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A new characterization of A_{12}

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Abstract

Purpose: It is well known that the conjugacy class sizes have an important influence on the structure of a group. This work is considering a different set of 'sizes', the number of elements of a given order.

Methods: By using the set nse(G) and the order of G, We prove that G is isomorphic to A_{12}.

Results: Thompson's conjecture is true for $A1_{12}$.

Conclusions: We proved that a finite group G is isomorphic to A_{12}, the alternating group A_{12} of degree 12 if, and only if, $|G| = |A_{12}|$ and nse(G)=nse(A_{12}).

Keywords: Finite group, Insoluble group, Simple group, Order of elements

Subject classification: 20D05, 20D06

Introduction

In this study, all groups are assumed to be finite. It is well known that the conjugacy class sizes have an important influence on the structure of a group. The relation between conjugacy class sizes and the structure of a group has been studied by many authors (for example, see [1-4]). In the present work, we are considering a different set of 'sizes', the number of elements of a given order.

Most notations are standard (see [5,6]). We introduce some which may be unfamiliar to the reader. Let $\omega(G)$ denote the set of element orders of G. Let $m_i(G) := |\{g \in G|\text{the order of } g \text{ is } i\}|$ (m_i for short) be the number of elements of order i, and let nse$(G) := \{m_i(G)|i \in \omega(G)\}$ be the set of sizes of elements with the same order. $n_p(G)$ denotes the number of Sylow p-subgroup of G, namely, $n_p(G) = |\mathrm{Syl}_p(G)|$. $\pi(G)$ denotes the set of all prime divisors of $|G|$. A_{12} is the alternating group of degree 12. We use $a|b$ to mean that a divides b; if p is a prime, then $p^n||b$ means $p^n|b$ but $p^{n+1} \nmid b$. $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$ denotes the set of positive integers. $\pi(G)$ denotes the set of prime divisors of $|G|$ and $|\pi(G)|$, the number of the element of the set $\pi(G)$. nse(G) denotes the number of elements of a given order of G.

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Methods

Thompson's problem

For the set nse(G), the most important problem is related to Thompson's problem. In 1987, JG Thompson put forward the following problem. For each finite group G and each integer $d \geq 1$, let $G(d) = \{x \in G|x^d = 1\}$. Defining G_1 and G_2 is of the same order type if, and only if, $|G_1(d)| = |G_2(d)|$, $d = 1, 2, 3, \cdots$. Suppose G_1 and G_2 are of the same order type. If G_1 is solvable, is G_2 necessarily solvable?

Professor WJ Shi in [7] made the above problem public in 1989. Unfortunately, no one can solve it or give a counterexample until now, and it remains open. The influence of nse(G) on the structure of finite groups was studied by some authors (see [8-11]). In the present work, we show that a condition related to the order type characterizes A_{12}; explicitly, we prove the following:

Main theorem. *Let G be a group, and then $G \cong A_{12}$ if, and only if, $|G| = |A_{12}|$ and nse(G)=nse(A_{12}).*

Preliminary

Lemma 1. *If G is a soluble group of order mn, where m is prime to n ([12] (p. 99)), then*

1. *G possesses at least one subgroup of order m,*
2. *Any two subgroups of G of order m are conjugate,*
3. *Any subgroup of G whose order divides m belongs to some subgroup of order m,*

4. *The number h_m of subgroups of G of order m may be expressed as a product of factors, each of which (1) is congruent to 1 modulo some prime factor of m, (2) is a power of a prime and divides one of the chief factors of G.*

Definition 1. *A finite group G is called a simple K_n-group if G is a simple group with $|\pi(G)| = n$ (see [13] (p. 658) or [14]).*

We will give need a description of the simple K_n-groups for $n \leq 5$.

Remark 1. *If G is a simple K_1-group, then G is a cyclic of prime-order.*

Remark 2. *If $|G| = p^a q^b$ with p and q distinct primes and a, b nonnegative integers, then by Burnside's pq-theorem (see [6] section (10.2.1)), G is soluble. In particular, there are no simple K_2-groups.*

Lemma 2. *If G is a simple K_3-group, then G is isomorphic to one of the groups (see Theorem 2 of [15]): $A_5(2^2 \cdot 3 \cdot 5), A_6(2^3 \cdot 3^2 \cdot 5), L_2(7)(2^3 \cdot 3 \cdot 7), L_2(8)(2^3 \cdot 3^2 \cdot 7), L_2(17)(2^4 \cdot 3^2 \cdot 17), L_3(3)(2^4 \cdot 3^3 \cdot 13), U_3(3)(2^5 \cdot 3^3 \cdot 7)$, or $U_4(2)(2^6 \cdot 3^4 \cdot 5)$, where $*(*)$ means the group (the order of G).*

Lemma 3. *Let G be a simple K_4-group, and then G is isomorphic to one of the following groups (see Theorem 1 of [13] or Theorem 2 of [14]):*

1. A_7, A_8, A_9, *or* A_{10}.
2. M_{11}, M_{12}, *or* J_2.
3. *One of the following:*
   (a) $L_2(r)$, *where r is a prime and* $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ *with* $a \geq 1, b \geq 1, c \geq 1$, *and v is a prime greater than 3.*
   (b) $L_2(2^m)$, *where* $2^m - 1 = u$, $2^m + 1 = 3t^b$ *with* $m \geq 2$, *u, t are primes,* $t > 3$, $b \geq 1$;
   (c) $L_2(3^m)$, *where* $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ *or* $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, *with* $m \geq 2$, *u, t are odd primes,* $b \geq 1, c \geq 1$;
   (d) $L_2(16), L_2(25), L_2(49), L_2(81), L_3(4), L_3(5), L_3(7), L_3(8), L_3(17), L_4(3), S_4(4), S_4(5), S_4(7), S_4(9), S_6(2), O_8^+(2), G_2(3), U_3(4), U_3(5), U_3(7), U_3(8), U_3(9), U_4(3), U_5(2), Sz(8), Sz(32), {}^3D_4(2)$, *or* ${}^2F_4(2)'$.

Lemma 4. *Each simple K_5-group is isomorphic to one of the following simple groups (see Theorem A of [16]):*

1. $L_2(q)$ *where q satisfies* $|\pi(q^2 - 1)| = 4$;
2. $L_3(q)$ *where q satisfies* $|\pi(q^2 - 1)(q^3 - 1)| = 4$;
3. $U_3(q)$ *where* $|\pi(q^2 - 1)(q^3 + 1)| = 4$;
4. $O_5(q)$ *where* $|\pi(q^4 - 1)| = 4$;
5. $Sz(2^{2^m+1})$ *where* $|\pi((2^{2^m+1} - 11)(2^{2^{4m+2}} + 1))| = 4$;
6. $R(q)$ *where q is an odd power of 3 and* $|\pi(q^2 - 1)| = 3$;
7. *Following 30 simple groups:* $A_{11}, A_{12}, M_{22}, J_3, HS, He, McL, L_4(4), L_4(5), L_4(7), L_5(2), L_5(3), L_6(2), O_7(3), O_9(2), PSp_6(3), PSp_8(2), U_4(4), U_4(5), U_4(7), U_4(9), U_5(3), U_6(2), O_8^+(3), O_8^-(2), {}^3D_4(3), G_2(4), G_2(5), G_2(7)$, *or* $G_2(9)$.

Lemma 5. *Let G be a finite group, $P \in Syl_p(G)$, where $p \in \pi(G)$. Suppose that G has a normal series $K \trianglelefteq L \trianglelefteq G$ and $p \nmid |K|$, and then following statements hold [17]:*

1. $N_{G/K}(PK/K) = N_G(P)K/K$.
2. *If* $P \leq L$, *then* $|G : N_G(P)| = |L : N_L(P)|$, *namely,* $n_p(G) = n_p(L)$.
3. *If* $P \leq L$, *then* $|L/K : N_{L/K}(PK/K)|t = |G : N_G(P)| = |L : N_L(P)|$, *namely,* $n_p(L/K) = t = n_p(G) = n_p(L)$. *In particular,* $|N_K(P)|t = |K|$.

Lemma 6. *Let G be a simple K_5-group and $3^5 \big| |G| \big| 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$; then $G \cong A_{12}$.*

Proof. Assume G is isomorphic to $L_2(q)$ in Lemma 4, then 7 or 11 $||L_2(q)|$. □

Case 1. $7\, ||L_2(q)|$.

If $q = 7$, then $|\pi(q^2 - 1)| = 2$, which contradicts $|\pi(q^2 - 1)| = 4$.

If $q = 2^m$, then $7 \big| 2^{2m} - 1$, so we have $5|m$ and $19\, ||L_2(q)|$, which is a contradiction.

If $q = 3, 3^2$, or 5, then $|\pi(q^2 - 1)| < 4$, which is a contradiction.

If $q = 3^3$, or 5^2, then $13 \big| (q^2 - 1) \big| |G|$, which is a contradiction.

If $q = 3^4$, then $41\, ||L_2(q)|$, which is a contradiction.

If $q = 3^5$, then $61\, ||L_2(q)|$, which is a contradiction.

Case 2. $11\, ||L_2(q)|$.

If $q = 11$, then $|\pi(11^2 - 1)| = 3$, which contradicts $|\pi(q^2 - 1)| = 4$.

If $q = 2^m$, then $11 \big| 2^{2m} - 1$, so we have $5|m$ and $31\, ||L_2(q)|$, which is a contradiction.

If $q = 3, 3^2, 5$, or 7, then $|\pi(q^2 - 1)| < 4$, which is a contradiction.

If $q = 3^3$, or 5^2, then $13 \big| (q^2 - 1) \big| |G|$, which is a contradiction.

If $q = 3^4$, then $41\, ||L_2(q)|$, which is a contradiction.

If $q = 3^5$, then $61\, ||L_2(q)|$, which is a contradiction.

Thus, G is not isomorphic to $L_2(q)$.

Similarly, G is not isomorphic to $L_3(q), U_3(q), O_5(q)$, $Sz(2^{2m+1})$, and $R(q)$.

In view of item number 7 of Lemma 4, we get that $G \cong A_{12}$.

This completes the proof. □

Results and discussion

The proof of the main theorem

In this section, we will give the proof of the main theorem. We rewrite the main theorem here:

Main theorem. *Let G be a group, and then $G \cong A_{12}$ if, and only if, $|G| = |A_{12}|$ and nse(G)=nse(A_{12})={1, 63855, 570240, 2154900, 3825360, 3991680, 4809024, 6652800, 8553600, 11404800, 11975040, 13685760, 21621600, 25530120, 25945920, 26611200, 29937600, 43545600.}*

Proof. If $G \cong A_{12}$, from [18] (pp. 91 to 92), we easily get the results. □

Then, we assume that $|G| = |A_{12}|$, and nse(G) =nse(A_{12}).

We prove $G \cong A_{12}$ by first proving that G is insoluble, and then showing that it must be isomorphic to A_{12}.

Step 1. G is insoluble.

If G is soluble, let H be a $\{3, 5, 7, 11\}$-Hall subgroup of G. By item number 2 of Lemma 1, all $\{3, 5, 7, 11\}$-Hall subgroups of G are conjugate in G; hence, by I and Exercise 2 of [5], the number of $\{3, 5, 7, 11\}$-Hall subgroup of G is $|G : N_G(H)| \big| 2^9$.

By I and Theorem 2.9 of [5], $n_{11}(H) = 1, 12, 111, 925, 1442$ in H. Thus, we have the following cases:

If $n_{11}(H) = 1$, then the number m of elements of order 11 in G is $10 < m < 5120$ and $10 \mid m$, but $m \notin$ nse(G), which is a contradiction.

If $n_{11}(H) = 12$, then the number m of elements of order 11 in G is $120 < m < 61440$ and $10 \mid m$, but $m \notin$ nse(G), which is a contradiction.

If $n_{11}(H) = 111$, then the number m of elements of order 11 in G is $1110 < m < 568320$ and $10 \mid m$, but $m \notin$ nse(G), which is a contradiction.

If $n_{11}(H) = 925$, or 1442, then the number m of elements of order 11 in G is $9250 < m < 4736000$ or $14420 < m < 7383040$ and $10 \mid m$. Since $m \in$nse(G), $m = 776600, \ 3825360, \ 570240$, so $11k + 1 = 776600, \ 3825360, \ 570240$, but none of these equations have solutions in $\mathbb{N}$.

Thus, G is insoluble.

Step 2. $G \cong A_{12}$.

Through Step 1, it has been proven that G is insoluble. Since $p \,||G|$, where $p \in \{7, 11\}$, then G has a normal series: $1 \trianglelefteq K \trianglelefteq L \trianglelefteq G$ such that L/K is a simple K_i-group where $i = 3, \ 4$, or 5. Thus, we will prove this through the following three cases due to Remarks 1 and 2.

Case 1. L/K is a simple K_3-group.

Since 7 or $11 \mid |L/K| \mid 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$, we have known, from Lemma 2, that $L/K \cong L_2(7), L_2(8)$, or $U_3(3)$.

In the following, let P be the Sylow 7-subgroup of G. Then, $PK/K \in$Syl$_7(L/K)$, and by Lemma 5, $n_7(L/K)t = n_7(G)$ for some integer $t \in \mathbb{N}$.

Subcase 1.1. $L/K \cong L_2(7)(2^3 \cdot 3 \cdot 7)$.

From [18], $n_7(L/K) = n_7(L_2(7)) = 8$. Hence, $n_7(G) = 8t$ and $7 \nmid t$, so the number of elements of order 7 in G is $m = 8t \cdot 6 = 48t$.

Since $m \in$nse(G) and $7 \nmid t$, then $m = 4809024, \ 570240$ and $t = 100188, \ 11880$.

1. Let $m = 4809024$ and $t = 100188$. By Lemma 5 and since $|G| = |G : L||L : K||K|$, $100188|N_K(P)| = |K|$, so $2^2 \cdot 3^2 \cdot 11^2 \cdot 23 \big| |K|$. However, $23 \notin \pi(G)$, which is a contradiction.
2. Let $m = 570240$ and $t = 11880$. By Lemma 5 and since $|G| = |G : L||L : K||K|$, $11880|N_K(P)| = |K|$, so $2^3 \cdot 3^3 \cdot 5 \cdot 11 \big| |K| \big| 2^6 \cdot 3^4 \cdot 7 \cdot 11$, which is a contradiction.

Subcase 1.2. $L/K \cong L_2(8)(2^3 \cdot 3^2 \cdot 7)$.

From [18], $n_7(L/K) = n_7(L_2(8)) = 36$. Hence, $n_7(G) = 8t$ and $7 \nmid t$, so the number of elements of order 7 in G is $m = 36t \cdot 6 = 216t$.

Since $m \in$nse(G) and $7 \nmid t$, then $m = 4809024, 570240$ and $t = 22264, 2640$.

1. Let $m = 4809024$ and $t = 22264$. By Lemma 5 and since $|G| = |G : L||L : K||K|$, $22264|N_K(P)| = |K|$, so $2^3 \cdot 11^2 \cdot 23 \big| |K|$. However, $23 \notin \pi(G)$, which is a contradiction.
2. Let $m = 570240$ and $t = 2640$. By Lemma 5 and since $|G| = |G : L||L : K||K|$, $11880|N_K(P)| = |K|$. As $|K| \big| 2^6 \cdot 3^3 \cdot 5^2 \cdot 11$, $n_{11}(K) = 1, 12, 45, 100, 144, 320, 540, 1200, 1728$, so the number of elements of order 11 in G is 10, 120, 450, 1000, 1440, 3200, 12000, 17280 $\notin$nse(G). Also, we get a contradiction.

Subcase 1.3. $L/K \cong U_3(3)(2^5 \cdot 3^3 \cdot 7)$.

From [18], $n_7(L/K) = n_7(U_3(3)) = 288$. Hence, $n_7(G) = 288t$ and $7 \nmid t$, so the number of elements of order 7 in G is $m = 288t \cdot 6 = 1728t$.

Since $m \in$ nse(G) and $7 \nmid t$, then $m = 4809024, 570240$ and $t = 2783, 330$.

1. Let $m = 4809024$ and $t = 2783$. By Lemma 5 and since $|G| = |G : L||L : K||K|$, $2783|N_K(P)| = |K|$, so $11^2 \cdot 23 \mid |K|$. However, $23 \notin \pi(G)$, which is a contradiction.

2. Let $m = 570240$ and $t = 330$. By Lemma 5 and since $|G| = |G : L||L : K||K|$, $330|N_K(P)| = |K|$. As $|K| \mid 2^4 \cdot 3^2 \cdot 5^2 \cdot 11$, $n_{11}(K) = 1, 12, 45, 100, 144, 1200$, so the number of elements of order 11 in G is 10, 120, 450, 1000, 1440, $12000 \notin \mathrm{nse}(G)$. Also, we get a contradiction.

Case 2. L/K is a simple K_4-group.

Since 7 or $11 \mid |L/K| \mid 2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$, and then by Lemma 3, we have the following subcases:

Subcase 2.1. $L/K \cong A_i$, where $i = 7, 8, 9, 10$.

If $L/K \cong A_7$, then from [18], $n_7(L/K) = n_7(A_7) = 120$. Hence, $n_7(G) = 120t$ and $7 \nmid t$. The number of elements of order 7 in G is $m = 120t \cdot 6 = 720t$.

Since $m \in \mathrm{nse}(G)$ and $7 \nmid t$, then $m = 570240, 8553600, 11404800, 13685760$ and $t = 792,\ 11880,\ 15840,\ 19008$.

Let $m = 570240$ and $t = 792$. By Lemma 5 and since $|G| = |G : L||L : K||K|$, $792|N_K(P)| = |K|$. As $|K| \mid 2^6 \cdot 3^3 \cdot 5 \cdot 11$, $n_{11}(K) = 1, 12, 45, 100, 144, 320, 540, 1728$, so the number of elements of order 11 in G is 10, 120, 450, 1000, 1440, 3200, 5400, $17280 \notin \mathrm{nse}(G)$. Also, we get a contradiction.

For $m = 8553600,\ 11404800, 13685760$ and $t = 11880,\ 15840,\ 19008$, the proofs are similar to $m = 570240$ and $t = 792$. We get contradictions.

For A_i, where $i = 8, 9$, or 10, we also get a contradiction.

Subcase 2.2. $L/K \cong M_{11}, M_{12}, J_2$.

The proof is similar to Subcase 2.1.

Subcase 2.3. L/K is isomorphic to one of the group as listed in item number 3 of Lemma 3.

The proof is similar to Subcase 2.1.

Case 3. L/K is a simple K_5-group.

In this case, $\pi(L/K) = \{2, 3, 5, 7, 11\}$. From Lemma 6, we have $G \cong A_{12}$.

This completes the proof. □

Conclusions

We proved that a finite group G is isomorphic to A_{12}, the alternating group A_{12} of degree 12 if, and only if, $|G| = |A_{12}|$ and nse(G)=nse(A_{12}).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

SL and RZ carried out the studies and participated in drafting the manuscript. Both authors read and approved the final manuscript.

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References

1. Chillag, D, Herzog, M: On the length of the conjugacy classes of finite groups. Journal of Algebra. **131**(1), 110–125 (1990)
2. Conway, JH, Wang, Y: Remarks on the length of conjugacy classes of finite groups. Communications in Algebra. **27**(9), 4347–4353 (1999)
3. Dolfi, S, Moretó, A, Navarro, G: The groups with exactly one class of size a multiple of p. Journal of Group Theory. **12**, 219–234 (2009)
4. Guo, S, Liu, S, Shi, W: A new characterization of alternating group A_{13}. Far East Journal of Mathematical Sciences. **62**(1), 15–28 (2012)
5. Gorenstein, D: Finite Groups. 2nd edition. Chelsea Publishing Company, New York (1980)
6. Kurzweil, H, Stellmacher, B: The Theory of Finite Groups: An Introduction. Springer-Verlag, New York (2004)
7. Shi, WJ: A new characterization of the sporadic simple groups in group theory. In *Proceeding of the 1987 Singapore Group Theory Conference*. Walter de Gruyter, Berlin, 531-540 (1989)
8. Khatami, M, Khosravi, B, Akhlaghi, Z: A new characterization for some linear groups. Monatsch Math. **169**, 39–50 (2011)
9. Shao, C, Jiang, Q: A new characterization of A_{11}. Journal of Suzhou University (Natural Science Edition). **24**(2), 11-14 (in Chinese) (2008)
10. Shao, C, Shi, W, Jiang, Q: A characterization of simple K_3-groups. Advances in Mathematics. **38**(3), 327-330 (in Chinese) (2009)
11. Shen, R, Shao, C, Jiang, Q, Shi, W, Mazurov, V: A new characterization of A_5. Monatsch Math. **160**, 337–341 (2010)
12. Hall, P: A note on soluble groups. Journal of the London Mathematical Society series 1. **3**, 98–105 (1928)
13. Bugeaud, Y, Cao, Z, Mignotte, M: On simple K_4-groups. Journal of Algebra. **241**, 658–668 (2001)
14. Shi, WJ: On simple K_4-groups. Chinese Science Bulletin. **36**(17), 1281-1283 (in Chinese) (1991)
15. Herzog, M: Finite simple groups divisible by only three primes. Journal of Algebra. **10**, 383–388 (1968)
16. Jafarzadeh, A, Iranmanesh, A: On simple K_n-groups for $n = 5, 6$(In Campbell, CM, Quick, MR, Robertson, EF, Smith, GC, eds.) Groups St. Andrews 2005. London Mathematical Society lecture note series, vol. 2, pp. 668-680, Cambridge University Press , Cambridge, (2007)
17. Shao, C, Jiang, Q: A new charcterization of Mathieu groups. Archivum Mathematicum (BRNO). **46**, 13–23 (2010)
18. Conway, JH, Curtis, RT, Norton, SP, Parker, RA, Wilson, RA: Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups. Clarendon Press, Oxford (1985)

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