

ORIGINAL RESEARCH

Open Access

Approximation by complex Favard-Szász-Mirakjan-Stancu operators in compact disks

Vijay Gupta^{1*} and Durvesh K Verma²

Abstract

Purpose: The purpose of the present paper is to study the Stancu-type generalization of complex Favard-Szász-Mirakjan operators and establish some approximation results in complex domain.

Methods: It is observed that the complex Favard-Szász-Mirakjan-Stancu operators can be written in the form of divided differences. Thus, it is possible to study such operators in complex domain. We use analytical method to obtain our results.

Results: We have estimated the order of simultaneous approximation, Voronovskaja-type results with quantitative estimates for the complex Favard-Szász-Mirakjan-Stancu operators attached to analytic functions on compact disks. Also, some estimates on the rate of convergence are given.

Conclusions: The results proposed here are new and have better rate of convergence.

Keywords: Complex Favard-Szász-Mirakjan-Stancu operators, Voronovskaja-type result, Exact order of approximation in compact disks, Simultaneous approximation

Mathematical Classification Subject: 30E10; 41A25; 41A28

Introduction

The Favard-Szász-Mirakjan operators are important and have been studied intensively, in connection with different branches of analysis, such as numerical analysis, approximation theory statistics etc. For a real function f of real variable $f : [0, \infty) \rightarrow \mathbb{R}$, the Favard-Szász-Mirakjan operators are defined as follows:

$$S_n(f, x) = e^{-nx} \sum_{v=0}^{\infty} \frac{(nx)^v}{v!} f\left(\frac{v}{n}\right), \quad x \in [0, \infty),$$

where the convergence of $S_n(f, x) \rightarrow f(x)$ under the exponential growth condition on f that is $|f(x)| \leq Ce^{Bx}$ for all $x \in [0, +\infty)$, with $C, B > 0$ was established in [1]. The actual construction of the Szász-Mirakjan operators and its various modifications require estimations of infinite series which, in a certain sense, restrict their usefulness

from the computational point of view. Recently Walczak [2] proposed and studied the Szász-Mirakjan operators by considering a finite sum.

Recently, Gal in his famous book [3] estimated Voronovskaja-type results with quantitative estimates for several complex operators. He also established exact order of simultaneous approximation by such complex operators. The Durrmeyer-type operators in complex domain were recently established in [4-12], etc. In [13], a Stancu-type generalization of the complex Durrmeyer operator was treated.

Also, Gal et. al [14] introduced complex Baskakov-Stancu operators and studied exact quantitative estimates and quantitative Voronovskaja-type results for these operators. Motivated by the recent study on this important topic, here, we deal with the following complex form for the Favard-Szász-Mirakjan-Stancu operator

*Correspondence: vijaygupta2001@hotmail.com

¹School of Applied Sciences, Netaji Subhas Institute of Technology, Sector 3 Dwarka, New Delhi, 110078, India

Full list of author information is available at the end of the article

$$S_n^{\alpha, \beta}(f, z) = \sum_{v=0}^{\infty} \left[\frac{\alpha}{n + \beta}, \frac{\alpha + 1}{n + \beta}, \dots, \frac{\alpha + v}{n + \beta}; f \right] z^v,$$

which for $\alpha = 0 = \beta$ was studied in [3]. Here, $[x_0, x_1, \dots, x_m; f]$ denotes the divided difference of the function f on the knots x_0, x_1, \dots, x_m . We may note here that such a formula was first established by Lupas [15] for the special case $\alpha = \beta = 0$ and for functions of real variable. This formula holds for complex setting too, since only algebraic calculations were used in [15].

The aim of the present paper is to study the rate of approximation of analytic functions without exponential growth conditions and the Voronovskaja-type result for the Favard-Szász-Mirakjan-Stancu operator $S_n^{\alpha, \beta}(f, z)$. Also, the exact order of approximation by this operator is obtained.

Methods

The principal methods used in the present work involve application of the theory of functions to analyze and study the order of simultaneous approximation, Voronovskaja-type results with quantitative estimates for the complex Favard-Szász-Mirakjan-Stancu operators attached to analytic functions on compact disks.

Results and discussion

In the sequel, we need the following auxiliary results:

Lemma 1. For all $n, k \in \mathbb{N} \cup \{0\}$, $0 \leq \alpha \leq \beta$, $z \in \mathbb{C}$, let us define

$$S_n^{\alpha, \beta}(e_k, z) = \sum_{\nu=0}^{\infty} \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, \dots, \frac{\alpha+\nu}{n+\beta}; e_k \right] z^\nu,$$

where $e_k(z) = z^k$. Then, $S_n^{\alpha, \beta}(e_0, z) = 1$, and we have the following recurrence relation

$$S_n^{\alpha, \beta}(e_{k+1}, z) = \frac{z}{n+\beta} (S_n^{\alpha, \beta}(e_k, z))' + \frac{nz+\alpha}{n+\beta} S_n^{\alpha, \beta}(e_k, z). \quad (1)$$

Consequently,

$$\begin{aligned} S_n^{\alpha, \beta}(e_1, z) &= \frac{nz+\alpha}{n+\beta}, \quad S_n^{\alpha, \beta}(e_2, z) \\ &= \frac{n^2 z^2}{(n+\beta)^2} + \frac{nz(1+2\alpha)}{(n+\beta)^2} + \frac{\alpha^2}{(n+\beta)^2}. \end{aligned} \quad (2)$$

Proof. First, note that we have

$$S_n^{\alpha, \beta}(e_k, z) = \sum_{\nu=0}^k \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, \dots, \frac{\alpha+\nu}{n+\beta}; e_k \right] z^\nu.$$

Simple calculation shows that the recurrence relation in the statement is equivalent to

$$\begin{aligned} & \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, \dots, \frac{\alpha+\nu}{n+\beta}; e_{k+1} \right] \\ &= \frac{\alpha+\nu}{n+\beta} \cdot \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, \dots, \frac{\alpha+\nu}{n+\beta}; e_k \right] \\ &+ \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, \dots, \frac{\alpha+\nu-1}{n+\beta}; e_k \right], \end{aligned} \quad (3)$$

which is an immediate consequence of the well-known relation (see [16], Exercise 4.9)

$$[x_0, x_1, \dots, x_m; f \cdot g] = \sum_{i=0}^m [x_0, x_1, \dots, x_i; f] \cdot [x_i, \dots, x_m; g],$$

by taking $m = \nu$, $f = e_k$, $g = e_1$ and $x_i = \frac{\alpha+i}{n+\beta}$. \square

Lemma 2. Let α, β be satisfying $0 \leq \alpha \leq \beta$. Denoting $e_j(z) = z^j$ and $S_n^{0,0}(e_j)$ by $S_n(e_j)$ for all $n, k \in \mathbb{N} \cup \{0\}$, the following recursive relation for the images of the monomials e_k under $S_n^{\alpha, \beta}$ in terms of $S_n(e_j)$, $j = 0, 1, 2, \dots, k$,

$$S_n^{\alpha, \beta}(e_k, z) = \sum_{j=0}^k \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n+\beta)^k} S_n(e_j, z).$$

Lemma 3. For all $n, k \in \mathbb{N} \cup \{0\}$, $0 \leq \alpha \leq \beta$ and $|z| \leq r$, $r \geq 1$, we have

$$|S_n^{\alpha, \beta}(e_k, z)| \leq (2r)^k.$$

Proof. Denoting $e_k(z) = z^k$ for any $k \in \mathbb{N}$, we have

$$S_n^{\alpha, \beta}(e_k, z) = \sum_{\nu=0}^k \left[\frac{\alpha}{n+\beta}, \frac{\alpha+1}{n+\beta}, \dots, \frac{\alpha+\nu}{n+\beta}; e_k \right] z^\nu.$$

Using the mean value theorem in complex analysis, we get

$$\begin{aligned} |S_n^{\alpha, \beta}(e_k, z)| &\leq \sum_{\nu=1}^k \frac{k(k-1)\dots(k-\nu+1)}{\nu!} r^{k-\nu} r^\nu \\ &\leq r^k \sum_{\nu=1}^k \frac{k(k-1)\dots(k-\nu+1)}{\nu!} \\ &= r^k \sum_{\nu=1}^k \binom{k}{\nu} \leq (2r)^k, \end{aligned}$$

which proves the lemma. \square

Main results

Our first main result is the following theorem for upper estimates:

Theorem 1. For $2 < R < +\infty$, let $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \rightarrow \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in \mathbb{D}_R , that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.

(a) Suppose that $0 \leq \alpha \leq \beta$ and $1 \leq r < \frac{R}{2}$ are arbitrarily fixed. Then, for all $|z| \leq r$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} |S_n^{\alpha, \beta}(f, z) - f(z)| &\leq \frac{\alpha + \beta r}{n + \beta} \sum_{k=1}^{\infty} |c_k| r^{k-1} \\ &\quad + \frac{A_r(f)}{n + \beta} + \frac{\alpha B_r(f)}{n + \beta} + \frac{\beta C_r(f)}{n + \beta}, \end{aligned} \tag{4}$$

where $\sum_{k=1}^{\infty} |c_k| r^{k-1} < +\infty$,
 $B_r(f) = \sum_{k=1}^{\infty} |c_k| k r^{k-1} < +\infty$,
 $C_r(f) = \sum_{k=1}^{\infty} |c_k| k r^k < +\infty$ and
 $A_r(f) = 2 \sum_{k=1}^{\infty} |c_k| (k-1) (2r)^{k-1} < +\infty$.

(b) Suppose that $0 \leq \alpha \leq \beta$ and $1 \leq r < r_1 < \frac{R}{2}$, then for all $|z| \leq r$ and $n \in \mathbb{N}$, we have

$$|[S_n^{\alpha, \beta}(f, z)]^{(p)} - f^{(p)}(z)| \leq \frac{p! r_1}{(r_1 - r)^{p+1}} \cdot \frac{M_{r_1}(f)}{n + \beta},$$

where $M_{r_1}(f) = (\alpha + \beta r_1) \sum_{k=1}^{\infty} |c_k| \cdot r_1^{k-1} + A_{r_1}(f) + B_{r_1}(f) + C_{r_1}(f)$.

Proof. (a) Using the recurrence (1) in Lemma 1, we get

$$\begin{aligned} S_n^{\alpha, \beta}(e_k, z) - z^k &= \frac{z}{n + \beta} (S_n^{\alpha, \beta}(e_{k-1}, z))' \\ &\quad + \frac{nz + \alpha}{n + \beta} (S_n^{\alpha, \beta}(e_{k-1}, z) - z^{k-1}) \\ &\quad + \frac{nz + \alpha}{n + \beta} z^{k-1} - z^k \end{aligned} \tag{5}$$

and

$$\begin{aligned} |S_n^{\alpha, \beta}(e_k, z) - z^k| &\leq \frac{|z|}{n + \beta} |(S_n^{\alpha, \beta}(e_{k-1}, z))'| \\ &\quad + \frac{n|z| + \alpha}{n + \beta} |S_n^{\alpha, \beta}(e_{k-1}, z) - z^{k-1}| \\ &\quad + \frac{\alpha}{n + \beta} |z|^{k-1} + \frac{\beta}{n + \beta} |z|^k. \end{aligned} \tag{6}$$

Clearly, $S_n^{\alpha, \beta}(e_0, z) - e_0 = 0$ and

$$|S_n^{\alpha, \beta}(e_1, z) - e_1(z)| = \left| \frac{nz + \beta}{n + \beta} - z \right| = \left| \frac{\alpha - \beta z}{n + \beta} \right| \leq \frac{\alpha + \beta r}{n + \beta}. \tag{7}$$

Using Lemma 3 and Bernstein's inequality for the polynomial $S_n^{\alpha, \beta}(e_{k-1}, z)$ of degree $\leq k-1$, we have

$$\begin{aligned} |(S_n^{\alpha, \beta}(e_{k-1}, z))'| &\leq \frac{k-1}{r} \max\{|S_n^{\alpha, \beta}(e_{k-1}, z)| : |z| \leq r\} \\ &\leq \frac{k-1}{r} (2r)^{k-1} = 2(k-1)(2r)^{k-2}. \end{aligned}$$

Therefore, it follows

$$\begin{aligned} &|S_n^{\alpha, \beta}(e_k, z) - z^k| \\ &\leq \frac{k-1}{n + \beta} (2r)^{k-1} + \frac{nr + \alpha}{n + \beta} |S_n^{\alpha, \beta}(e_{k-1}, z) - z^{k-1}| \\ &\quad + \frac{\alpha}{n + \beta} r^{k-1} + \frac{\beta}{n + \beta} r^k \\ &\leq r |S_n^{\alpha, \beta}(e_{k-1}, z) - z^{k-1}| + \frac{k}{n + \beta} (2r)^{k-1} \\ &\quad + \frac{\alpha}{n + \beta} r^{k-1} + \frac{\beta}{n + \beta} r^k. \end{aligned} \tag{8}$$

Taking above $k = 2$, we obtain

$$\begin{aligned} |S_n^{\alpha, \beta}(e_2, z) - e_2(z)| &\leq r \cdot \frac{\alpha + \beta r}{n + \beta} + \frac{1}{n + \beta} (2r) \\ &\quad + \frac{\alpha}{n + \beta} r^1 + \frac{\beta}{n + \beta} r^2. \end{aligned} \tag{10}$$

Then, for $k = 3$, it follows

$$\begin{aligned} |S_n^{\alpha, \beta}(e_3, z) - e_3(z)| &\leq r^2 \cdot \frac{\alpha + \beta r}{n + \beta} + \frac{1 \cdot 2^1}{n + \beta} r^2 + \frac{\alpha}{n + \beta} r^2 \\ &\quad + \frac{\beta}{n + \beta} r^3 + \frac{2 \cdot 2^2}{n + \beta} r^2 + \frac{\alpha}{n + \beta} r^2 \\ &\quad + \frac{\beta}{n + \beta} r^3 = r^2 \cdot \frac{\alpha + \beta r}{n + \beta} \\ &\quad + \frac{1}{n + \beta} [(1 \cdot 2^1) + (2 \cdot 2^2)] r^2 \\ &\quad + \frac{2\alpha}{n + \beta} r^2 + \frac{2\beta}{n + \beta} r^3. \end{aligned} \tag{11}$$

Reasoning by recurrence for any $k \geq 2$, we finally get

$$\begin{aligned} |S_n^{\alpha, \beta}(e_k, z) - e_k(z)| &\leq r^{k-1} \cdot \frac{\alpha + \beta r}{n + \beta} + \frac{1}{n + \beta} \cdot \left[\sum_{j=1}^{k-1} j \cdot 2^j \right] \cdot r^{k-1} \\ &\quad + \frac{(k-1)\alpha}{n + \beta} r^{k-1} + \frac{(k-1)\beta}{n + \beta} r^k \\ &\leq r^{k-1} \cdot \frac{\alpha + \beta r}{n + \beta} + \frac{2(k-1)}{n + \beta} \cdot (2r)^{k-1} \\ &\quad + \frac{k\alpha}{n + \beta} r^{k-1} + \frac{k\beta}{n + \beta} r^k. \end{aligned} \tag{12}$$

Since the formula $\sum_{j=1}^{k-1} j 2^j = (k-2)2^k + 2$ can easily be proved by mathematical induction, clearly, this inequality is valid for $k = 1$ too.

Now, reasoning exactly as in the case of complex Favard-Szász-Mirakjan operators in Remark 2 in [3], we can write

$$S_n^{\alpha,\beta}(f, z) = \sum_{k=0}^{\infty} c_k S_n^{\alpha,\beta}(e_k, z),$$

which implies

$$\begin{aligned} |S_n^{\alpha,\beta}(f, z) - f(z)| &\leq \sum_{k=1}^{\infty} |c_k| \cdot |S_n^{\alpha,\beta}(e_k, z) - z^k| \\ &\leq \frac{\alpha + \beta r}{n + \beta} \sum_{k=1}^{\infty} |c_k| \cdot r^{k-1} \\ &\quad + \frac{2}{n + \beta} \sum_{k=1}^{\infty} |c_k| (k-1) \cdot (2r)^{k-1} \\ &\quad + \frac{\alpha}{n + \beta} \sum_{k=1}^{\infty} |c_k| k r^{k-1} + \frac{\beta}{n + \beta} \sum_{k=1}^{\infty} |c_k| k r^k \\ &= \frac{\alpha + \beta r}{n + \beta} \sum_{k=1}^{\infty} |c_k| \cdot r^{k-1} \\ &\quad + \frac{A_r(f)}{n + \beta} + \frac{\alpha B_r(f)}{n + \beta} + \frac{\beta C_r(f)}{n + \beta}. \end{aligned} \tag{13}$$

Note here that by the analyticity of f , we clearly get $\sum_{k=1}^{\infty} |c_k| r^{k-1} < +\infty$, $B_r(f) < +\infty$, $C_r(f) < +\infty$ and $A_r(f) = 2 \sum_{k=1}^{\infty} |c_k| \cdot (k-1) \cdot (2r)^{k-1} < +\infty$, which proves (a).

(b) Denoting by γ , the circle of radius $r_1 > r$ with center 0. For any $|z| \leq r$ and $v \in \gamma$, we have $|v - z| \geq r_1 - r$ and by Cauchy's formula for all $|z| \leq r$, it follows

$$\begin{aligned} |[S_n^{\alpha,\beta}(f, z)]^{(p)} - f^{(p)}(z)| &= \frac{p!}{2\pi} \left| \int_{\gamma} \frac{S_n^{\alpha,\beta}(f, z) - f(z)}{(v - z)^{p+1}} dz \right| \\ &\leq \frac{p! r_1}{(r_1 - r)^{p+1}} \\ &\quad \cdot \left[\frac{\alpha + \beta r_1}{n + \beta} \sum_{k=1}^{\infty} |c_k| \cdot r_1^{k-1} + \frac{A_{r_1}(f)}{n + \beta} \right. \\ &\quad \left. + \frac{\alpha B_{r_1}(f)}{n + \beta} + \frac{\beta C_{r_1}(f)}{n + \beta} \right], \end{aligned} \tag{14}$$

which proves (b) and the theorem. \square

The next main result is a Voronovskaja-type asymptotic formula.

Theorem 2. For $2 < R < +\infty$, let $f : [R, +\infty) \cup \overline{\mathbb{D}}_R \rightarrow \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in \mathbb{D}_R , that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$. Also, let $1 \leq r < \frac{R}{2}$ and $0 \leq \alpha \leq \beta$. Then, for all $|z| \leq r$ and $n \in \mathbb{N}$, we have the following

Voronovskaja-type result

$$\begin{aligned} \left| S_n^{\alpha,\beta}(f, z) - f(z) - \frac{\alpha - \beta z}{n + \beta} f'(z) \right. \\ \left. - \frac{z}{2n} f''(z) \right| &\leq \frac{M_{1,r}(f)}{n^2} + \frac{\sum_{j=2}^6 M_{j,r}(f)}{(n + \beta)^2}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} M_{1,r}(f) &= 26 \sum_{k=3}^{\infty} |c_k| (k-1)^2 (k-2) (2r)^{k-2} < +\infty, M_{2,r}(f) \\ &= \left(\frac{\alpha^2}{2} + 2\alpha \right) \cdot \sum_{k=2}^{\infty} |c_k| \cdot k(k-1) (2r)^{k-2} < +\infty, \\ M_{3,r}(f) &= \frac{\beta^2}{2} \sum_{k=2}^{\infty} |c_k| k(k-1) (2r)^k < +\infty, M_{4,r}(f) \\ &= \beta \sum_{k=2}^{\infty} |c_k| k(k-1) (2r)^{k-1} < +\infty, \\ M_{5,r}(f) &= \alpha\beta \sum_{k=0}^{\infty} |c_k| k(k-1) r^{k-1} < +\infty, M_{6,r}(f) \\ &= \beta^2 \sum_{k=0}^{\infty} |c_k| k(k-1) r^k < +\infty. \end{aligned} \tag{16}$$

Proof. For all $z \in \mathbb{D}_R$, let us consider

$$\begin{aligned} S_n^{\alpha,\beta}(f, z) - f(z) - \frac{\alpha - \beta z}{n + \beta} f'(z) - \frac{z}{2n} f''(z) \\ = S_n(f, z) - f(z) - \frac{z}{2n} f''(z) + S_n^{\alpha,\beta}(f, z) - S_n(f, z) - \frac{\alpha - \beta z}{n + \beta} f'(z). \end{aligned} \tag{17}$$

Taking $f(z) = \sum_{k=0}^{\infty} c_k z^k$, we get

$$\begin{aligned} S_n^{\alpha,\beta}(f, z) - f(z) - \frac{\alpha - \beta z}{n + \beta} f'(z) - \frac{z}{2n} f''(z) \\ = \sum_{k=2}^{\infty} c_k \left(S_n(e_k, z) - z^k - \frac{z}{2n} k(k-1) z^{k-2} \right) \\ + \sum_{k=2}^{\infty} c_k \left(S_n^{\alpha,\beta}(e_k, z) - S_n(e_k, z) - \frac{\alpha - \beta z}{n + \beta} k z^{k-1} \right). \end{aligned}$$

To estimate the first sum, we use the Voronovskaja-type result for the Favard-Szász-Mirakjan operators obtained in [3], Theorem 1.8.5.

$$\left| S_n(f, z) - f(z) - \frac{z}{2n} f''(z) \right| \leq \frac{26}{n^2} \sum_{k=3}^{\infty} |c_k| (k-1)^2 (k-2) (2r)^{k-2}. \tag{18}$$

Next, to estimate the second sum, using Lemma 2, we obtain

$$\begin{aligned}
 S_n^{\alpha, \beta}(e_k, z) - S_n(e_k, z) - \frac{\alpha - \beta z}{n + \beta} kz^{k-1} \\
 = \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n + \beta)^k} S_n(e_j, z) + \left(\frac{n^k}{(n + \beta)^k} - 1 \right) \\
 S_n(e_k, z) - \frac{\alpha - \beta z}{n + \beta} kz^{k-1} \\
 = \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n + \beta)^k} S_n(e_j, z) + \frac{kn^{k-1}\alpha}{(n + \beta)^k} S_n(e_{k-1}, z) \\
 - \sum_{j=0}^{k-1} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n + \beta)^k} S_n(e_k, z) \\
 - \frac{\alpha - \beta z}{n + \beta} kz^{k-1} \\
 = \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n + \beta)^k} S_n(e_j, z) + \frac{kn^{k-1}\alpha}{(n + \beta)^k} [S_n(e_{k-1}, z) - z^{k-1}] \\
 - \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n + \beta)^k} S_n(e_k, z) \\
 - \frac{kn^{k-1}\beta}{(n + \beta)^k} [S_n(e_k, z) - z^k] \\
 + \left(\frac{n^{k-1}}{(n + \beta)^{k-1}} - 1 \right) \frac{k\alpha}{n + \beta} z^{k-1} \\
 + \left(1 - \frac{n^{k-1}}{(n + \beta)^{k-1}} \right) \frac{k\beta}{n + \beta} z^k. \tag{19}
 \end{aligned}$$

Now, using the inequalities

$$1 - \frac{n^k}{(n + \beta)^k} \leq \sum_{j=1}^k \left(1 - \frac{n}{n + \beta} \right) = \frac{k\beta}{n + \beta}, \tag{20}$$

$$\begin{aligned}
 \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{n^j \alpha^{k-2-j}}{(n + \beta)^{k-2}} &= \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{n^j}{(n + \beta)^j} \\
 &\cdot \frac{\alpha^{k-2-j}}{(n + \beta)^{k-2-j}} = \left(\frac{n + \alpha}{n + \beta} \right)^{k-2} \leq 1, \tag{21}
 \end{aligned}$$

$$|S_n(e_j, z)| \leq (2r)^k \text{ (see Lemma 3 for } \alpha = \beta = 0 \text{)},$$

$$|S_n(e_k, z) - z^k| \leq \frac{k-1}{n} \cdot (2r)^{k-1},$$

for this last inequality take $\alpha = \beta = 0$ in the proof of the above Theorem 1

and

$$\begin{aligned}
 \left| \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n + \beta)^k} S_n(e_j, z) \right| &\leq \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n + \beta)^k} |S_n(e_j, z)| \\
 &= \sum_{j=0}^{k-2} \frac{k(k-1)}{(k-j)(k-j-1)} \binom{k-2}{j} \frac{n^j \alpha^{k-j}}{(n + \beta)^k} |S_n(e_j, z)| \\
 &\leq \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n + \beta)^2} (2r)^{k-2} \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{n^j \alpha^{k-2-j}}{(n + \beta)^{(k-2)}} \\
 &\leq \frac{k(k-1)}{2} \cdot \frac{\alpha^2}{(n + \beta)^2} (2r)^{k-2}, \tag{22}
 \end{aligned}$$

it follows

$$\begin{aligned}
 \left| S_n^{\alpha, \beta}(e_k, z) - S_n(e_k, z) - \frac{\alpha - \beta z}{n + \beta} kz^{k-1} \right| \\
 \leq \left| \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \alpha^{k-j}}{(n + \beta)^k} S_n(e_j, z) \right| \\
 + \frac{kn^{k-1}\alpha}{(n + \beta)^k} |S_n(e_{k-1}, z) - z^{k-1}| \\
 + \left| \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n + \beta)^k} S_n(e_k, z) \right| \\
 + \frac{kn^{k-1}\beta}{(n + \beta)^k} |S_n(e_k, z) - z^k| \\
 + \left| \left(\frac{n^{k-1}}{(n + \beta)^{k-1}} - 1 \right) \right| \frac{k\alpha}{n + \beta} |z|^{k-1} \\
 + \left| \left(1 - \frac{n^{k-1}}{(n + \beta)^{k-1}} \right) \right| \frac{k\beta}{n + \beta} |z|^k \\
 \leq \frac{k(k-1)\alpha^2}{2(n + \beta)^2} (2r)^{k-2} + \frac{kn^{k-1}\alpha}{(n + \beta)^k} \cdot \frac{(2r)^{k-2}(k-2)}{n} \\
 + (2r)^k \cdot \sum_{j=0}^{k-2} \binom{k}{j} \frac{n^j \beta^{k-j}}{(n + \beta)^k} \\
 + \frac{kn^{k-1}\beta}{(n + \beta)^k} \cdot \frac{(2r)^{k-1}(k-1)}{n} + \frac{k(k-1)\alpha\beta}{(n + \beta)^2} r^{k-1} \\
 + \frac{k(k-1)\beta^2}{(n + \beta)^2} r^k,
 \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{k(k-1)\alpha^2}{2(n+\beta)^2} (2r)^{k-2} \\
 &\quad + \frac{k(k-2)\alpha}{(n+\beta)^2} \cdot (2r)^{k-2} \\
 &\quad + \frac{\beta^2 k(k-1)}{2(n+\beta)^2} (2r)^k \\
 &\quad + \frac{k(k-1)\beta}{(n+\beta)^2} (2r)^{k-1} \\
 &\quad + \frac{k(k-1)\alpha\beta}{(n+\beta)^2} r^{k-1} + \frac{k(k-1)\beta^2}{(n+\beta)^2} r^k \\
 &\leq \frac{k(k-1)}{(n+\beta)^2} \left(\frac{\alpha^2}{2} + 2\alpha \right) (2r)^{k-2} \\
 &\quad + \frac{\beta^2 k(k-1)}{2(n+\beta)^2} (2r)^k + \frac{k(k-1)\beta}{(n+\beta)^2} (2r)^{k-1} \\
 &\quad + \frac{k(k-1)\alpha\beta}{(n+\beta)^2} r^{k-1} + \frac{k(k-1)\beta^2}{(n+\beta)^2} r^k,
 \end{aligned} \tag{23}$$

which immediately proves the theorem. \square

Now, we will give the exact order of approximation by the operators $V_n^{\alpha,\beta}$.

Theorem 3. For $2 < R < +\infty$, $1 \leq r < \frac{R}{2}$, let $f : [R, +\infty) \cup \mathbb{D}_R \rightarrow \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in \mathbb{D}_R , that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$.

Let $0 < \alpha \leq \beta$ and suppose that f is not a polynomial of degree ≤ 0 . Then, for all $n \in \mathbb{N}$ and $|z| \leq r$, we have

$$\left| S_n^{\alpha,\beta}(f, z) - f(z) \right| \geq \frac{C_r(f)}{n},$$

where the constant $C_r(f)$ depends only on f, α, β and r .

Proof. For all $|z| \leq r$ and $n \in \mathbb{N}$, we can write

$$\begin{aligned}
 S_n^{\alpha,\beta}(f, z) - f(z) &= \frac{1}{n} \left[\frac{n}{n+\beta} (\alpha - \beta z) f'(z) + \frac{z}{2} f''(z) \right. \\
 &\quad \left. + \frac{1}{n} \cdot n^2 \left(S_n^{\alpha,\beta}(f, z) - f(z) - \frac{\alpha - \beta z}{n+\beta} f'(z) - \frac{z}{2n} f''(z) \right) \right] \\
 &= \frac{1}{n} \left[(\alpha - \beta z) f'(z) + \frac{z}{2} f''(z) \right. \\
 &\quad \left. + \frac{1}{n} \cdot n^2 \left(S_n^{\alpha,\beta}(f)(z) - f(z) - \frac{\alpha - \beta z}{n+\beta} f'(z) - \frac{z}{2n} f''(z) - \frac{\beta(\alpha - \beta z) f'(z)}{n(n+\beta)} \right) \right].
 \end{aligned} \tag{24}$$

Applying the inequality

$$\|F + G\| \geq \| \|F\| - \|G\| \| \geq \|F\| - \|G\|,$$

we obtain

$$\begin{aligned}
 \|S_n^{\alpha,\beta}(f) - f\|_r &\geq \frac{1}{n} \left[\left\| (\alpha - \beta e_1) f' + \frac{e_1}{2} f'' \right\|_r \right. \\
 &\quad \left. - \frac{1}{n} \cdot n^2 \left\| S_n^{\alpha,\beta}(f) - f - \frac{\alpha - \beta e_1}{n+\beta} f' - \frac{e_1}{2n} f'' - \frac{\beta(\alpha - \beta e_1) f'}{n(n+\beta)} \right\|_r \right].
 \end{aligned} \tag{25}$$

Since f is not a polynomial of degree ≤ 0 in \mathbb{D}_R , we get $\|(\alpha - \beta e_1) f' + \frac{e_1}{2} f''\|_r > 0$. Indeed, supposing the contrary, it follows that

$$(\alpha - \beta z) f'(z) + \frac{z}{2} f''(z) = 0, \text{ for all } z \in \mathbb{D}_R.$$

Denoting $y(z) = f'(z)$, seeking $y(z)$ in the form $y(z) = \sum_{k=0}^{\infty} b_k z^k$ and replacing in the above differential equation, we easily get $b_k = 0$ for all $k = 0, 1, \dots$, (we can make here similar reasonings with those in [17]; see also [3]). Thus, we get that $f(z)$ is a constant function, which is a contradiction.

Now, since by Theorem 2 it follows

$$\begin{aligned}
 &n^2 \left\| S_n^{\alpha,\beta}(f) - f - \frac{\alpha - \beta e_1}{n+\beta} f' - \frac{e_1}{2n} f'' - \frac{\beta(\alpha - \beta e_1) f'}{n(n+\beta)} \right\|_r \\
 &\leq n^2 \left\| S_n^{\alpha,\beta}(f) - f - \frac{\alpha - \beta e_1}{n+\beta} f' - \frac{e_1}{2n} f'' \right\|_r + \|\beta(\alpha - \beta e_1) f'\|_r \\
 &\leq \sum_{j=1}^6 M_{j,r}(f) + \beta(\alpha + \beta r) \|f'\|_r,
 \end{aligned} \tag{26}$$

there exists $n_1 > n_0$ (depending on f, α, β and r only) such that for all $n \geq n_1$, we have

$$\begin{aligned}
 &\left\| (\alpha - \beta e_1) f' + \frac{e_1}{2} f'' \right\|_r \\
 &\quad - \frac{1}{n} \cdot n^2 \left\| S_n^{\alpha,\beta}(f) - f - \frac{\alpha - \beta e_1}{n+\beta} f' - \frac{e_1}{2n} f'' - \frac{\beta(\alpha - \beta e_1) f'}{n(n+\beta)} \right\|_r \geq \frac{1}{2} \left\| (\alpha - \beta e_1) f' + \frac{e_1}{2} f'' \right\|_r,
 \end{aligned} \tag{27}$$

which implies that

$$\|S_n^{\alpha,\beta}(f) - f\|_r \geq \frac{1}{2n} \left\| (\alpha - \beta e_1) f' + \frac{e_1}{2} f'' \right\|_r$$

for all $n \geq n_1$.

For $n \in \{n_0 + 1, \dots, n_1\}$, we get $\|S_n^{\alpha,\beta}(f) - f\|_r \geq \frac{1}{n} A_r(f)$ with $A_r(f) = n \cdot \|S_n^{\alpha,\beta}(f) - f\|_r > 0$, which implies that

$$\|S_n^{\alpha, \beta}(f) - f\|_r \geq \frac{C_r(f)}{n} \text{ for all } n \geq n_0, \text{ with}$$

$$C_r(f) = \min \left\{ A_{r, n_0+1}(f), \dots, A_{r, n_1}(f), \right. \\ \left. \frac{1}{2} \left\| (\alpha - \beta e_1)f' + \frac{e_1}{2} f'' \right\|_r \right\}, \quad (28)$$

which proves the theorem. \square

Remark 1. By Theorems 1 and 3, it easily follows that if f is not a constant function, then the exact order in the approximation by the complex Favard-Szász-Mirakjan-Stancu operator $S_n^{\alpha, \beta}$ is $\frac{1}{n}$.

Concerning the simultaneous approximation, we present the following:

Theorem 4. For $2 < R < +\infty$, $1 \leq r < \frac{R}{2}$, let $f : [R, +\infty) \cup \mathbb{D}_R \rightarrow \mathbb{C}$ be bounded on $[0, +\infty)$ and analytic in \mathbb{D}_R , that is $f(z) = \sum_{k=0}^{\infty} c_k z^k$, for all $z \in \mathbb{D}_R$ and let $1 \leq r < r_1 < \frac{R}{2}$ and $p \in \mathbb{N}$ be fixed.

Let $0 < \alpha \leq \beta$ and suppose that f is not a polynomial of degree $\leq p - 1$. Then, for all $n \in \mathbb{N}$ and $|z| \leq r$, we have

$$\| [S_n^{\alpha, \beta}(f)]^{(p)} - f^{(p)} \|_r \sim \frac{1}{n},$$

where the constants in the equivalence depend only on f (that is on M and A), α , β , p and r .

Proof. Denoted by γ , the circle of radius r_1 with $r < r_1 < \min\{n_0/2, 1/A\}$ and center 0. Since for $|z| \leq r$ and $v \in \gamma$ we have $|v - z| \geq r_1 - r$, by the Cauchy's formula, for all $|z| \leq r$ and $n > n_0$, we obtain

$$\left| [S_n^{\alpha, \beta}(f, z)]^{(p)} - f^{(p)}(z) \right| = \frac{p!}{2\pi} \cdot \left| \int_{\gamma} \frac{S_n^{\alpha, \beta}(f, v) - f(v)}{(v - z)^{p+1}} dv \right| \\ \leq \frac{C_{r, r_1, \alpha, \beta}}{n} \cdot \frac{p!}{2\pi} \cdot \frac{2\pi r_1}{(r_1 - r)^{p+1}}, \quad (29)$$

which proves one of the inequalities in the equivalence. Note here that r_1 depends in fact on r , A and n_0 ; therefore, it depends in fact on r and f .

To prove the converse inequality in the equivalence, we start from the relationship for $S_n^{\alpha, \beta}(f, v) - f(v)$ in (1) (in the proof of Theorem 3, with v instead of z there), replaced in the Cauchy's formula

$$[S_n^{\alpha, \beta}(f, z)]^{(p)} - f^{(p)}(z) = \frac{p!}{2\pi i} \cdot \int_{\gamma} \frac{S_n^{\alpha, \beta}(f, v) - f(v)}{(v - z)^{p+1}} dv.$$

By standard reasonings as those for the case of classical complex Favard-Szász-Mirakjan operator (see the proof of Theorem 1.8.6 in [3]), combined with those for the Bernstein-Stancu polynomials (see [17] or the proof of Theorem 1.6.5 in [3]), the present proof finally reduces to

the proof of the fact that $\| [(\alpha - \beta z)f'(z) + \frac{z}{2} f''(z)]^{(p)} \|_r > 0$. However, this can be shown by following exactly the lines in [17]; see also [3]. As the reasonings are standard, we omit the details. \square

Conclusions

For the parameters α and β , our results have better rate convergence for the complex Favard-Szász-Mirakjan-Stancu operators which include $\alpha = 0 = \beta$, as special case. In special case, the Theorems 1, 2, 3 and 4 become the results in [7]; see also [3].

Competing interests

Both authors declare that they have no competing interests.

Authors' contributions

VG and DKV contributed equally to this work. Both authors read and approved the final manuscript.

Authors' information

VG is a professor at the School of Applied Sciences, Netaji Subhas Institute of Technology, Sector 3 Dwarka, New Delhi, India. DKV is a research fellow at the Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, India.

Acknowledgements

DKV is thankful to the 'Ministry of Human Resource and Development, India' for the financial support to carry out the above work. Authors are thankful to one of the reviewers for making valuable suggestions leading to the improvements in the paper.

Author details

¹School of Applied Sciences, Netaji Subhas Institute of Technology, Sector 3 Dwarka, New Delhi, 110078, India. ²Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, 247667, India.

Received: 21 May 2012 Accepted: 24 July 2012

Published: 17 August 2012

References

- Favard, J: Sur les multiplicateurs d'interpolation. *J. Math. Pures Appl. Ser.* **9**(23), 219-247 (1944)
- Walczak, Z: On a class of Szász-Mirakjan operators. *Czech. Math. J.* **58**(3), 705-716 (2008)
- Gal, SG: Approximation by Complex Bernstein and Convolution Type Operators. World Scientific Publ. Co, Singapore (2009)
- Anastassiou, GA, Gal, SG: Approximation by complex Bernstein-Schurer and Kantorovich-Schurer polynomials in compact disks. *Comput. Math. Appl.* **58**(4), 734-743 (2009)
- Anastassiou, GA, Gal, SG: Approximation by complex Bernstein-Durrmeyer polynomials in compact disks. *Mediterr. J. Math.* **7**(4), 471-482 (2010)
- Gal, SG: Approximation by complex genuine Durrmeyer type polynomials in compact disks. *Appl. Mathematics Comput.* **217**, 1913-1920 (2010)
- Gal, SG: Approximation of analytic functions without exponential growth conditions by complex Favard-Szász-Mirakjan operators. *Rendiconti del Circolo Matematico di Palermo.* **59**(3), 367-376 (2010)
- Gal, SG: Approximation by complex Bernstein-Durrmeyer polynomials with Jacobi weights in compact disks. *Math. Balkanica (N.S.)* **24**(1-2), 103-119 (2010)
- Gal, SG, Gupta, V: Approximation by certain integrated Bernstein-type operators in compact disks. *Lobachevski J. Mathematics.* **33**(1), 39-46 (2012)
- Gal, SG, Gupta, V: Approximation by a Durrmeyer-type operator in compact disks. *Ann. Univ. Ferrara.* **57**(2), 261-274 (2011)
- Gupta, V: Approximation properties by a Bernstein-Durrmeyer type operator. *Complex Anal. OperTheory* (2011). doi:10.1007/s11785-011-0167-9

12. Gal, SG, Gupta, V: Approximation by complex beta operators of first kind in strips of compact disks. *Mediterranean J. Mathematics* (2011). doi:10.1007/s00009-011-0164-2
13. Mahmudov, NI, Gupta, V: Approximation by genuine Durrmeyer-Stancu polynomials in compact disks. *Math. Comput. Modell.* **55**, 278–285 (2012)
14. Gal, SG, Gupta, V, Verma, DK, Agrawal, PN: Approximation by complex Baskakov-Stancu operators in compact disks. *Rendiconti del Circolo Matematico di Palermo.* **61**(2), 153–165 (2012)
15. Lupas, A: Some properties of the linear positive operators I. *Math. (Cluj).* **9**(32), 77–83 (1967)
16. Stancu, DD: Course in Numerical Analysis (in Romanian). Faculty of Mathematics and Mechanics, Babes-Bolyai University Press, Cluj (1977)
17. Gal, SG: Exact orders in simultaneous approximation by complex Bernstein-Stancu polynomials. *Revue d' Anal. Numér. Théor. L'Approx(Cluj-Napoca).* **37**(1), 47–52 (2008)

doi:10.1186/2251-7456-6-25

Cite this article as: Gupta and Verma: Approximation by complex Favard-Szász-Mirakjan-Stancu operators in compact disks. *Mathematical Sciences* 2012 **6**:25.

Archive of SID

Submit your manuscript to a SpringerOpen[®] journal and benefit from:

- ▶ Convenient online submission
- ▶ Rigorous peer review
- ▶ Immediate publication on acceptance
- ▶ Open access: articles freely available online
- ▶ High visibility within the field
- ▶ Retaining the copyright to your article

Submit your next manuscript at ▶ springeropen.com
