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Euler-related sums

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Abstract

Purpose: The purpose of this paper is to develop a set of identities for Euler type sums of products of harmonic numbers and reciprocal binomial coefficients.

Method: We use analytical methods to obtain our results.

Results: We obtain identities for variant Euler sums of the type $\sum_{n=1}^{\infty} \frac{H_{2n}}{n\binom{n+k}{k}}$, and its finite counterpart, which generalize some results obtained by other authors.

Conclusions: Identities are successfully achieved for the sums under investigation. Some published results have been successfully generalized.

Keywords: Harmonic numbers, Binomial coefficients and gamma function, Polygamma function, Combinatorial series identities and summation formulas, Partial fraction approach

PAC Codes: MSC (2000), primary: 05A10, 05A19, 11B65; secondary: 11B83, 11M06, 33B15, 33D60, 33C20

Background and preliminaries

In the spirit of Euler, we shall investigate the summation of some variant Euler sums. In common terminology, let, as usual,

$$H_n = \gamma + \psi(n+1) = \sum_{r=1}^{n} \frac{1}{r} = \int_0^\infty \frac{1-t^n}{1-t} dt$$

be the n^{th} harmonic number, γ denotes the Euler-Mascheroni constant, $\psi(z) := d \log \Gamma(z)/dz$ is the digamma function and $\Gamma(z)$ is the well-known gamma function. Let also, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ denote, respectively, the sets of real, complex and natural numbers. A generalized binomial coefficient $\binom{w}{z}$ may be defined by

$$\binom{w}{z} := \frac{\Gamma(w+1)}{\Gamma(z+1)\Gamma(w-z+1)} \quad (w, z \in \mathbb{C})$$

and in the special case when $z = n, n \in \mathbb{N}$, we have

$$\binom{w}{n} := \frac{w(w-1)\dots(w-n+1)}{n!} = \frac{(-1)^n(-w)_n}{n!},$$

where

$$(w)_\lambda := \frac{\Gamma(w+\lambda)}{\Gamma(w)} = \begin{cases} 1 \ (\lambda=0;\ w \in \mathbb{C}\setminus\{0\}) \\ w(w+1)\dots(w+\lambda-1)(w \in \mathbb{C},\ \lambda \in \mathbb{N}) \end{cases}$$

with $(0)_0 := 1$ is known as the Pochhammer symbol. Some well-known Euler sums are (see, e.g., [1])

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3), \quad \sum_{n=1}^{\infty} \left(\frac{H_n}{n}\right)^2 = \frac{17}{4}\zeta(4);$$

recently, Chen [2] obtained

$$\sum_{n=1}^{\infty} \frac{H_{2n}}{(2n+1)^2} = \frac{7}{16}\zeta(3), \quad \sum_{n=1}^{\infty} \frac{H_{2n}-\frac{1}{2}H_n}{4^n n} = \frac{1}{4}(\ln 3)^2.$$

In [3], we have, for $k \geq 1$,

$$\sum_{n-1}^{\infty} \frac{H_n}{n\binom{n+k}{k}} = 2\zeta(2) + \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \times \left(H_{r-1}^2 + H_{r-1}^{(2)}\right) \tag{1}$$

and in [4],

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$$\sum_{n-1}^{\infty} \frac{H_n}{n\binom{n+k}{k}} = H_p^{(2)} + H_p^2 + \sum_{r=1}^{k} (-1)^r \binom{k}{r} \times \begin{pmatrix} H_pH_{p+r} - H_pH_{r-1} \\ -\sum_{m=1}^{r-1} \left(\frac{H_m - H_{m+p}}{m}\right) \end{pmatrix}, \tag{2}$$

where $H_n^{(r)}$ denotes the *generalized* n^{th} *harmonic number* in power r defined by

$$H_n^{(r)} := \sum_{m=1}^{n} \frac{1}{m^r} \quad (n, r \in \mathbb{N}).$$

We study, in this paper, $\sum_{n=1}^{\infty} \frac{H_{2n}}{n\binom{n+k}{k}}$ and its finite counterpart. Analogous results of Euler type for infinite series have been developed by many authors, see for example [5,6] and references therein. Many finite versions of harmonic number sum identities also exist in the literature, for example in [7], we have

$$\sum_{n=0}^{p} (-1)^n \binom{p}{n} H_{n+b}^2 = \frac{2H_{p-1} - H_{p+b} - H_b}{p\binom{p+b}{b}},$$

and in [8],

$$\sum_{n=0}^{p} (-1)^n \binom{p+n}{n}\binom{p}{n} nH_n = (-1)^p p(p+1)(2H_p - 1).$$

Also, from the study of Prodinger [9],

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} (-1)^{n-k} H_k = \sum_{m=1}^{n} \frac{2}{m} = 2H_n.$$

Further work in the summation of harmonic numbers and binomial coefficients has also been done by Sofo [10]. The works of [11-17] and references therein also investigate various representations of binomial sums and zeta functions in a simpler form by the use of the beta function and by means of certain summation theorems for hypergeometric series.

Lemma 1. *Let n and r be positive integers. Then we have*

$$\sum_{r=1}^{n} \frac{1}{2r-1} = \frac{1}{2} H_{n-\frac{1}{2}} + \ln 2 \tag{3}$$

$$= H_{2n} - \frac{1}{2} H_n; \tag{4}$$

$$\sum_{m=1}^{r-1} \frac{H_m}{2m+1} = \frac{1}{2} H_{r-1} H_{r-\frac{1}{2}} - \frac{1}{2} \sum_{s=1}^{r-1} \frac{H_{s-\frac{1}{2}}}{s}; \tag{5}$$

$$\sum_{m=1}^{r} \left(\frac{H_m}{2m+1} + \frac{H_{2m}}{m}\right) = H_r H_{2r+1} + \frac{1}{4}\left(H_r^{(2)} - H_r^2\right). \tag{6}$$

Proof. From the definition of harmonic numbers and the digamma function,

$$H_{n-\frac{1}{2}} = \gamma + \psi\left(n + \frac{1}{2}\right) = \gamma + 2\sum_{r=1}^{n} \frac{1}{2r-1} - \gamma - 2\ln 2$$

and Equation 3 follows. From the double argument identity of the digamma function

$$\psi(2n) = \frac{1}{2}\psi(n) + \frac{1}{2}\psi\left(n + \frac{1}{2}\right) + \ln 2$$

$$H_{2n-1} - \gamma = \frac{1}{2}(H_{n-1} - \gamma) + \frac{1}{2}\left(H_{n-\frac{1}{2}} - \gamma\right) + \ln 2,$$

using Equation 3 and rearranging, we obtain Equation 4. For Equation 5, we first note that for an arbitrary sequence $\{X_{k,l}\}$, the following identity holds:

$$\sum_{k=1}^{n}\sum_{l=1}^{k} X_{k,l} = \sum_{l=1}^{n}\sum_{k=l}^{n} X_{k,l};$$

hence,

$$\sum_{m=1}^{r-1} \frac{H_m}{2m+1} = \sum_{m=1}^{r-1}\sum_{s=1}^{m} \frac{1}{s(2m+1)} = \sum_{s=1}^{r-1}\sum_{m=s}^{r-1} \frac{1}{s(2m+1)}$$

$$= \sum_{s=1}^{r-1} \frac{1}{2s}\left(H_{r-\frac{1}{2}} - H_{s-\frac{1}{2}}\right) = \frac{1}{2} H_{r-1} H_{r-\frac{1}{2}} - \frac{1}{2}\sum_{s=1}^{r-1} \frac{H_{s-\frac{1}{2}}}{s}.$$

The interesting identity (Equation 6) follows from Equation 5 and substituting

$$\frac{1}{2} H_{n-\frac{1}{2}} = H_{2n} - \frac{1}{2} H_n - \ln 2 \tag{7}$$

so that

$$\sum_{m=1}^{r-1} \frac{H_m}{2m+1} = H_{r-1}\left(H_{2r} - \frac{1}{2}H_r - \ln 2\right) - \sum_{m=1}^{r-1} \frac{\left(H_{2m} - \frac{1}{2}H_m - \ln 2\right)}{m}$$

$$
\begin{aligned}
&= H_{r-1}\left(H_{2r}-\frac{1}{2}H_r-\ln 2\right)+H_{r-1}\ln 2\\
&\quad+\frac{1}{2}\sum_{m=1}^{r-1}\frac{H_m}{m}-\sum_{m=1}^{r-1}\frac{H_{2m}}{m}\\
&= H_{r-1}\left(H_{2r}-\frac{1}{2}\left(\frac{1}{r}+H_{r-1}\right)\right)+\frac{1}{4}\\
&\quad\times\left(H_{r-1}^2+H_{r-1}^{(2)}\right)-\sum_{m=1}^{r-1}\frac{H_{2m}}{m}\sum_{m=1}^{r-1}\\
&\quad\times\left(\frac{H_m}{2m+1}+\frac{H_{2m}}{m}\right)=H_{r-1}H_{2r-1}\\
&\quad+\frac{1}{4}\left(H_{r-1}^{(2)}-H_{r-1}^2\right)
\end{aligned}
$$

replacing the counter, we obtain Equation 6. □

Main results and discussion

We now prove the two following theorems:

Theorem 1. *Let $k \in \mathbb{N}$. Then we have*

$$
\sum_{n=1}^{\infty}\frac{H_{2n}-\frac{1}{2}H_n}{n\binom{n+k}{k}}=\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\times\begin{pmatrix}H_{r-1}H_{2r-1}+\frac{1}{4}\left(H_{r-1}^{(2)}-H_{r-1}^2\right)\\+2\ln 2\left(H_{2r}-\frac{1}{2}H_r\right)-\sum_{j=1}^{r-1}\frac{H_{2j}}{j}\end{pmatrix}. \tag{8}
$$

Proof. Let $h_n = H_{2n} - \frac{1}{2}H_n$ and consider the following expansion:

$$
\sum_{n=1}^{\infty}\frac{h_n}{n\binom{n+k}{k}}=\sum_{n=1}^{\infty}\frac{k!\,h_n}{n\prod_{r=1}^{k}(n+r)}=\sum_{n=1}^{\infty}\frac{k!\,h_n}{n\,(n+1)_{k+1}}.
$$

Now,

$$
\sum_{n=1}^{\infty}\frac{h_n}{n\binom{n+k}{k}}=\sum_{n=1}^{\infty}\frac{k!\,h_n}{n}\sum_{r=1}^{k}\frac{A_r}{n+r}, \tag{9}
$$

where

$$
A_r=\lim_{n\to -r}\frac{n+r}{\prod_{r=1}^{k}n+r}=\frac{(-1)^{r+1}\,r}{k!}\binom{k}{r}. \tag{10}
$$

For an arbitrary positive sequence $\{X_{k,p}\}$, the following identity holds:

$$
\sum_{k=0}^{\infty}\sum_{p=0}^{n}X_{p,k}=\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}X_{p,k+p};
$$

hence, from Equations 4 and 9,

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{k!\,h_n}{n}\sum_{r=1}^{k}\frac{A_r}{n+r}&=\sum_{r=1}^{k}(-1)^{r+1}\,r\binom{k}{r}\\
&\quad\times\sum_{n=1}^{\infty}\frac{1}{n\,(n+r)}\sum_{j=1}^{n}\frac{1}{2j-1}\\
&=\sum_{r=1}^{k}(-1)^{r+1}\,r\binom{k}{r}\sum_{j=1}^{\infty}\frac{1}{2j-1}\\
&\quad\times\sum_{n=0}^{\infty}\frac{1}{(n+j)\,(n+j+r)}\\
&=\sum_{r=1}^{k}(-1)^{r+1}\,r\binom{k}{r}\\
&\quad\times\sum_{j=1}^{\infty}\frac{1}{2j-1}\left(\frac{\psi\,(j+r)-\psi\,(r)}{r}\right).
\end{aligned}
$$

Since we notice that

$$
\frac{\psi\,(j+r)-\psi\,(r)}{r}=\frac{1}{j}+\sum_{m=1}^{r-1}\frac{1}{m+j},
$$

we get

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{h_n}{n\binom{n+k}{k}}&=\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\sum_{j=1}^{\infty}\frac{1}{2j-1}\\
&\quad\times\left(\frac{1}{j}+\sum_{m=1}^{r-1}\frac{1}{m+j}\right)=\sum_{r=1}^{k}(-1)^{r+1}\\
&\quad\times\binom{k}{r}\left(2\ln 2+\sum_{m=1}^{r-1}\frac{2\ln 2+H_m}{2m+1}\right)\\
&=\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\left(2\ln 2+2\ln 2\right.\\
&\quad\left.\times\left(-1+\ln 2+\frac{1}{2}H_{r-\frac{1}{2}}\right)+\sum_{m=1}^{r-1}\frac{H_m}{2m+1}\right)\\
&=2\ln^2(2)+\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\\
&\quad\times\left(\ln 2H_{r-\frac{1}{2}}+\sum_{m=1}^{r-1}\frac{H_m}{2m+1}\right).
\end{aligned}
$$

Now,

$$\sum_{n=1}^{\infty}\frac{h_n}{n\binom{n+k}{k}} = \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r} \times \begin{pmatrix} \ln 2H_{r-\frac{1}{2}}+\frac{1}{2}H_{r-1}H_{r-\frac{1}{2}} \\ -\frac{1}{2}\sum_{s=1}^{r-1}\frac{H_{s-\frac{1}{2}}}{s}\end{pmatrix} + 2\ln^2(2)$$

$$= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r} \times \begin{pmatrix} \frac{1}{2}H_{r-1}H_{r-\frac{1}{2}}-\frac{1}{2}\sum_{s=1}^{r-1}\frac{H_{s-\frac{1}{2}}}{s} \\ +\ln 2\left(-2\ln 2+2\sum_{j=1}^{r}\frac{1}{2j-1}\right)\end{pmatrix}$$

$$+ 2\ln^2(2) = \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r} \times \left(\frac{1}{2}H_{r-1}H_{r-\frac{1}{2}}+\sum_{j=1}^{r}\frac{2\ln 2}{2j-1} -\frac{1}{2}\sum_{j=1}^{r-1}\frac{H_{j-\frac{1}{2}}}{j}\right);$$

substituting Equation 7 and simplifying, we have

$$= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r} \times \begin{pmatrix} H_{r-1}\left(H_{2r}-\frac{1}{2}\left(H_{r-1}+\frac{1}{r}\right)\right)+2\ln 2\left(H_{2r}-\frac{1}{2}H_r\right) \\ +\frac{1}{4}\left(H^2_{r-1}+H^{(2)}_{r-1}\right)-\sum_{j=1}^{r-1}\frac{H_{2j}}{j}\end{pmatrix};$$

hence, the identity (Equation 8) follows. ☐

Corollary 1. *From Equation 8 and using Equations 3 and 4, we obtain the results,*

$$\sum_{n=1}^{\infty}\frac{H_{2n}}{n\binom{n+k}{k}} = \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r} \times \begin{pmatrix} H_{r-1}H_{2r-1}+\frac{1}{4}\left(5H^{(2)}_{r-1}+3H^2_{r-1}\right) \\ +2\ln 2\left(H_{2r}-\frac{1}{2}H_r\right)-\sum_{j=1}^{r-1}\frac{H_{2j}}{j}\end{pmatrix} + \frac{\zeta(2)}{2} \tag{11}$$

and

$$\sum_{n=1}^{\infty}\frac{H_{n-\frac{1}{2}}}{n\binom{n+k}{k}} = 2\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r} \times \begin{pmatrix} H_{r-1}H_{2r-1}+\frac{1}{4}\left(H^{(2)}_{r-1}-H^2_{r-1}\right) \\ +2\ln 2\left(H_{2r}-\frac{1}{2}H_r\right)-\sum_{j=1}^{r-1}\frac{H_{2j}}{j}\end{pmatrix} - \frac{2\ln 2}{k}. \tag{12}$$

Proof. We can use Equations 3 and 4 and also note that

$$\sum_{n=1}^{\infty}\frac{\ln 2}{n\binom{n+k}{k}} = \frac{\ln 2}{k}. \tag{13}$$

From the rearrangement of $\sum_{n=1}^{\infty}\frac{H_{2n}-\frac{1}{2}H_n}{n\binom{n+k}{k}}$ and Equation 1, we can obtain Equation 11; and from the rearrangement of $\sum_{n=1}^{\infty}\frac{\frac{1}{2}H_{n-\frac{1}{2}}+\ln 2}{n\binom{n+k}{k}}$ and Equation 13, we can obtain Equation 12. ☐

Example 1. *For k=3 and 5,*

$$\sum_{n=1}^{\infty}\frac{H_{n-\frac{1}{2}}}{n\binom{n+3}{3}} = \frac{22\ln 2}{15}-\frac{11}{15}, \sum_{n=1}^{\infty}\frac{H_{2n}}{n\binom{n+3}{3}} = \frac{\zeta(2)}{2}+\frac{16\ln 2}{15}-\frac{119}{120}$$

$$\sum_{n=1}^{\infty}\frac{H_{n-\frac{1}{2}}}{n\binom{n+5}{5}} = \frac{386\ln 2}{315}-\frac{1321}{1890}, \sum_{n=1}^{\infty}\frac{H_{2n}}{n\binom{n+5}{5}} = \frac{\zeta(2)}{2}+\frac{256\ln 2}{315}-\frac{32093}{30240}$$

$$\sum_{n=1}^{\infty}\frac{H_{2n}-\frac{1}{2}H_n}{n\binom{n+3}{3}} = \frac{16\ln 2}{15}-\frac{11}{30}, \sum_{n=1}^{\infty}\frac{H_{2n}-\frac{1}{2}H_n}{n\binom{n+5}{5}} = \frac{256\ln 2}{315}-\frac{1321}{3780}.$$

Now, we consider the following finite version of Theorem 1:

Theorem 2. *Let $k, p \in \mathbb{N}$. Then we have*

$$\sum_{n=1}^{p} \frac{H_{2n} - \frac{1}{2} H_n}{n \binom{n+k}{k}} = \sum_{n=1}^{p} \frac{\frac{1}{2} H_{n-\frac{1}{2}} + \ln 2}{n \binom{n+k}{k}}$$

$$= H_p \left(H_{2p} - \frac{1}{2} H_p - 1 \right) + \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \begin{pmatrix} \left(H_{2p} - \frac{1}{2} H_p\right) \left(2H_{2r} - H_r - H_{p+r}\right) \\ + \sum_{m=1}^{r-1} \left(\frac{H_m - H_{m+p}}{2m+1} \right) \end{pmatrix} \tag{14}$$

Proof. To prove Equation 14, we may write

$$\sum_{n=1}^{p} \frac{h_n}{n \binom{n+k}{k}} = \sum_{n=1}^{p} \frac{k!\; h_n}{n} \sum_{r=1}^{k} \frac{A_r}{n+r},$$

where A_r is given by Equation 10, and by a rearrangement of sums,

$$\sum_{n=1}^{p} \frac{k!\; h_n}{n} \sum_{r=1}^{k} \frac{A_r}{n+r} = \sum_{r=1}^{k} (-1)^{r+1} r \binom{k}{r} \sum_{j=1}^{p} \sum_{n=j}^{p} \left(\frac{1}{n(n+r)(2j-1)} \right)$$

$$= \sum_{r=1}^{k} (-1)^{r+1} r \binom{k}{r} \sum_{j=1}^{p} \frac{1}{r(2j-1)} \begin{pmatrix} \psi(r+j) - \psi(j) \\ -\left(\psi(p+1+j) - \psi(p+1)\right) \end{pmatrix}$$

$$= \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \sum_{j=1}^{p} \frac{1}{(2j-1)} \begin{pmatrix} \frac{1}{j} + \sum_{m=1}^{r-1} \frac{1}{m+j} \\ -\frac{1}{p+1} + \sum_{m=1}^{r-1} \frac{1}{m+p+1} \end{pmatrix}$$

$$= \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \begin{pmatrix} 2\ln 2 + H_{p-\frac{1}{2}} - H_p - \frac{1}{p+1} \left(\ln 2 + \frac{1}{2} H_{p-\frac{1}{2}} \right) \\ + \sum_{m=1}^{r-1} \begin{pmatrix} \frac{1}{2m+1} \left(2\ln 2 + H_m + H_{p-\frac{1}{2}} - H_{m+p} \right) \\ -\frac{1}{p+m+1} \left(\ln 2 + H_{p-\frac{1}{2}} \right) \end{pmatrix} \end{pmatrix}$$

$$= 2\ln 2 + H_{p-\frac{1}{2}} - H_p - \frac{1}{p+1} \left(\ln 2 + \frac{1}{2} H_{p-\frac{1}{2}} \right)$$

$$+ \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \begin{pmatrix} \left(2\ln 2 + H_{p-\frac{1}{2}} \right) \left(-1 + \ln 2 + \frac{1}{2} H_{r-\frac{1}{2}} \right) \\ - \left(\ln 2 + \frac{1}{2} H_{p-\frac{1}{2}} \right) \left(H_{p+r} - H_{p+1} \right) \\ + \sum_{m=1}^{r-1} \left(\frac{1}{2m+1} \left(H_m - H_{m+p} \right) \right) \end{pmatrix}$$

$$= \ln 2 \left(2\ln 2 + H_{p-\frac{1}{2}} \right) - H_p - \frac{1}{p+1} \left(\ln 2 + \frac{1}{2} H_{p-\frac{1}{2}} \right) + H_{p+1} \left(\ln 2 + \frac{1}{2} H_{p-\frac{1}{2}} \right)$$

$$+ \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} \begin{pmatrix} \left(H_{r-\frac{1}{2}} - H_{p+r} \right) \left(\ln 2 + \frac{1}{2} H_{p-\frac{1}{2}} \right) \\ + \sum_{m=1}^{r-1} \left(\frac{1}{2m+1} \left(H_m - H_{m+p} \right) \right) \end{pmatrix}. \tag{15}$$

Substituting Equation 7 into Equation 15 and after simplification, Equation 14 follows. ☐

Corollary 2. *Let $k, p \in \mathbb{N}$. Then we obtain*

$$\sum_{n=1}^{p} \frac{H_{n-\frac{1}{2}}}{n\binom{n+k}{k}} = 2\sum_{r=1}^{k} (-1)^{r+1}\binom{k}{r} \times \begin{pmatrix} \left(H_{2p}-\frac{1}{2}H_p\right)\left(2H_{2r}-H_r-H_{p+r}\right) \\ +\sum_{m=1}^{r-1}\left(\frac{H_m-H_{m+p}}{2m+1}\right) \end{pmatrix} + 2H_p\left(H_{2p}-\frac{1}{2}H_p-1\right) - \frac{2\ln 2}{k} \times \left(1-\frac{1}{\binom{p+k}{p}}\right) \tag{16}$$

and

$$\sum_{n=1}^{p} \frac{H_{2n}}{n\binom{n+k}{k}} = H_p\left(H_{2p}-1\right) + \frac{1}{2}H_p^{(2)} + \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r} \times \begin{pmatrix} \left(H_{2p}-\frac{1}{2}H_p\right)\left(2H_{2r}-H_r-H_{p+r}\right) \\ +\frac{H_p}{2}\left(H_{p+r}-H_{r-1}\right)-\sum_{m=1}^{r-1}\frac{(H_m-H_{m+p})}{2m(2m+1)} \end{pmatrix}. \tag{17}$$

Proof. It is straightforward to show that

$$\sum_{n=1}^{p} \frac{\ln 2}{n\binom{n+k}{k}} = \frac{\ln 2}{k}\left(1-\frac{1}{\binom{p+k}{p}}\right); \tag{18}$$

then rearranging Equation 14 and using Equation 18, we obtain Equation 16. Rearranging Equation 14 and using Equation 2, we obtain Equation 17. ☐

Example 2. *Some examples are*

$$\sum_{n=1}^{p} \frac{H_{2n}-\frac{1}{2}H_n}{n(n+1)} = \frac{2p+1}{p+1}H_{2p} - \frac{4p+3}{2(p+1)}H_p,$$

$$\sum_{n=1}^{p} \frac{H_{2n}}{n(n+1)} = \frac{2p+1}{p+1}H_{2p} - 2H_p + \frac{H_p^{(2)}}{2},$$

$$\sum_{n=1}^{p} \frac{H_{2n}-\frac{1}{2}H_n}{n\binom{n+2}{2}} = \frac{(2p+1)(2p+5)}{3(p+1)(p+2)}H_{2p} - \frac{\left(8p^2+24p+13\right)}{6(p+1)(p+2)}H_p - \frac{p}{3(p+1)},$$

$$\sum_{n=1}^{p} \frac{H_{n-\frac{1}{2}}}{n\binom{n+2}{2}} = \frac{(2p+1)(2p+5)}{3(p+1)(p+2)}H_{2p} - \frac{\left(8p^2+24p+13\right)}{6(p+1)(p+2)}H_p - \frac{p}{3(p+1)} - \frac{p(p+3)}{(p+1)(p+2)}\ln 2,$$

$$\sum_{n=1}^{p} \frac{H_{2n}}{n\binom{n+2}{2}} = \frac{(2p+1)(2p+5)}{3(p+1)(p+2)}H_{2p} + \frac{H_p^{(2)}}{2} - \frac{4H_p}{3} - \frac{5p}{6(p+1)}$$

and

$$\sum_{n=1}^{p} \frac{H_{2n}-\frac{1}{2}H_n}{n\binom{n+3}{3}} = \frac{2(2p+1)\left(4p^2+22p+33\right)}{15(p+1)(p+2)(p+3)}H_{2p} - \frac{\left(16p^3+96p^2+176p+81\right)}{15(p+1)(p+2)(p+3)}H_p - \frac{p(11p+25)}{30(p+1)(p+2)}.$$

Conclusions

The author has generalized some results on variant Euler sums and specifically obtained identities for $\sum_{n=1}^{\infty} \frac{H_{2n}}{n\binom{n+k}{k}}$ and its finite counterpart.

Methods

Analytical techniques have been employed in the analysis of our results. We have used many relations of the polygamma functions together with results of reordering of double sums and partial fraction decomposition.

Competing interests

The author declares that he has no competing interests.

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Author's information

Professor Anthony Sofo is a Fellow of the Australian Mathematical Society.

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