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Graded coprime submodules

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Abstract

Let G be a group. Let R be a G-graded commutative ring with identity, and let M be a G-graded module over R. Two graded submodules N and K of graded module M are called graded coprime whenever $N + K = M$. In this paper, some properties of graded coprime submodules are discussed. For example, we show that if M is a graded finitely generated module, then two graded submodules N and K of M are graded coprime if and only if $grad_M(N)$ and $grad_M(K)$ are graded coprime.

Keywords: Graded multiplication module, Graded coprime submodule, Graded cancelation module

MSC: 13A02; 16W50

Introduction

We define a G-graded ring R and a G-graded module over R in the same way as in [1] and [2]. Let G be a group with identity e and R be a commutative ring. Then, R is a G-graded ring if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$; here, $R_g R_h$ denotes the additive subgroup of R consisting of all finite sums of elements $r_g k_h$ with $r_g \in R_g$ and $k_h \in R_h$. Moreover, R_e is a subring of R and $1_R \in R_e$. We denote this by (R, G). The elements of R_g are called homogeneous of degree g. If $x \in R$, then x can be written uniquely as $\Sigma_{g\in G} x_g$, where x_g is the component of x in R_g. Also, we write $h(R) = \cup_{g\in G} R_g$.

Let R be a G-graded ring and M be an R-module. We say that M is a graded R-module if there exists a family of submodules $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$, and we write $h(M) = \cup_{g\in G} M_g$.

Throughout this paper, G is a group, R is a G-graded commutative ring with identity, and M is a G-graded module over R. Also, for basic properties of coprime ideals, one may refer to [3].

The concept of multiplication module has been studied by various authors (see, for example, [4,5]). Also, the notion of the product of two submodules of a multiplication module has been studied in [6].

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We define a graded multiplication module and the product of two graded submodules of a graded multiplication module in the same way as in [7].

Let R be a graded ring. A graded R-module M is said to be a *graded multiplication module* if for every graded submodule N of M, there exists a graded ideal I of $h(R)$ such that $N = IM$. Assume that M is a graded multiplication R-module. If N and K are graded submodules of M, then there exist graded ideals I and J of $h(R)$ such that $N = IM$ and $K = JM$. Then, *the product* of N and K is defined to be $(IJ)M$ and is denoted by $N * K$. In fact, IJ is a graded ideal of R by [7, Lemma 1.1], and $N * K$ is well-defined and is independent of the choices of I and J by [6, Theorem 3.4], and [2, Theorem 4]. Also, for every positive integer t, N^t is defined to be

$$\overbrace{N * N * \ldots * N}^{t\text{ times}}.$$

Lemma 1.1. *[7, Lemma 1.2], Let R be a graded ring and M be a graded R-module.*

(i) If N and K are graded submodules of M, then $N + K$ and $N \cap K$ are graded submodules of M.
(ii) If a is an element of $h(R)$ and x is an element of $h(M)$, then aM and Rx are graded submodules of M.
(iii) If N and K are graded submodules of M, then $(N :_R K)$ is a graded ideal of R.

Definition 1.2. Let R be a graded ring and M be a graded module over R. Let P be a proper graded submodule of M.

(i) P is called a *graded prime submodule* of M whenever $am \in P$ implies that $m \in P$ or $a \in (P :_R M)$ where $a \in h(R)$ and $m \in h(M)$.

(ii) P is called a *graded semiprime submodule* of M whenever $I^n K \subseteq P$ implies that $IK \subseteq P$ where $I \subseteq h(R)$ and $K \subseteq h(M)$.

(iii) P is called a *graded maximal submodule* of M if there is no graded submodule K of M such that $P \subset K \subset M$.

Theorem 1.3. *[2, Theorem 5], Let R be a graded ring and M be a graded multiplication R-module. Let N be a proper graded submodule of M. Then, N is graded prime if and only if $K * L \subseteq N$ implies that $K \subseteq N$ or $L \subseteq N$ for graded submodules K and L of M.*

Theorem 1.4. *[7, Theorem 2.1], Let R be a graded ring and M be a graded multiplication R-module. Let N be a graded submodule of M. Then, N is graded semiprime if and only if $(Rx)^n \subseteq N$ implies that $x \in N$ for each $x \in h(M)$ and positive integer n.*

The graded radical of a graded submodule N of a graded module M is the intersection of all graded prime submodules of M containing N and is denoted by $grad_M(N)$. If there is no graded prime submodule of M containing N, then we say $grad_M(N) = M$. Also, $grad_M(M) = M$. It is easy to show that if M is a graded multiplication module, then $grad_M(N)$ is the set of all elements m of $h(M)$ such that $(Rm)^k \subseteq N$ for some positive integer k.

Results and discussion

Let $G = (\mathbb{Z}, +)$ and $R = (\mathbb{Z}, +, \cdot)$. Clearly, R is a G-graded ring. Let $M = \mathbb{Z} \times \mathbb{Z}$. So, M is a G-graded R-module. Let $x, y \in h(M)$. Consider the graded submodules $N = (Rx) \times 0$ and $K = (Ry) \times 0$ of M. Then, $N + K$ is the graded submodule generated by the greatest common factor of x and y.

Definition 2.1. Let R be a graded ring and M be a graded module over R; two graded submodules N and K of M are called graded coprime whenever $N + K = M$.

Clearly, two distinct graded maximal submodules of a graded module are graded coprime.

Proposition 2.2. *Let R be a graded ring and M be a graded multiplication module over R. Let N_1 and K_1 be two graded coprime submodules of M. Let N_2 and K_2 be two graded submodules of M such that every element of N_1 (resp. K_1) has a power in N_2 (resp. K_2). Then, N_2 and K_2 are graded coprime.*

Proof. Under the given hypothesis, every graded prime submodule which contains N_2 (resp. K_2) contains N_1 (resp. K_1). Let P be a graded prime submodule of M which contains both N_2 and K_2. So, P contains N_1 and K_1, that is, $N_1 + K_1 \subseteq P$, which is absurd. Hence, no graded prime submodule contains both N_2 and K_2. Therefore, N_2 and K_2 are graded coprime. □

Proposition 2.3. *Let R be a graded ring and M be a graded multiplication module over R. Let N and K be graded coprime submodules of M. Then, $N * K = N \cap K$.*

Proof. Similar to the proof of [6, Proposition 3.5]. □

Proposition 2.4. *Let R be a graded ring and M be a graded multiplication module over R with this property that every graded submodule of M is graded semiprime. Let $N_1, N_2, ..., N_t$ be graded submodules of M such that N_i and N_j are graded coprime whenever $i \neq j$. Then, for each $i (1 \leq i \leq t)$,*

$$\begin{aligned} &N_i + (N_1 * ... * N_{i-1} * N_{i+1} * ... * N_t) \\ &\quad = N_i + (N_1 \cap ... \cap N_{i-1} \cap N_{i+1} \cap ... \cap N_t) = M. \end{aligned}$$

Furthermore,

$$N_1 * N_2 * ... * N_t = N_1 \cap N_2 \cap ... \cap N_t.$$

Proof. Since

$$N_1 * ... * N_{i-1} * N_{i+1} * ... * N_t \subseteq N_1 \cap ... \cap N_{i-1} \cap N_{i+1} \cap ... \cap N_t,$$

it is enough to show that

$$N_i + (N_1 * ... * N_{i-1} * N_{i+1} * ... * N_t) = M.$$

Without loss of generality, let $i = 1$. Suppose that $2 \leq j \leq t$. Let m be an arbitrary element of $h(M)$. So, there exist elements $x_{1j} \in N_1, x_j \in N_j$ such that $x_{1j} + x_j = m$. So,

$$\begin{aligned} (Rm)^t &= R(x_{12} + x_2) * R(x_{13} + x_3) * ... * R(x_{1t} + x_t) \\ &= Rx + (Rx_2 * Rx_3 * ... * Rx_t), \end{aligned}$$

where $x \in N_1$. Therefore, $(Rm)^t \subseteq N_1 + (N_2 * N_3 * ... * N_t)$. Now, $N_1 + (N_2 * N_3 * ... * N_t)$ graded semiprime implies that $m \in N_1 + (N_2 * N_3 * ... * N_t)$, as required.

We prove the next result by induction on t. When $t = 1$, there is nothing to prove, and when $t = 2$, the required result follows from Proposition 2.3. Now, assume that $t > 2$ and the relation has been proved for all smaller values of the inductive variable. So,

$$N_1 * N_2 * ... * N_{t-1} = N_1 \cap N_2 \cap ... \cap N_{t-1}.$$

Therefore,

$$N_1 * N_2 * ... * N_{t-1} * N_t = (N_1 \cap N_2 \cap ... \cap N_{t-1}) * N_t.$$

But we showed that $N_1 \cap N_2 \cap ... \cap N_{t-1}$ and N_t are graded coprime. Hence, by applying Proposition 2.3, we find that

$$(N_1 \cap N_2 \cap ... \cap N_{t-1}) * N_t = N_1 \cap N_2 \cap ... \cap N_{t-1} \cap N_t.$$

Accordingly,

□

$$N_1 * N_2 * ... * N_t = N_1 \cap N_2 \cap ... \cap N_t.$$

The following result can be obtained by the above proposition easily.

Corollary 2.5. *Let R be a graded ring and M be a graded multiplication module over R. Let N and K be graded coprime submodules of M. Then, N and K^t are graded coprime for every positive integer t.*

Proposition 2.6. *With the assumption in Proposition 2.4, let $\varphi : M \to \Pi_{i=1}^{t}(M/N_i)$ be a homomorphism by the rule $\varphi(m) = (m + N_1, ..., m + N_t)$. Then, φ is injective if and only if $N_1 * N_2 * ... * N_t = 0$.*

Proof. It is enough to show that $ker\varphi = N_1 * N_2 * ... * N_t$. Let $x \in ker\varphi$. So, $x \in N_i$ for every $i(1 \leq i \leq t)$. Thus, $(Rx)^t \subseteq N_1 * N_2 * ... * N_t$ and $N_1 * N_2 * ... * N_t$ graded semiprime implies that $x \in N_1 * N_2 * ... * N_t$, that is, $ker\varphi \subseteq N_1 * N_2 * ... * N_t$.

Conversely, let $x \in N_1 * N_2 * ... * N_t$. Therefore, $x \in N_1 \cap N_2 \cap ... \cap N_t$. Hence, $x + N_i = N_i$ for every $i(1 \leq i \leq t)$. Now, we conclude that $\varphi(x) = (N_1, ..., N_t)$, that is, $N_1 * N_2 * ... * N_t \subseteq ker\varphi$. □

Proposition 2.7. *Let R be a graded ring and M be a graded multiplication module over R. Let P be a graded maximal submodule of M. Then, for every positive integer n, the only graded prime submodule containing P^n is P.*

Proof. Let $\acute{P}$ be a graded prime submodule of M containing P^n. So, by Theorem 1.3, $P \subseteq \acute{P}$ as P is grade maximal; $P = \acute{P}$. □

Proposition 2.8. *Let R be a graded ring and M be a graded module over R. Let $N = IM$ and $K = JM$ be graded submodules of M where I and J are graded coprime ideals of R. Then, N and K are graded coprime.*

Proof. Since I and J are graded coprime ideals in R, we have

$$M = RM = (I + J)M \subseteq IM + JM = N + K.$$

□

Definition 2.9. Let R be a graded ring and M be a graded module over R. Let I and J be graded ideals of R. Then, M is called *graded cancelation* whenever $IM = JM$ gives $I = J$.

Theorem 2.10. *Let R be a graded ring and M be a graded finitely generated cancelation module over R. Let N and K be graded submodules of M with graded presentation ideals I and J, respectively. Then, I and J are graded coprime in R if and only if N and K are graded coprime in M.*

Proof. One direction is proved in Proposition 2.8.

Since N and K are graded coprime submodules in a graded finitely generated module M, we have

$$RM = M = N + K = IM + JM = (I + J)M.$$

Now, M graded cancelation implies that $R = I + J$, as required. □

Corollary 2.11. *Let M be a graded finitely generated module. Then, every proper graded submodule of M is contained in a graded prime submodule.*

Proof. Similar to the proof of [8, Corollary 1]. □

Proposition 2.12. *Let R be a graded ring and M be a graded module over R. Let N and K be graded submodules of M. then,*

(i) $N \subseteq grad_M(N)$.
(ii) $grad_M(grad_M(N)) = grad_M(N)$.
(iii) *If* $M = N$, *then* $grad_M(N) = M$.

Moreover, if M is finitely generated, then $grad_M(N) = M$ if and only if $N = M$.

(iv) $grad_M(N + K) = grad_M(grad_M(N) + grad_M(K))$.

Also, if M is a graded multiplication module, then

(v) $grad_M(N * K) = grad_M(N \cap K) = grad_M(N) \cap grad_M(K)$.
(vi) *Let N be graded prime. Then $grad_M(N^t) = N$ for every positive integer t.*

Proof. (i,ii) (i) and (ii) are trivial by definition of $grad_M(N)$.

(iii) Suppose that $N = M$. So, $grad_M(N) = grad_M(M) = M$. Also, let M be finitely generated and $grad_M(N) = M$. Suppose to the contrary that $N \neq M$. Hence, by Corollary 2.11,

$grad_M(N) \subseteq P$ for some graded prime submodule P of M. Therefore, $grad_M(N) \neq M$, a contradiction.

(iv) $N \subseteq grad(N)$ and $K \subseteq grad(K)$ implies that $N + K \subseteq grad(N) + grad(K)$. Therefore, $grad(N + K) \subseteq grad(grad(N) + grad(K))$. Also, $N, K \subseteq N + K$, hence $grad(N), grad(K) \subseteq grad(N + K)$. So, $grad(N) + grad(K) \subseteq grad(N + K)$, and by (ii), $grad(grad(N) + grad(K)) \subseteq grad(N + K)$.

(v) We prove the first equality. Suppose that $x \in grad(N * K)$. Then, $(Rx)^t \subseteq N * K \subseteq N \cap K$ for some positive integer t. So, $x \in grad_M(N \cap K)$. Conversely, suppose $x \in grad_M(N \cap K)$. Then, $(Rx)^t \subseteq N$ and $(Rx)^t \subseteq K$ for some positive integer t. Hence, $(Rx)^{2t} = (Rx)^t * (Rx)^t \subseteq N * K$, that is, $x \in grad_M(N * K)$.
Now, we prove the second equality. Let $x \in grad_M(N \cap K)$. So, $(Rx)^t \subseteq N \cap K \subseteq grad_M(N) \cap grad_M(K)$ for some positive integer t. Thus, $(Rx)^t \subseteq grad_M(N)$ and $(Rx)^t \subseteq grad_M(K)$. Hence, $x \in grad_M(N) \cap grad_M(K)$.
Conversely, if $x \in grad_M(N) \cap grad_M(K)$, then $(Rx)^n \subseteq N$ and $(Rx)^k \subseteq K$ for some positive integers n, k. Therefore, $(Rx)^{nk} \subseteq N \cap K$, giving us that in fact $x \in grad_M(N \cap K)$.

(vi) Note that $N \subseteq grad_M(N^t)$. Now, suppose $x \in grad_M(N^t)$. Then, $(Rx)^n \subseteq N^t \subseteq N$, for some positive integer n. Now, N graded prime implies that $x \in N$, as needed.

□

Proposition 2.13. *Let R be a graded ring and M be a graded finitely generated module over R. Let N and K be graded submodules of M. Then, N and K are graded coprime if and only if $grad_M(N)$ and $grad_M(K)$ are graded coprime.*

Proof. By using (iii) and (iv) above, we have

$$N + K = M \Leftrightarrow grad_M(N + K) = M$$

$$\Leftrightarrow grad_M(grad_M(N) + grad_M(K)) = M$$

$$\Leftrightarrow grad_M(N) + grad_M(K) = M.$$

□

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