - x_0))$$

$$- 2203488\cosh(4k(x - x_0)) + 152637\cosh(6k(x - x_0))$$

$$- 2036\cosh(8k(x - x_0)) + \cosh(10k(x - x_0))],$$

$$\vdots$$

Continuing this process the complete solution $u(x,t) = \lim_{k\to\infty}\varphi_k$ found by means of n-term approximation $\varphi_k = \sum_{n=0}^{k} v_n$. The solution $u(x,t)$ in a series form and in a close form by $u(x,t) = 2k^2(2 - 3\tanh^2(k(x - 56k^4t - x_0)))$. This result can be verified through substitution.

Example 2. We second consider the Sawada-Kotera equation [19, 21] with the initial condition given by:

$$\begin{cases} u_t + 45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx} = 0, \\ u(x,0) = 2k^2(2 - 3\text{sech}^2(k(x-x_0))). \end{cases} \tag{4.3}$$

Since initial value is known and decompose the unknown function $u(x,t)$ a sum of components defined by the decomposition series $u = \sum_{n=0}^{\infty} v_n$, with u_0 identified as $u(x,0)$.

A homotopy can be readily constructed as follows:

$$u(x,t) - h(x,t) - p\int_0^t (45u^2u_x + 15u_xu_{xx} + 15uu_{xxx} + u_{xxxxx})dt = 0. \tag{4.4}$$

Substituting (2.7) into (4.4), and equating the terms with identical powers of p, we have:

$p^0 : v_0(x,t) = h(x,t) \Rightarrow \; v_0(x,t) = 2k^2(2 - 3\text{sech}^2(k(x-x_0))),$

$$p^1 : v_1(x,t) = \int_0^t [45v_0^2(v_0)_x + 15(v_0)_x(v_0)_{xx} + 15v_0(v_0)_{xxx} + L_x(v_0)]dt \Rightarrow$$

$v_1(x,t) = 4k^7t\text{sech}^7(k(x-x_0))[302\sinh(k(x-x_0)) - 57\sinh(3k(x-x_0)) + \sinh(5k(x-x_0))],$

$$p^2 : v_2(x,t) = \int_0^t [30(2v_0v_1(v_0)_x + v_0^2(v_1)_x) + 30((v_0)_x(v_1)_{xx} + (v_1)_x(v_0)_{xx})$$

$$+ 10(v_0(v_1)_{xxx} + v_1(v_0)_{xxx}) + L_x(v_1)]dt$$

$$\Rightarrow \quad v_2(x,t) = 2k^{12}t^2\text{sech}^{12}(k(x-x_0))[-7862124 + 9738114\cosh(2k(x-x_0))$$

$$- 2203488\cosh(4k(x-x_0)) + 152637\cosh(6k(x-x_0))$$

$$-2036\cosh(8k(x-x_0))+\cosh(10k(x-x_0))],$$

$$\vdots$$

Continuing this process the complete solution $u(x,t)=\lim_{k\to\infty}\varphi_k$ found by means of n-term approximation $\varphi_k=\sum_{n=0}^{k}v_n$. The solution $u(x,t)$ in a series form and in a close form by $u(x,t)=2k^2\text{sech}^2(k(x-16k^4t-x_0)))$ which can be easily verified.

5 Numerical experiments

In this section, we consider two gfKdV equations for numerical comparisons. Based on the **HPM**, we constructed the solution $u(x,t)$ as $u\approx\varphi_k=\sum_{n=0}^{k}v_n$, where $u=\lim_{k\to\infty}\varphi_k$. In this Letter, we demonstrate how the approximate solutions of the gfKdV equations are close to exact solutions. In order to verify numerically whether the proposed methodology lead to higher accuracy, we can evaluate the numerical solutions using the n-term approximation. Tables 1 and 2 show the difference of the analytical solution and numerical solution of the absolute errors. It is to be note that 10 terms only were used in evaluating the approximate solutions. We achieved a very good approximation with the actual solution of the equations by using 10 terms only of the decomposition derived above. It is evident that the overall errors can be made smaller by adding new terms of the decomposition series.

6 Conclusion

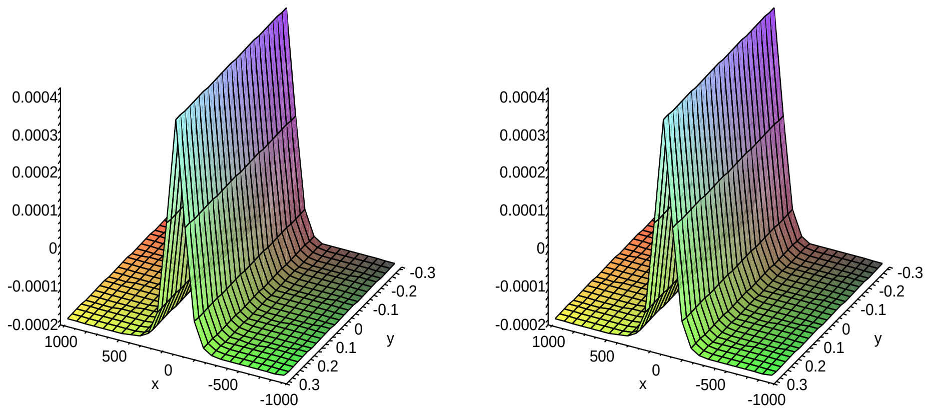
In this work, we successfully apply the homotopy perturbation method to approximate the solution of fifth order KdV equations. It gives a simple and a powerful mathematical tool for nonlinear problems. In our work, we use the Maple Package to calculate the series obtained from the iteration method.

Table 1: Numerical results for $|u(x,t) - \varphi_{10}(x,t)|$ where $u(x,t) = 2k^2(2 - 3\tanh^2(k(x - 56k^4t - x_0)))$ for Eq. (4.1).

$t_i \backslash x_i$	0.1	0.2	0.3	0.4	0.5
0.1	9.60e-12	9.60e-12	9.60e-12	9.50e-12	9.60e-12
0.2	1.92e-11	1.92e-11	1.92e-11	1.91e-11	1.92e-11
0.3	2.88e-11	2.88e-11	2.88e-11	2.87e-11	2.88e-11
0.4	3.84e-11	3.84e-11	3.84e-11	3.83e-11	3.84e-11
0.5	4.78e-11	4.80e-11	4.80e-11	4.79e-11	4.80e-11

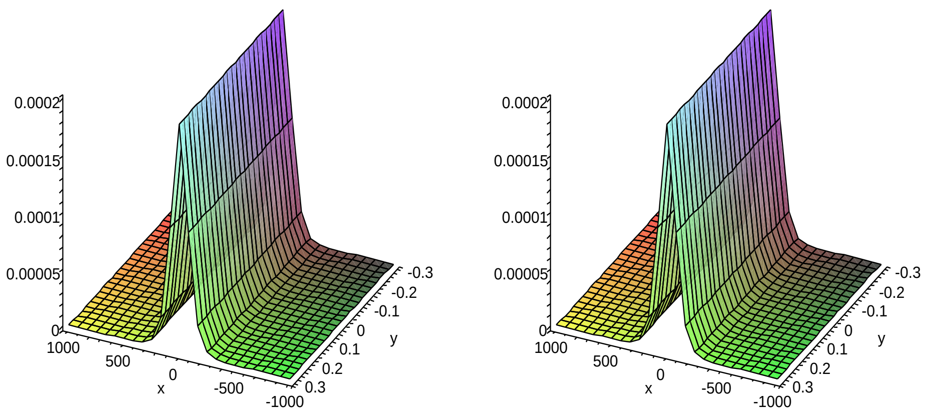
Table 2: Numerical results for $|u(x,t) - \varphi_{10}(x,t)|$ where $u(x,t) = 2k^2 sech^2(k(x - 16k^4t - x_0)))$ for Eq. (4.3).

$t_i \backslash x_i$	0.1	0.2	0.3	0.4	0.5
0.1	4.800E-16	9.600E-16	1.440E-15	1.920E-15	2.400E-14
0.2	9.600E-16	1.920E-15	2.880E-15	3.840E-15	4.800E-14
0.3	1.440E-15	2.880E-15	4.320E-15	5.760E-15	7.200E-14
0.4	1.920E-15	3.840E-15	5.760E-15	7.680E-15	9.600E-14
0.5	2.400E-15	4.800E-15	7.200E-15	9.599E-15	1.200E-14



(*a*) (*b*)

Figure 1: The numerical results for $\varphi_{10}(x,t)$: (a) for $x_0 = 0$ and $k = 0.01$ in comparison with the analytical solutions $u(x,t)$: (b) for the solution with the equation (4.1)



(*a*) (*b*)

Figure 2: The numerical results for $\varphi_{10}(x,t)$: (a) for $x_0 = 0$ and $k = 0.01$ in comparison with the analytical solutions $u(x,t)$: (b) for the solution with the equation (4.3)

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