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An efficient computational method for the system of linear Volterra integral equations by means of hybrid functions

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Abstract

In this paper hybrid functions which consist of Block-Pulse functions plus Legendre polynomials are developed to approximate the solution of the linear Volterra integral equations system that arises in the elastodynamic problems. Properties of these hybrid functions are first presented, the operational matrix of integration and the product operational matrix are utilized to reduce the computation of Volterra integral equations system to some algebraic equations. Finally numerical results which we compared them with some existed method are given to showing the profit and efficiency of the proposed method.

Keywords: Hybrid function, system of Volterra integral equations, Operational matrix of integration, product operational matrix, Block-Pulse functions, Legendre polynomials.

1 Introduction

Mathematical modeling for many problems in different disciplines, such as engineering, chemistry, physics and biology leads to integral equation, or system of integral equations. It's the reason of great interest for solving these equations.

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We consider the following system of linear Volterra integral equations:

$$\mathcal{G}(x)\mathcal{U}(x) + \int_0^x \mathcal{K}(x,s)\mathcal{U}(s)ds = \mathcal{F}(x); \quad i = 1,2,...,q, \tag{1}$$

where

$$\mathcal{U}(x) = [u_1(x),...,u_q(x)]^T, \mathcal{F}(x) = [f_1(x),...,f_q(x)]^T, \tag{2}$$

and

$$\mathcal{G}(x) = \begin{bmatrix} g_{11}(x) & \cdots & g_{1q}(x) \\ \vdots & \ddots & \vdots \\ g_{q1}(x) & \cdots & g_{qq}(x) \end{bmatrix},$$

$$\mathcal{K}(x,s) = \begin{bmatrix} k_{11}(x,s) & \cdots & k_{11}(x,s) \\ \vdots & \ddots & \vdots \\ k_{q1}(x,s) & \cdots & k_{qq}(x,s) \end{bmatrix},$$

where the functions $g_{ij}(t), f_i(x) \in L^2[0,1)$ and the kernels $k_{ij}(x,s) \in L^2([0,1) \times [0,1))$ for $i,j = 1,2,...,q$ are known and $u_i(x)$ for $i = 1,2,...,q$ are the solutions to be determined. The theory on existence and uniqueness of a continuous solution for such equations was already established by Volterra and Brunner [1, 2].

For some specific problems that lead to the system of linear Volterra integral equations, we can mention mathematical model of linear quasi-static visco-elasticity problem [2], magneto-electro-elastic dynamic problems [3] and the elastodynamic problems of piezoelectric [4]. Some existed numerical methods for approximating the solution of Eq.(1) are as follows. Maleknejad, Rabbani and Aghazadeh in [5] used expansion method to solve Volterra integral equations system of the second kind, in [6] Mirzaee obtained a numerical solution of these equations by using rationalized Haar functions, Saeed and Ahmed in [7] produced a method for numerical solution of the System of linear Volterra Integral equations of the second kind using Monte-Carlo method , in [8] Biazar and Pourabd solved these system of integral equations numerically based on Adomian decomposition method, Maleknejad and Salimi Shamloo in [9] solved singular Volterra integral equations system of convolution type by using operational matrices of Block-Pulse functions.

In this paper we use hybrid functions which consists of Block-Pulse functions and Legendre polynomials as basis. Beforehand this method has been used for system of Fredholm integral equations [10]. The main advantage of this basis is their efficiency and simple applicability that is based on some useful properties of hybrid functions, such as operational matrix, a special product matrix and a related coefficient matrix. These matrices are applied to convert these system of linear Volterra integral equations into linear algebraic equations and in this way, the solution procedures are either reduced or simplified, accordingly. The other advantage of hybrid functions is that the values of n and m are adjustable as well as being able to yield more accurate numerical solutions than the other function's method [11, 12].

The paper is organized as follows. In Section 2 we presented some properties of hybrid Legendre and Block-Pulse function and introduced some useful operational matrices of these functions. In Section 3 we implemented the hybrid function method on the system of linear Volterra integral equations and convert them to a linear algebraic system of equations. In Section 4 we give some computable error bounds for system of linear Volterra integral equations. Section 5 presents numerical examples that shows the efficiency and accuracy of proposed method in analogy to some existed method. Finally Section 6 concludes the paper.

2 Some properties of hybrid functions

2.1 Definition of hybrid function of Legendre and Block-Pulse

Consider the Legendre polynomials $L_m(x)$ on the interval $[-1,1]$: $L_0(x)=1, \;\; L_1(x)=x$,

$$L_{m+1}(x)=\frac{2m+1}{m+1}xL_m(x)-\frac{m}{m+1}L_{m-1}(x), \quad m=1,2,....$$

The set $\{L_m(x) \;:\; m=0,1,...\}$ in Hilbert space $L^2[-1,1]$ is a complete orthogonal set. A set of Block-Pulse functions $\phi_i(x), i=1,2,...,n$ and the orthogonal set of hybrid functions $h_{ij}(x),\; i=1,2,...,n$ and $j=0,1,...,m-1$, that produces by Legendre polynomials and Block-Pulse functions on $[0,1)$ are defined as follows:

$$\phi_i(x)=\begin{cases} 1, & \frac{i-1}{n}\le x<\frac{i}{n} \\ 0, & otherwise \end{cases}, \tag{3}$$

$$h_{ij}(x)=\begin{cases} L_j(2nx-2i+1), & \frac{i-1}{n}\le x<\frac{i}{n} \\ 0, & otherwise \end{cases}, \tag{4}$$

where n and m are the order of Block-Pulse functions and Legendre polynomials, respectively, and x is the normalized time.

2.2 Function approximation

Any function $u(x)\in L^2[0,1)$ can be expanded in hybrid functions as

$$u(x)=\sum_{i=1}^{\infty}\sum_{j=0}^{\infty}c_{ij}h_{ij}(x), \tag{5}$$

where the hybrid coefficients are given by $c_{ij}=\frac{(u(x),h_{ij}(x))}{(h_{ij}(x),h_{ij}(x))}$ for $i=1,2,...,\infty,\; j=0,1,...,\infty$, so that, $(\cdot,\cdot)$ denotes the inner product.

Usually, the series expansion Eq.(5) contains an infinite number of terms for a smooth $u(x)$. If $u(x)$ is piecewise constant or may be approximated as piecewise constant, then the sum in Eq.(5) may be terminated after nm terms, that is

$$u(x) \simeq \sum_{i=1}^{n} \sum_{j=0}^{m-1} c_{ij} h_{ij}(x) = C^T \mathbf{h}(x), \tag{6}$$

where

$$C = [c_{10}, ..., c_{1,m-1}, c_{20}, ..., c_{2,m-1}, ..., c_{n0}, ..., c_{n,m-1}]^T, \tag{7}$$

$$\mathbf{h}(x) = [h_{10}(x), ..., h_{1m-1}(x), h_{20}(x), ..., h_{2m-1}(x), ..., h_{nm-1}(x)]^T. \tag{8}$$

We can also approximate the function $k(x, s) \in L^2([0, 1) \times [0, 1))$ as follows

$$k(x, s) \simeq \mathbf{h}^T(x) K \mathbf{h}(s), \tag{9}$$

where K is an $nm \times nm$ matrix that $K_{ij} = \frac{\left(\mathbf{h}_{(i)}(x),\left(k(x,s),\mathbf{h}_{(j)}(s)\right)\right)}{\left(\mathbf{h}_{(i)}(x),\mathbf{h}_{(i)}(x)\right)\left(\mathbf{h}_{(j)}(s),\mathbf{h}_{(j)}(s)\right)}$ for $i, j = 1, 2, ..., nm$.

2.3 Operational matrix of integration

The integration of the vector $\mathbf{h}(x)$ defined in Eq.(8) is given by

$$\int_0^x \mathbf{h}(x')dx' \simeq P\mathbf{h}(x), \tag{10}$$

where P is the $nm \times nm$ operational matrix for integration and is given in [13] as

$$P = \begin{bmatrix} E & H & H & \dots & H \\ O & E & H & \dots & H \\ O & O & E & \dots & H \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \dots & E \end{bmatrix}, \tag{11}$$

that E and H are $m \times m$ matrices that have the following shapes,

$$H = \frac{1}{n} \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{12}$$

$$E=\frac{1}{2n}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & \frac{1}{5} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{7} & 0 & \frac{1}{7} & \cdots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{9} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2m-9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{-1}{2m-7} & 0 & \frac{1}{2m-7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-1}{2m-5} & 0 & \frac{1}{2m-5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{-1}{2m-3} & 0 & \frac{1}{2m-3} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{-1}{2m-1} & 0 \end{bmatrix}. \quad (13)$$

2.4 Product operational matrix

It is always necessary to evaluate the product of $\mathbf{h}(x)$ and $\mathbf{h}^T(x)$, that is called the product matrix of hybrid functions. Let

$$\mathbf{H}(x)=\mathbf{h}(x)\mathbf{h}^T(x), \quad (14)$$

where $\mathbf{H}(x)$ is an $nm\times nm$ matrix. By multiplying the matrix $\mathbf{H}(x)$ in the vector C that defined in Eq.(7) we obtain

$$\mathbf{H}(x)C=\widetilde{C}\mathbf{h}(x), \quad (15)$$

where $\widetilde{C}$ is an $nm\times nm$ matrix and is called the coefficient matrix. Basic multiplication properties of arbitrary two hybrid function $h_{ij}(x)$ and $h_{kl}(x)$ are described in [13]. When $n=2$ and $m=8$ we have the components of $\widetilde{C}$ as follows

$$\widetilde{C}=\begin{bmatrix} \widetilde{C_1} & O \\ O & \widetilde{C_2} \end{bmatrix},$$

where $C_i, i=1,2$ are 8×8 matrices given by

$$\widetilde{C_i} = \begin{bmatrix} c_{i0} & c_{i1} & c_{i2} & \cdots & c_{i7} \\ 1/3c_{i1} & \begin{array}{c} c_{i0} \\ +2/5c_{i2} \end{array} & \begin{array}{c} 2/3c_{i1} \\ +3/7c_{i3} \end{array} & \cdots & 7/13c_{i6} \\ 1/5c_{i2} & \begin{array}{c} 2/5c_{i1} \\ +9/35c_{i3} \end{array} & \begin{array}{c} c_{i0} \\ +2/7c_{i2} \\ +2/7c_{i4} \end{array} & \cdots & \begin{array}{c} 63/143c_{i5} \\ +56/221c_{i7} \end{array} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1/15c_{i7} & 7/65c_{i6} & \begin{array}{c} 21/143c_{i5} \\ +56/663c_{i7} \end{array} & \cdots & \begin{array}{c} c_{i0} \\ +56/221c_{i2} \\ +6804/46189c_{i4} \\ +5000/46189c_{i6} \end{array} \end{bmatrix}. \tag{16}$$

3 Implementation of hybrid function method on system of linear Volterra integral equations

Consider the system of linear Volterra integral equation (1), we can present that equation by the following form:

$$\sum_{j=1}^{q} g_{ij}(x)u_i(x) + \sum_{j=1}^{q} \int_0^x k_{ij}(x,s)u_j(s)ds = f_i(x); \quad i = 1, 2, ..., q. \tag{17}$$

We put

$$u_i(x) \simeq U_i^T \mathbf{h}(x), \quad i = 1, ..., q, \tag{18}$$

where U_i for $i = 1, ..., q$ are unknown nm–vectors and $\mathbf{h}(x)$ is given by Eq.(8). Likewise, $g_{ij}(x), k_{ij}(x,s)$ and $f_i(x)$ for $i, j = 1, ..., q$ are expanded into the hybrid functions as follows

$$k_{ij}(x,s) \simeq \mathbf{h}^T(x)K_{ij}\mathbf{h}(s), \; g_{ij}(x) \simeq G_{ij}^T \mathbf{h}(x), \quad i, j = 1, ..., q, \tag{19}$$

$$f_i(x) \simeq F_i^T \mathbf{h}(x), \quad i = 1, ..., q, \tag{20}$$

where K_{ij} for $i,j = 1,...,q$ are known $nm \times nm$–matrices and G_{ij}, F_i for $i,j = 1,...,q$, are known nm–vectors.

After substituting the approximate equations (18), (19), (20) in (17) we get

$$\sum_{j=1}^{q} \left(G_{ij}^T \mathbf{h}(x)\mathbf{h}^T(x)U_j \right) + \sum_{j=1}^{q} \int_0^x \mathbf{h}^T(x)K_{ij}\mathbf{h}(s)\mathbf{h}^T(s)U_j ds = \mathbf{h}^T(x)F_i, \; i = 1,...,q, \quad (21)$$

by using Eqs.(10) and (15) we can convert Eqs.(21) to the following equations,

$$\sum_{j=1}^{q} \mathbf{h}^T(x)\widetilde{G_{ij}}U_j + \sum_{j=1}^{q} \mathbf{h}^T(x)K_{ij}\widetilde{U_j}P\mathbf{h}(x) = \mathbf{h}^T(x)F_i, \; i = 1,...,q, \quad (22)$$

now we have q equations with $q \times n \times m$ unknowns $U_1, U_2, ..., U_q$ (each of these vectors have nm unknowns).

In order to find U_i for $i = 1,...,q$ we collocate each of Eqs.(22) in nm points $x_p, \; p = 1,...,nm$ in the interval $[0,1]$ that are roots of shifted Legendre polynomial $L_{nm}(2x-1)$ [14]. Then we have following system of linear equations

$$\sum_{j=1}^{q} \mathbf{h}^T(x_p)\widetilde{G_{ij}}U_j + \sum_{j=1}^{q} \mathbf{h}^T(x_p)K_{ij}\widetilde{U_j}P\mathbf{h}(x_p) = \mathbf{h}^T(x_p)F_i, \; i = 1,...,q, \; p = 1,\ldots,nm. \quad (23)$$

After solving above linear system we can achieve U_i for $i = 1,...,q$, then we will have our unknown $u_i(x)$ as $U_i^T\mathbf{h}(x)$ for $i = 1,...,q$, that are the approximate solution for our system of linear Volterra integral equations (1).

4 Error analysis

We assume throughout this paper, all functions are continuously differentiable finitely or infinitely.

Theorem 4.1 *Let $u(x) \in H^k(-1,1)$ (Sobolev space), $u_J(x) = \sum_{i=0}^{J} a_i L_i(x)$ be the best approximation polynomial of $u(x)$ in L^2, then*

$$\|u(x) - u_J(x)\|_{L^2[-1,1]} \leq C_0 J^{-k} \|u(x)\|_{H^k(-1,1)}, \quad (24)$$

where C_0 is a positive constant, which depends on the selected norm and is independent of $u(x)$ and J, see [15].

We denote the $u_{nm}(x)$ and $u(x)$ show the approximate and exact solutions of the integral equations respectively.

Theorem 4.2 *Let $u(x) \in H^k(0,1)$, $I_i = [\frac{i-1}{n}, \frac{i}{n}]$ then*

$$\|u(x) - u_{nm}(x)\|_{L^2[0,1]} \leq C_0(mn)^{-k}\max_{0\leq i\leq n}\|u(x)\|_{H^k(I_i)}, \tag{25}$$

Proof By using Theorem 4.1 it is obvious.

If $u(x)$ is approximated by $u_{nm}(x) \simeq \sum_{i=1}^{n}\sum_{j=0}^{m-1} c_{ij}h_{ij}(x)$ and we compute $\overline{c}_{ij}$ so that $\overline{c}_{ij}$ is approximation of c_{ij} and $\overline{u}_{nm}(x) \simeq \sum_{i=1}^{n}\sum_{j=0}^{m-1} \overline{c}_{ij}h_{ij}(x)$, then for $x \in [\frac{i-1}{n}, \frac{i}{n}]$ we get

$$\|u(x)-\overline{u}_{nm}(x)\| = \|u(x)-u_{nm}(x)+u_{nm}(x)-\overline{u}_{nm}(x)\| \leq \|u(x)-u_{nm}(x)\|+\|u_{nm}(x)-\overline{u}_{nm}(x)\|, \tag{26}$$

We have

$$\|u_{nm}(x)-\overline{u}_{nm}(x)\| = \Big[\int_0^1 (u_{nm}(x)-\overline{u}_{nm}(x))^2 dx\Big]^{\frac{1}{2}} = \Big[\int_0^1 (\sum_{i=1}^{n}\sum_{j=0}^{m-1}(c_{ij}-\overline{c}_{ij})h_{ij}(x))^2 dx\Big]^{\frac{1}{2}}. \tag{27}$$

The hybrid functions set $\{h_{ij}(x)\}$ is an orthogonal set and

$$\mathbf{D} = \int_0^1 \mathbf{h}(x)\mathbf{h}^T(x)dx = \begin{bmatrix} \mathbf{L} & 0 & \dots & 0 \\ 0 & \mathbf{L} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{L} \end{bmatrix}, \tag{28}$$

where $\mathbf{L}$ is a $m \times m$ diagonal matrix that is given by

$$\mathbf{L} = \frac{1}{n}\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & \frac{1}{3} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{2m-1} \end{bmatrix}. \tag{29}$$

Then from Eq.(27)and (28) we get

$$\|u_{nm}(x) - \overline{u}_{nm}(x)\| \leq \Big[\sum_{i=1}^{n}\sum_{j=0}^{m-1}\frac{1}{n(2j+1)}(c_{ij} - \overline{c}_{ij})\Big]^{\frac{1}{2}}. \quad (30)$$

If we use the Gaussian points that are the roots of shifted Legendre polynomials for approximation of c_{ij} [16, 17], we have

$$|c_{ij} - \overline{c}_{ij}| \leq C_1(nm)^{-(k-1)}, \quad (31)$$

we have the sum of $\sum_{j=0}^{m-1}\frac{1}{(2j+1)}$ as

$$\ln 2 + \frac{1}{2}(\gamma + \psi(\frac{1}{2} + m)),$$

where γ is the Euler's constant and its approximate solution is about 0.577216 and the function ψ is the logarithmic derivative of the gamma function that is defined as follows

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) + \frac{\Gamma'(x)}{\Gamma(x)},$$

and its maximum value on the interval $[0, 1]$, is one. So we have

$$\sum_{i=1}^{n}\sum_{j=0}^{m-1}\frac{1}{n(2j+1)} = \sum_{i=1}^{n}\frac{1}{n}\sum_{j=0}^{m-1}\frac{1}{(2j+1)} = \sum_{j=0}^{m-1}\frac{1}{(2j+1)} \leq \ln 2+\frac{1}{2}(0.575216+1) = 1.48176,$$

so by use of above relation and Eq. (30) we have

$$\|u_{nm}(x) - \overline{u}_{nm}(x)\| \leq C_2(nm)^{\frac{-k+1}{2}}(1.48176)^{\frac{1}{2}}, \quad (32)$$

then from Eqs.(25), (26) and (32) we found the error bound

$$\|u(x) - \overline{u}_{nm}(x)\| \leq C_3(nm)^{\frac{-k+1}{2}} + C_0(mn)^{-k}\max_{0\leq i\leq n}\|u(x)\|_{H^k(I_i)}. \quad (33)$$

Let $e^i_{nm}(x) = \|u_i(x) - u^i_{nm}(x)\|$ be the error function of approximate solution $u^i_{nm}(x)$ to the exact solution $u_i(x)$, then we consider

$$\sum_{j=1}^{q} g_{ij}(x)u^i_{nm}(x) + \sum_{j=1}^{q}\int_0^x k_{ij}(x,s)u^j_{nm}(s)ds = f_i(x) + R^i_{nm}(x); \quad i = 1, 2, ..., q, \quad (34)$$

$R^i_{nm}(x)$ is the perturbation functions that depends only on $u^i_{nm}(x)$ and we can obtain it by subtracting Eqs.(34) form (1),

$$\sum_{j=1}^{q} g_{ij}(x)e^i_{nm}(x) + \sum_{j=1}^{q}\int_0^x k_{ij}(x,s)e^j_{nm}(s)ds = R^i_{nm}(x); \quad i=1,2,...,q. \qquad (35)$$

If we assume $\mathcal{M}_{ij} = sup_{0\leq x,s\leq 1} \mid k_{ij}(x,s) \mid < \infty$ and $\mathcal{W}_{ij} = sup_{0\leq x\leq 1} \mid g_{ij}(x) \mid < \infty$ then each perturbation function $R^i_{nm}(x)$ is bounded

$$\|R^i_{nm}(x)\| \leq \sum_{j=1}^{q} \mathcal{W}_{ij}(x)e^i_{nm}(x) + \sum_{j=1}^{q} \mathcal{M}_{ij}e^j_{nm}(s); \quad i=1,2,...,q. \qquad (36)$$

and for $i=1,2,...,q$

$$\|R^i_{nm}(x)\| \leq \sum_{j=1}^{q}(\mathcal{W}_{ij}+\mathcal{M}_{ij})\Big(C_3(nm)^{\frac{-k+1}{2}} + C_0(mn)^{-k}\max_{0\leq i\leq n}\|u(x)\|_{H^k(I_i)}\Big). \qquad (37)$$

5 Numerical illustrations

Example 5.1 *For the first example, consider the following Volterra system of integral equations:*

$$\begin{cases} u_1(x) - \int_0^x u_2(s)ds = 1 - x^2 \\ u_2(x) - \int_0^x u_1(s)ds = x \end{cases},$$

the exact solution is $u_1(x) = 1$, $u_2(x) = 2x$. Table 1 given the absolute errors for $u_1(x)$ and $u_2(x)$ by hybrid function method with comparison by absolute errors of rationalized Haar function method [6].

Example 5.2 *For the second example, consider the following Volterra system of integral equations:*

$$\begin{cases} u_1(x) - \int_0^x (s^2 - x)(u_1(s)+u_2(s))ds = x + \frac{1}{2}x^3 + \frac{1}{12}x^4 - \frac{1}{5}x^5 \\ u_2(x) - \int_0^x s(u_1(s)+u_2(s))ds = x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 \end{cases},$$

with the exact solution $u_1(x) = x$, $u_2(x) = x^2$. Table 2 shows the absolute errors for our proposed method and the comparison with absolute errors by rationalized Haar function method.

Table 1. *Absolute errors for example 1.*

	Absolute errors for $u_1(x)$		*Absolute errors for* $u_2(x)$	
x	*Present method with* $n = 2, m = 8$	*Method in [6] with* $k = 32$	*Present method with* $n = 2, m = 8$	*Method in [6] with* $k = 32$
0.0	0.0×10^{-16}	0.0×10^{-4}	0.0×10^{-16}	0.0×10^{-4}
0.1	0.0×10^{-16}	0.9×10^{-4}	0.0×10^{-16}	0.1×10^{-4}
0.2	0.0×10^{-16}	0.6×10^{-4}	0.0×10^{-16}	0.7×10^{-4}
0.3	0.0×10^{-16}	0.2×10^{-4}	0.0×10^{-16}	0.4×10^{-4}
0.4	0.0×10^{-16}	0.3×10^{-4}	0.0×10^{-16}	0.5×10^{-4}
0.5	0.0×10^{-16}	0.2×10^{-4}	0.0×10^{-16}	0.1×10^{-4}
0.6	0.0×10^{-16}	0.6×10^{-4}	0.0×10^{-16}	0.2×10^{-4}
0.7	0.0×10^{-16}	0.9×10^{-4}	0.0×10^{-16}	0.8×10^{-4}
0.8	0.0×10^{-16}	0.1×10^{-4}	0.0×10^{-16}	0.8×10^{-4}
0.9	0.0×10^{-16}	0.9×10^{-4}	0.0×10^{-16}	0.2×10^{-4}
1.0	0.0×10^{-16}	0.1×10^{-4}	0.0×10^{-16}	0.8×10^{-4}

Table 2. *Absolute errors for example 2.*

	Absolute errors for $u_1(x)$		*Absolute errors for* $u_2(x)$	
x	*Present method with* $n = 2, m = 8$	*Method in [6] with* $k = 32$	*Present method with* $n = 2, m = 8$	*Method in [6] with* $k = 32$
0.0	0.0×10^{-16}	0.0×10^{-4}	0.0×10^{-16}	0.0×10^{-4}
0.1	0.0×10^{-16}	0.8×10^{-4}	0.0×10^{-16}	0.8×10^{-4}
0.2	0.0×10^{-16}	0.8×10^{-4}	0.0×10^{-16}	0.2×10^{-4}
0.3	0.0×10^{-16}	0.8×10^{-4}	0.0×10^{-16}	0.2×10^{-4}
0.4	0.0×10^{-16}	0.8×10^{-4}	0.0×10^{-16}	0.8×10^{-4}
0.5	0.0×10^{-16}	0.9×10^{-4}	0.0×10^{-16}	0.5×10^{-4}
0.6	0.0×10^{-16}	0.8×10^{-4}	0.0×10^{-16}	0.9×10^{-4}
0.7	0.0×10^{-16}	0.8×10^{-4}	0.0×10^{-16}	0.5×10^{-4}
0.8	0.0×10^{-16}	0.8×10^{-4}	0.0×10^{-16}	0.5×10^{-4}
0.9	0.0×10^{-16}	0.8×10^{-4}	0.0×10^{-16}	0.9×10^{-4}
1.0	0.0×10^{-16}	0.1×10^{-4}	0.0×10^{-16}	0.3×10^{-4}

6 Conclusion

In this work we solved a system of linear Volterra integral equations via hybrid legendre and Block-Pulse functions. By some useful properties of these hybrid functions such as, operational matrix, product matrix and coefficient matrix together with collocation method, a Volterra system of integral equations can be transformed to a linear system of algebraic equations. Illustrative examples are given to demonstrate the high validity and applicability of proposed method and our answers compared with the answers of some existed method.

References

[1] Volterra V. (1896) "Sulla inversione degli integrali multipli," Rend. R. Accad. Lincei, 5(5), 289-300.

[2] Brunner H., Collocation methods for Volterra integral and related functional equation, Cambridge University Press, Cambridge, 2004.

[3] Hao-jiang D., Hui-ming W. (2005) "Numerical approach for a system of second kind Volterra integral equations in magneto-electro-elastic dynamic problems," J. Zhejiang Univ SCI, 6A(9), 928-932.

[4] Hao-jiang D., Hui-ming W., Wei-qiu C. (2004) "Sulla inversione degli integrali multipli," Appl. Math. Mech., 25(1), 16-23.

[5] Rabbani M., Maleknejad K., Aghazadeh N. (2007) "Numerical computational solution of the Volterra integral equations system of the second kind by using an expansion method," Appl. Math. Comput., 1143-1146.

[6] Mirzaee F. (2010) "Numerical computational solution of the linear Volterra integral equations system via rationalized Haar functions," Journal of King Saud University (Science), 22, 265-268.

[7] Saeed R.K. , Ahmed C.S. (2008) "Numerical solution of the system of linear Volterra integral equations of the second kind using Monte- Carlo method," J. Appl. Sci. Res. 4(10), 1174-1181.

[8] Biazar J., Pourabd M. (2007) "A Maple Program for Solving Systems of Linear and Nonlinear Integral Equations by Adomian Decomposition Method," Int. J. Contemp. Math. Sciences 2(29), 1425-1432.

[9] Maleknejad K., Salimi Shamloo A. (2008) "Numerical solution of singular Volterra integral equations system of convolution type by using operational matrices," Appl. Math. Comput. 195, 500-505.

[10] Rabbani M., Nouri K. (2010) "Solution of Integral Equations by Using Block-Pulse Functions," Mathematical Sciences: Quarterly Journal (MSQJ) 4(1), 39-48.

[11] Maleknejad k., Basirat B., Hashemizadeh E. (2011) "Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra-Fredholm integro-differential equations," J.Comput. Math. Appl. 61(9), 2821-2828.

[12] Maleknejad k., Hashemizadeh E. (2011) "A numerical approach for Hammerstein integral equations of mixed type using operational matrices of hybrid functions," Scientific Bulletin, Series A Applied Mathematics and Physics 73(3), 95-104.

[13] Hsiao C.H. (2009) "Hybrid function method for solving Fredholm and Volterra integral equations of the second kind," J. Comput. Appl. Math. 230, 59-68.

[14] Saadatmandi A., Dehghan M. (2010) "A new operational matrix for solving fractional-order differential equations," J. Comput. Math. Appl. 59, 1326-1336.

[15] Canuto C., Hussaini M.Y., Quarteroni A., Zang T.A., Spectral Methods on Fluid Dynamics, Springer Verlag, 1988.

[16] Delves L.M., Mohamed J.L., Computational Methods for Integral Equations, Cambridge University Press, Cambridge, 1985.

[17] Maleknejad K., Sohrabi S., Derili H. (2010) "A new computational method for solution of non-linear Volterra-Fredholm integro-differential equations," Int. J. Appl. Math., 87(2), 327-338.