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Generalized composition operators on $Q_{K,\omega}(p,q)$ spaces

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Abstract

Purpose: Our aim in this paper is to study generalized composition operators on α- Bloch and $Q_{K,\omega}(p,q)$ spaces.

Methods: By the help of generalized composition operators, we act between several classes of weighted function spaces. Some important results obtained by using modified Nevanlinna counting function.

Results: The boundedness and compactness of the generalized composition operator C_ϕ^g acting between two different Möbius invariant spaces $Q_{K_1}(p,q)$ and $Q_{K_2}(p,q)$ are studied.

Conclusions: Our results in this paper extend, generalize and improve a lot of previous results.

Keywords: $Q_{K,\omega}(p,q)$ spaces, Holomorphic functions, and Weighted Bloch space

Introduction and preliminaries

Let ϕ be an analytic self-map of the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and let $dA(z)$ be the Euclidean area element on Δ. Associated with ϕ, the composition operator C_ϕ is defined by

$$C_\phi = f \circ \phi,$$

for f analytic on Δ. It maps analytic functions f to analytic functions. The problem of boundedness and compactness of C_ϕ has been studied in many function spaces. The first setting was in the Hardy space H^2, the space of functions analytic on Δ (see [1]). Madigan and Matheson (see [2]) gave a characterization of the compact composition operators on the Bloch space $\mathcal{B}$. Tjani (see [3]) gave a Carleson measure characterization of compact operators C_ϕ on Besov spaces $B_p(1 < p < \infty)$. Bourdon, Cima and Matheson in [4] and Smith in [5] investigated the same problem on *BMOA*. Li and Wulan in [6] gave a characterization of compact operators C_ϕ on Q_K and $F(p,q,s)$ spaces. Also, very recently in [7,8], there are some characterizations for the composition operators C_ϕ in holomorphic $F(p,q,s)$ spaces.

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For $a \in \Delta$ the Möbius transformations $\varphi_a(z)$ is defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \quad \text{for } z \in \Delta.$$

The following identity is easily verified:

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} = (1-|z|^2)|\varphi_a'(z)|. \tag{1}$$

Note that $\varphi_a(\varphi_a(z)) = z$ and thus $\varphi_a^{-1}(z) = \varphi_a(z)$. For $a, z \in \Delta$ and $0 < r < 1$, the pseudo-hyperbolic disc $\Delta(a,r)$ is defined by $\Delta(a,r) = \{z \in \Delta : |\varphi_a(z)| < r\}$. Denote by

$$g(z,a) = \log\left|\frac{1-\bar{a}z}{z-a}\right| = \log\frac{1}{|\varphi_a(z)|}$$

the Green's function of Δ with logarithmic singularity at $a \in \Delta$.

Definition 1.1. *[9] Let f be an analytic function in Δ and let $0 < \alpha < \infty$. If*

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\Delta}(1-|z|^2)^\alpha |f'(z)| < \infty,$$

then f belongs to the $\alpha-$Bloch space $\mathcal{B}^\alpha$. The space $\mathcal{B}^1$ is called the Bloch space $\mathcal{B}$. Now, given a reasonable function $\omega : (0,1] \to [0,\infty)$, the weighted Bloch space $\mathcal{B}_\omega$

(see [10]) is defined as the set of all analytic functions f on Δ satisfying

$$(1-|z|)|f'(z)| \leq C\omega(1-|z|), \quad z \in \Delta,$$

for some fixed $C = C_f > 0$. In the special case where $\omega \equiv 1, \mathcal{B}_\omega$ reduces to the classical Bloch space $\mathcal{B}$. Here, the word "reasonable" is a non-mathematical term; it was just intended to mean that the "not too bad" and the function satisfy some natural conditions.

Now, we introduce the following definitions:

Definition 1.2. *For a given reasonable function $\omega : (0,1] \to [0,\infty)$ and for $0 < \alpha < \infty$. An analytic function f on Δ is said to belong to the $\alpha-$weighted Bloch space $\mathcal{B}^\alpha_\omega$ if*

$$\|f\|_{\mathcal{B}^\alpha_\omega} = \sup_{z\in\Delta} \frac{(1-|z|)^\alpha}{\omega(1-|z|)}|f'(z)| < \infty.$$

Definition 1.3. *For a given reasonable function $\omega : (0,1] \to [0,\infty)$ and for $0 < \alpha < \infty$. An analytic function f on Δ is said to belong to the little weighted Bloch space $\mathcal{B}^\alpha_{\omega,0}$ if*

$$\|f\|_{\mathcal{B}^\alpha_{\omega,0}} = \lim_{|z|\to 1^-} \frac{(1-|z|)^\alpha}{\omega(1-|z|)}|f'(z)| = 0.$$

In this paper we study generalized compact composition operator on the spaces $Q_{K,\omega}(p,q)$, we will define and discuss properties of these spaces. A particular class of Möbius-invariant function spaces, the so-called $Q_{K,\omega}$ spaces, has attracted a lot of attention in recent years.

Definition 1.4. *For a nondecreasing function $K : [0,\infty) \to [0,\infty), 0 < p < \infty, -2 < q < \infty$ and for a given reasonable function $\omega : (0,1] \to (0,\infty)$, an analytic function f in Δ is said to belong to the space $Q_{K,\omega}(p,q)$ if*

$$\|f\|^p_{K,\omega,p,q} = \sup_{a\in\Delta} \int_\Delta \left|f'(z)\right|^p (1-|z|)^q \frac{K(g(z,a))}{\omega^p(1-|z|)} \times dA(z) < \infty.$$

Remark 1.1. *It should be remarked that our $Q_{K,\omega}(p,q)$ classes are more general than many classes of analytic functions. If $\omega \equiv 1$, we obtain $Q_K(p,q)$ type spaces (cf. [11,12]). If $q = p = 2$, and $\omega(t) = t$, we obtain Q_K spaces as studied recently in [12-17] and others. If $q = p = 2$, $\omega(t) = t$ and $K(t) = t^p$, we obtain Q_p spaces as studied in [18-20] and others. If $\omega \equiv 1$ and $K(t) = t^s$, then $Q_{K,\omega} = F(p,q,s)$ classes (cf. [7,21]).*

Definition 1.5. *[22,23] Let f be an analytic function in Δ and let $1 < p < \infty$. If*

$$\|f\|^p_{B_p} = \sup_{z\in\Delta} \int_\Delta |f'(z)|^p (1-|z|^2)^{p-2} dA(z) < \infty,$$

then f belongs to the Besov space B_p.

In [21] Zhao gave the following definition:

Definition 1.6. *Let f be an analytic function in Δ and let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If*

$$\|f\|^p_{F(p,q,s)} = \sup_{a\in\Delta} \int_\Delta |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) < \infty,$$

then $f \in F(p,q,s)$. Moreover, if

$$\lim_{|a|\to 1} \int_\Delta |f'(z)|^p (1-|z|^2)^q g^s(z,a) dA(z) = 0,$$

then $f \in F_0(p,q,s)$.

The spaces $F(p,q,s)$ were intensively studied by Zhao in [21] and Rättyä in [24]. It is known from ([21], Theorem 2.10) that, for $p \geq 1$, the spaces $F(p,q,s)$ are Banach spaces under the norm

$$\|f\| = \|f\|_{F(p,q,s)} + |f(0)|.$$

Now, we define the following definition

Definition 1.7. *Let f be an analytic function in Δ, $\omega : (0,1] \to (0,\infty)$ and let $0 < p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$. If*

$$\|f\|^p_{F_\omega(p,q,s)} = \sup_{a\in\Delta} \int_\Delta |f'(z)|^p (1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} \times dA(z) < \infty,$$

then $f \in F_\omega(p,q,s)$. Moreover, if

$$\lim_{|a|\to 1^-} \int_\Delta |f'(z)|^p (1-|z|^2)^q \frac{g^s(z,a)}{\omega(1-|z|)} dA(z) = 0,$$

then $f \in F_{\omega,0}(p,q,s)$.

Li and Stević in [25] defined the generalized composition operator C^g_ϕ as the follows:

$$(C^g_\phi)(z) = \int_0^z f'(\phi(\xi))g(\xi)d\xi.$$

When $g = \phi'$, we see that this operator is essentially the composition operator C_ϕ. Therefore, C^g_ϕ is a generalization of the composition operator C_ϕ.

We assume throughout this paper that

$$\int_0^1 K\left(\log\frac{1}{r}\right) \frac{r}{(1-r^2)^2}\, dr < \infty\,. \tag{2}$$

The author [26] collected the following immediate relations of $Q_{K,\omega}(p,q)$ and $Q_{K,\omega,0}(p,q)$

(i) $Q_{K,\omega}(p,q) \subset \mathcal{B}_\omega^{\frac{q+2}{p}}$ and

(ii) $Q_{K,\omega}(p,q) = \mathcal{B}_\omega^{\frac{q+2}{p}}$, iff

$$\int_0^1 K\left(\log\frac{1}{r}\right)\frac{r}{(1-r^2)^2}\,dr<\infty.$$

(iii) $F_\omega(p,q,0)=Q_{K,\omega}(p,q)$, *if* $K(0)>0$.

The following lemma is useful for our study (see [26]).

Lemma 1.1. *Let* $K:[0,\infty)\to[0,\infty)$, $0<p<\infty$, $-2<q<\infty$ *and* $\omega:(0,1]\to(0,\infty)$. *Then*

(i) $f\in\mathcal{B}_\omega^{\frac{q+2}{p}}$ *if and only if there exists* $R\in(0,1)$ *such that*

$$\sup_{a\in\Delta}\int_{\Delta(a,R)}|f'(z)|^p(1-|z|)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)<\infty, \tag{3}$$

(ii) $f\in\mathcal{B}_{\omega,0}^{\frac{q+2}{p}}$ *if and only if there exists* $R\in(0,1)$ *such that*

$$\lim_{|a|\to1^-}\int_{\Delta(a,R)}|f'(z)|^p(1-|z|)^q\frac{K(g(z,a))}{\omega^p(1-|z|)}dA(z)=0. \tag{4}$$

Recall that a linear operator $T:X\to Y$ *is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y of the space of all analytic functions* $H(\Delta)$, *we call that T is compact from X to Y if and only if for each bounded sequence* $\{x_n\}$ *in X, the sequence* $(Tx_n)\in Y$ *contains a subsequence converging to some limit in Y.*

Results and discussions

Composition operators $C_\phi^g: Q_{K_1,\omega}(p,q)\to Q_{K_2,\omega}(p,q)$

In this section, we characterize boundedness and compactness of the generalized composition operator C_ϕ^g from $Q_{K_1,\omega}(p,q)$ spaces to $Q_{K_2,\omega}(p,q)$ spaces.

Now we are ready to state and prove the main results in this section.

Theorem 2.1. *Let* $\omega:(0,1]\to(0,\infty)$, *let* $g\in H(\Delta)$ *and* ϕ *be an analytic self-map of* Δ. *If* $C_\phi^g(Q_{K_1,\omega}(p,q))\subset Q_{K_2,\omega}(p,q)$. *Then* $C_\phi^g: Q_{K_1,\omega}(p,q)\to Q_{K_2,\omega}(p,q)$ *is compact if and only if*

$$\lim_{t\to1^-}\sup_{a\in\Delta}\int_{|\phi(z)|>t}|f'(\phi(z))\phi'(z)g(z)|^p(1-|z|^2)^q \times\frac{K(g(z,a))}{\omega(1-|z|)}\times dA(z)=0,\quad \textit{where } f\in X_{Q_{K_1,\omega}(p,q)}. \tag{5}$$

Proof. First assume that (5) holds. To show that C_ϕ^g is compact we consider $\{f_n\}\subset X_{Q_{K_1}(p,q)}$. It suffices to prove that $\{C_\phi^g f_n\}$ has a subsequence which converges in $Q_{K_2,\omega}(p,q)$. Since $f_n\subset Q_{K_1,\omega}(p,q)\subset\mathcal{B}_\omega^{\frac{q+2}{p}}$ (cf. [26]), for $z\in\Delta$.

$$\begin{aligned}\left|f_n(z)-f_n(0)\right| &= \left|\int_0^1 f'(zt)zdt\right|\le\int_0^1|f'(zt)||z|dt\\ &\le\|f_n\|_{\mathcal{B}_\omega^{\frac{q+2}{p}}}\int_0^1\frac{\omega(1-|tz|)|z|dt}{(1-t^2|z|^2)^{\frac{q+2}{p}}}\\ &\le C\|f_n\|_{\mathcal{B}_\omega^{\frac{q+2}{p}}}\\ &\le\frac{C\omega(1-r)}{\pi r^2K(\log\frac{1}{r})}\|f\|_{Q_{K_1,\omega}(p,q)}.\end{aligned}$$

We know that $\{f_n\}$ is a normal family. By passing to a subsequence, we may assume, without loss of generality, that $\{f_n\}$ converges to 0 uniformly on compact subsets of Δ. We must show that $\{C_\phi^g f_n\}$ converges to 0 in the norm $\|\cdot\|_{Q_{K_2,\omega}(p,q)}$. Given $\epsilon\in(0,1)$, by (5), there is a $t\in(0,1)$ such that for all functions f_n and for all $a\in\Delta$,

$$\int_{|\phi(z)|>t}|f_n'(\phi(z))\phi'(z)g(z)|^p(1-|z|^2)^q\frac{K_2(g(z,a))}{\omega(1-|z|)}\times dA(z)<\epsilon \tag{6}$$

By (5) and the fact that $\Delta_t=\{z\in\Delta:|z|\le t\}$ is a compact subset of Δ, we see that $\phi\in Q_{K_2,\omega}(p,q)$, since $z\in Q_{K_1,\omega}(p,q)$, and also that $\{f_n'\}$ converges to 0 uniformly on Δ_t. Therefore, there exists an integer $N>1$ such that for $n\ge N$,

$$\int_{|\phi(z)|\le t}|f_n'(\phi(z))\phi'(z)g(z)|^p(1-|z|^2)^q\frac{K_2(g(z,a))}{\omega(1-|z|)}\times dA(z)<\epsilon\|\phi\|^p_{Q_{K_2,\omega}(p,q)}. \tag{7}$$

Thus (6) and (7) give

$$\int_{|\phi(z)|\le t}|f_n'(\phi(z))\phi'(z)|^p|g(z)|^p(1-|z|^2)^q\frac{K_2(g(z,a))}{\omega(1-|z|)}\times dA(z)<\epsilon\left(1+\|\phi\|^p_{Q_{K_2,\omega}(p,q)}\right),$$

when $n\ge N$. That is, $\|C_\phi^g f_n\|_{Q_{K_2,\omega}(p,q)}\to0$ as $n\to\infty$.

Now suppose that $C_\phi^g: Q_{K_1,\omega}(p,q)\to Q_{K_2,\omega}(p,q)$ is compact. To verify (5) consider $f\in X_{Q_{K_1,\omega}(p,q)}$ and let $f_s(z)=f(sz)$ for $s\in(0,1)$ and $z\in\Delta$. Note that $f_s\to f$ uniformly on compact subsets of Δ as $s\to1$. By [27] we know that $\{f_s, 0<s<1\}$ is bounded in $Q_{K_1,\omega}(p,q)$. Since C_ϕ is compact, $\|C_\phi^g f_s-C_\phi f\|_{Q_{K_2,\omega}(p,q)}\to0$ as $s\to1$. That is, for given $\epsilon>0$ there exists $s_0\in(0,1)$ such that

$$\sup_{a\in\Delta}\int_\Delta|f'_{s_0}(\phi(z))-f'(\phi(z))|^p|\phi'(z)|^p|g(z)|^p(1-|z|^2)^q\times\frac{K_2(g(z,a))}{\omega(1-|z|)}dA(z)<\epsilon.$$

For $t \in (0,1)$ and the above s_0 the triangle inequality gives

$$\sup_{a\in\Delta}\int_{|\phi(z)|>t}|f'(\phi(z))|^p|\phi'(z)|^p|g(z)|^p(1-|z|^2)^q \times \frac{K_2(g(z,a))}{\omega(1-|z|)}dA(z) \le \epsilon + \|f'_{s_0}\|^p_\infty \sup_{a\in\Delta}\int_{|\phi(z)|>t} \times \left|\phi'(z)\right|^p|g(z)|^p\left(1-|z|^2\right)^q\frac{K_2(g(z,a))}{\omega(1-|z|)}dA(z). \quad (8)$$

We know that

$$\sup_{a\in\Delta}\int_{|\phi(z)|>t}\left|\phi'(z)\right|^p|g(z)|^p\left(1-|z|^2\right)^q\frac{K_2(g(z,a))}{\omega(1-|z|)} \times dA(z) \le \|\phi\|^p_{Q_{K,\omega}(p,q)} < \infty$$

since $C^g_\phi(Q_{K_1,\omega}(p,q)) \subset Q_{K_2,\omega}(p,q)$. It will be shown that for given $\epsilon > 0$ and $\|f'_{s_0}\|^p_\infty > 0$ there exists a $\delta \in (0,1)$ such that for $\delta < t < 1$

$$\|f'_{s_0}\|^p_\infty \sup_{a\in\Delta}\int_{|\phi(z)|>t}\left|\phi'(z)\right|^p|g(z)|^p\left(1-|z|^2\right)^q \times \frac{K_2(g(z,a))}{\omega(1-|z|)}dA(z) < \epsilon.$$

Let $n = 2^j, j = 1,2,\ldots$. Choose $h_n(z) = \frac{n^{\frac{-1}{2}}z^n}{\omega(1-|z|)}$, and we know that $h_n \in \mathcal{B}_\omega^{\frac{q+2}{p}}$. It is easy to check that $\{h_n\}$ is a bounded family in $Q_{K_1,\omega}(p,q)$ since $\mathcal{B}_\omega^{\frac{q+2}{p}} \subseteq Q_{K_1,\omega}(p,q)$ (see [26]). Since C^g_ϕ is compact and h_n converges uniformly to 0 on compact subsets of Δ, we have

$$\lim_{n\to\infty}\|h_n\circ\phi\|_{Q_{K_2,\omega}(p,q)} = 0.$$

Thus, for any given $\epsilon > 0$, there exists an integer $N > 1$ such that for all $a \in \Delta$

$$n\int_{|\phi(z)|>t}\left|\phi'(z)\right|^p|\phi(z)|^{pn-p}\left(1-|z|^2\right)^q|g(z)|^p \times \frac{K_2(g(z,a))}{\omega(1-|z|)}dA(z) < \epsilon \quad (9)$$

whenever $n \ge N$. Given $t \in (0,1)$, (8) yields

$$Nt^{pN-p}\int_{|\phi(z)|>t}\left|\phi'(z)\right|^p|g(z)|^p\left(1-|z|^2\right)^q \times \frac{K_2(g(z,a))}{\omega(1-|z|)}dA(z) < \epsilon \quad (10)$$

Taking $t = e^{-\frac{\log N}{P(N-1)}}$, we get

$$\|f'_{s_0}\|^p_\infty \sup_{a\in\Delta}\int_{|\phi(z)|>t}\left|\phi'(z)\right|^p\left(1-|z|^2\right)^q|g(z)|^p \times \frac{K_2(g(z,a))}{\omega(1-|z|)}dA(z) < \epsilon.$$

Hence by (8) and (9) we have already proved that for any $\epsilon > 0$ and for $f \in X_{Q_{K_1,\omega}}(p,q)$, there exists a $\delta = \delta(\epsilon,f)$ such that

$$\sup_{a\in\Delta}\int_{|\phi(z)|>t}\left|(f\circ\phi)'(z)g(z)\right|^p\left(1-|z|^2\right)^q \times \frac{K_2(g(z,a))}{\omega(1-|z|)}dA(z) < \epsilon$$

whenever $\delta < t < 1$.

the above $\delta = \delta(\epsilon,f)$, in fact, is independent of $f \in X_{Q_{K_1,\omega}(p,q)}$. Since $C^g_\phi : Q_{K_1,\omega}(p,q) \to Q_{K_2,\omega}(p,q)$ is compact, $C^g_\phi(X_{Q_{K_1,\omega}(p,q)})$ is a relatively compact subset of $Q_{K_2,\omega}(p,q)$. It means that there is a finite collection of functions $f_1,f_2,\ldots,f_n$ in $X_{Q_{K_1,\omega}(p,q)}$ such that for any $\epsilon > 0$ and $f \in X_{Q_{K_1,\omega}(p,q)}$ there is a $k, 1 \le k \le n$, satisfying

$$\sup_{a\in\Delta}\int_\Delta |f'(\phi(z)) - f'_k(\phi(z))|^p|\phi'(z)|^p|g(z)|^p(1-|z|^2)^q \times \frac{K_2(g(z,a))}{\omega(1-|z|)}dA(z) < \epsilon\,. \quad (11)$$

On the other hand, if $\rho = \max_{1\le k\le n}\delta(\epsilon,f_k) < t < 1$, we have from the previous observation that for all $k = 1,2,\ldots,n$,

$$\sup_{a\in\Delta}\int_{|\phi(z)|>t}|f'_k(\phi(z))|^p|\phi'(z)|^p|g(z)|^p(1-|z|^2)^q \times \frac{K_2(g(z,a))}{\omega(1-|z|)}dA(z) < \epsilon\,. \quad (12)$$

The triangle inequality, together with (11) and (12), gives

$$\sup_{a\in\Delta}\int_{|\phi(z)|>t}|(f\circ\phi)'(z)g(z)|^p(1-|z|^2)^q \times \frac{K_2(g(z,a))}{\omega(1-|z|)}dA(z) < 2\epsilon$$

whenever $\rho < t < 1$. The proof is complete.

Although Theorem 2.1 can be viewed as a characterization of compact composition operators $C^g_\phi : Q_{K_1,\omega}(p,q) \to Q_{K_2,\omega}(p,q)$, by condition (5) it is not easy to check compactness of C^g_ϕ. The following theorem gives a characterization of C^g_ϕ directly in terms of ϕ. □

Theorem 2.2. *Let $\omega : (0,1] \to (0,\infty)$, let $g \in H(\Delta)$, ϕ be an analytic self-map of Δ and $C^g_\phi : Q_{K_1,\omega}(p,q) \subset Q_{K_2,\omega}(p,q)$. Let two functions $K_1, K_2 : [0,\infty) \to [0,\infty)$ be right-continuous and nondecreasing, satisfying*

$$\int_0^1(1-r^2)^{-2}K_1(\log\frac{1}{r})rdr < \infty. \quad (13)$$

If

$$\lim_{t\to1^-}\sup_{a\in\Delta}\int_{|\phi(z)|>t}\frac{|\phi'(z)|^p|g(z)|^p}{(1-|\phi(z)|^2)^{2p}}(1-|z|^2)^q \times \frac{K_2(g(z,a))}{\omega(1-|z|)}dA(z) = 0. \quad (14)$$

Then, $C^g_\phi : Q_{K_1,\omega}(p,q) \to Q_{K_2,\omega}(p,q)$ is compact. Conversely, if $C^g_\phi : Q_{K_1,\omega}(p,q) \to Q_{K_2,\omega}(p,q)$ is compact, then (14) holds.

Proof. Consider $\{f_n\} \in X_{Q_{K_1,\omega}(p,q)}$ which converges to 0 uniformly on compact subsets of Δ. We must show that $\{C^g_\phi f_n\}$ converges to 0 in the norm $\| \cdot \|_{Q_{K,\omega}(p,q)}$. Thus

$$\begin{aligned}\|C^g_\phi f_n\|^p_{Q_{K,\omega}(p,q)} &= \sup_{a\in\Delta} \int_\Delta \left|(f\circ\phi)'(z)g(z)\right|^p \left(1-|z|^2\right)^q \\ &\quad \times \frac{K_2(g(z,a))}{\omega(1-|z|)} dA(z) \\ &= \sup_{a\in\Delta}\left(\int_{|\phi(z)|\le t} + \int_{|\phi(z)|>t}\right)\left|f_n'(\phi)(z)\right|^p \\ &\quad \times |\phi'(z)|^p|g(z)|^p\left(1-|z|^2\right)^q \frac{K_2(g(z,a))}{\omega(1-|z|)} \\ &\quad \times dA(z) \\ &\le \sup\{|f_n'(w)g(w)|^p : |w|\le t\}\|\phi\|^p_{Q_{K_2,\omega}(p,q)} \\ &\quad + const.\|f_n\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \int_{|\phi(z)|>t} \\ &\quad \times |\frac{|\phi'(z)g(z)|^p}{(1-|\phi(z)|^2)^{2p}} K_2(g(z,a))dA(z) \\ &= I_1 + I_2.\end{aligned}$$

Since $\{f_n\}$ converges to 0 uniformly on compact sets and $\phi \in Q_{K_2,\omega}(p,q)$, we have $I_1 \to 0$ as $n \to \infty$. In the second term I_2 we know that

$$\|f_n\|^p_{\mathcal{B}_\omega^{\frac{q+2}{p}}} \le C\|f_n\|^p_{Q_{K_1,\omega}(p,q)}$$

since every function in $Q_{K_1,\omega}(p,q)$ must be weighted $\frac{q+2}{p}$−Bloch. Thus, I_2 goes to 0 when $t \to 1$ by our assumption. Therefore, C^g_ϕ is compact.

Conversely, let $C^g_\phi : Q_{K_1,\omega}(p,q) \to Q_{K_2,\omega}(p,q)$ be compact. By [28] we know that (13) ensures

$$\begin{aligned}f_\theta(z) &= \frac{1}{\omega(1-|z|)} \log\frac{1}{1-e^{-i\theta}z} \\ &\in Q_{K_1,\omega}(p,q) \quad \text{for all } \theta \in [0,2\pi).\end{aligned}$$

By Theorem 2.1,

$$\begin{aligned}\lim_{t\to1^-} \int_{|\phi(z)|>t} &\frac{|\phi'(z)|^p|g(z)|^p}{(1-|\phi(z)|^2)^{2p}\omega(1-|z|)} \\ &\times K_2(g(z,a))dA(z) = 0\end{aligned}$$

holds for all $a \in \Delta$ and $\theta \in [0,2\pi)$. Thus, we obtain (14) by integrating with respect to θ, the Fubini theorem and the Poisson formula. □

Composition operators $C^g_\phi : Q_{K_1,\omega}(p,q) \to Q_{K_2,\omega,0}(p,q)$

In this section, we consider compactness of the generalized composition operators $C^g_\phi : Q_{K_1,\omega}(p,q) \to Q_{K_2,\omega,0}(p,q)$, where $Q_{K,\omega,0}(p,q)$ is a subspace of $Q_{K,\omega}(p,q)$ satisfying

$$\lim_{|a|\to1^-} \int_\Delta \left|f'(z)\right|^p \left(1-|z|^2\right)^q \frac{K(g(z,a))}{\omega(1-|z|)} dA(z) = 0.$$

By [26], we know that $Q_{K,\omega,0}(p,q) \subset \mathcal{B}_{\omega,0}^{\frac{q+2}{p}}$ and that $Q_{K,\omega,0}(p,q) = \mathcal{B}_{\omega,0}^{\frac{q+2}{p}}$ if and only if

$$\int_0^1 (1-r^2)^{-2} K(\log\frac{1}{r}) r dr < \infty.$$

We should mention that the generalized composition operator C^g_ϕ is compact from $Q_{K_1,\omega}(p,q)$ to $Q_{K_2,\omega,0}(p,q)$ if $\phi \in Q_{K_2,\omega,0}(p,q)$ and C^g_ϕ is compact from $Q_{K_1,\omega}(p,q)$ to $Q_{K_2,\omega}(p,q)$.

Theorem 3.1. *Let $\omega : (0,1] \to (0,\infty)$, $g \in H(\Delta)$ and ϕ be an analytic self-map of Δ such that*

$$C^g_\phi(Q_{K_1,\omega}(p,q)) \subset Q_{K_2,\omega,0}(p,q).$$

Then $C^g_\phi : Q_{K_1,\omega}(p,q) \to Q_{K_2,\omega,0}(p,q)$ is compact if and only if

$$\begin{aligned}\lim_{|a|\to1^-} \sup_{\|f\|_{Q_{K_1,\omega}(p,q)}<1} \int_\Delta &\left|f'(\phi(z))\right|^p |\phi'(z)|^p |g(z)|^p \\ &\times (1-|z|^2)^q \frac{K_2(g(z,a))}{\omega(1-|z|)} dA(z) = 0.\end{aligned} \tag{15}$$

Proof. First suppose that $C^g_\phi : Q_{K_1,\omega}(p,q) \to Q_{K_2,\omega,0}(p,q)$ is compact. Then $A = cl(\{(f\circ\phi)g \in Q_{K_2,\omega,0}(p,q) : \|f\|_{Q_{K_1,\omega,p,q}} < 1\})$, the $Q_{K_2,\omega,0}(p,q)$ closure of the image under C^g_ϕ of the unit ball of $Q_{K_1,\omega}(p,q)$, is a compact subset of $Q_{K_2,\omega,0}(p,q)$. For given $\epsilon > 0$, since a compact set in a metric space is completely bounded, there exist $f_1, f_2, \ldots, f_N \in Q_{K_1,\omega}(p,q)$ such that each function f in A lies at most ϵ distant from

$$B = \{(f_1\circ\phi)g, (f_2\circ\phi)g, (f_3\circ\phi)g, \ldots, (f_N\circ\phi)g\}.$$

That is, there exists $j \in J = \{1,2,\ldots,N\}$ such that

$$\|(f\circ\phi)g - (f_j\circ\phi)g\|_{Q_{K_2,\omega}(p,q)} < \frac{\epsilon}{4}. \tag{16}$$

On the other hand, since $\{(f_j\circ\phi)g : j \in J\} \subset Q_{K,\omega,0}(p,q)$, there exists a $\delta > 0$ such that for all $j \in J$ and $|a| > 1-\delta$,

$$\int_\Delta \left|(f_j\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q \frac{K_2(g(z,a))}{\omega(1-|z|)} dA(z) < \frac{\epsilon}{4}. \tag{17}$$

Therefore by (16) and (17), we obtain that for each $|a| > 1-\delta$ and $f \in Q_{K_1,\omega}(p,q)$ with $\|f\|_{Q_{K_1,\omega,p,q}} < 1$ there exists

$j \in J$ such that

$$\int_{\Delta} \left|(f\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q \frac{K_2(g(z,a))}{\omega(1-|z|)} dA(z)$$
$$\leq 2\int_{\Delta} \left|(f\circ\phi - f_j\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q \frac{K_2(g(z,a))}{\omega(1-|z|)} dA(z)$$
$$+\int_{\Delta} \left|(f_j\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q \frac{K_2(g(z,a))}{\omega(1-|z|)} dA(z) < \epsilon.$$

This proves (15).

Now let (15) hold and let $\{f_n\}$ be a sequence in the unit ball of $Q_{K_1,\omega}(p,q)$. By Montel's theorem, there exists a subsequence $\{f_{n_k}\}$ which converges to a function f analytic in Δ and both $f_{n_k} \to f$ and $f'_{n_k} \to f'$ uniformly on compact subsets of Δ. By hypothesis and Fatou's lemma, we see that $C_\phi^g \in Q_{K_2,\omega,0}(p,q)$. Since $z \in Q_{K_1,\omega}(p,q)$, $\phi \in Q_{K_2,\omega,0}(p,q)$. Thus we remark that C_ϕ^g is a compact composition operator by showing that

$$\|C_\phi^g(f_{n_k} - f)\|_{Q_{K_2,\omega}(p,q)} \to 0 \quad \text{as} \quad k\to\infty.$$

In order to simplify the notation we additionally assume, without loss of generality, that $f = 0$. Hence it remains to show that

$$\lim_{|n|\to\infty} \|C_\phi^g f_n\|_{Q_{K_2,\omega}(p,q)} = 0.$$

Let $\epsilon > 0$. By (15), we can choose $r \in (0,1)$ for all n,

$$\sup_{r<|a|<1} \int_{\Delta} \left|(f\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q \frac{K_2(g(z,a))}{\omega(1-|z|)} \times dA(z) < \epsilon. \tag{18}$$

For $a \in \Delta$ and $t \in (0,1)$, define $t\Delta = \{z \in \Delta : |z| \leq t\}$ and set

$$I_t(a) = \int_{\Delta\setminus t\Delta} \left|(f\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q \frac{K_2(g(z,a))}{\omega(1-|z|)} \times dA(z).$$

By using the same way as in [6] we know that for each $t \in (0,1)$, $I_t(a)$ is a continuous function of a. Since

$$\int_{\Delta} \left|(f_n\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q \frac{K_2(g(z,a))}{\omega(1-|z|)} dA(z) < \infty$$

for each $a \in \Delta$, we can choose $t(a) \in (r,1)$ such that $I_{t(a)}(a) < \frac{\epsilon}{2}$. Moreover, there is a neighborhood $U(a) \subset \Delta$ of a such that $I_{t(a)}(b) < \epsilon$ for every $b \in U(a)$, by the continuity of $I_t(a)$. Thus, using the compactness of $\{a : |a| \leq r\}$, there exists $t_0 \in (0,1)$ such that $I_{t_0}(a) < \epsilon$ if $|a| \leq r$, and so

$$\sup_{|a|\leq r} \int_{\Delta\setminus t_0\Delta} \left|(f_n\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q \times \frac{K_2(g(z,a))}{\omega(1-|z|)} dA(z) < \epsilon. \tag{19}$$

Also, by the uniform convergence of $\{(f'_n\circ\phi)g\}$ to 0 on compact subsets of Δ, there exists N such that,

$$\int_{t_0\Delta} \left|(f_n\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q \frac{K_2(g(z,a))}{\omega(1-|z|)} dA(z) < \epsilon,$$

if $n \geq N$. Thus, for any such n, we have

$$\sup_{|a|\leq r} \int_{\Delta} \left|(f_n\circ\phi)'(z)g(z)\right|^p (1-|z|^2)^q \frac{K_2(g(z,a))}{\omega(1-|z|)} dA(z) < 2\epsilon. \tag{20}$$

Combining (18) and (21), we obtain that

$$\lim_{|n|\to\infty} \|C_\phi^g f_n\|_{Q_{K_2,\omega}(p,q)} = 0.$$

The proof of Theorem 3.1 is complete. ☐

Theorem 3.2. *Let $\omega : (0,1] \to (0,\infty)$, $g \in H(\Delta)$ and ϕ be an analytic self-map of Δ such that*

$$C_\phi^g(Q_{K_1,\omega}(p,q)) \subseteq Q_{K_2,\omega,0}(p,q).$$

Assume that

$$\int_0^1 (1-r^2)^{-2} K_1(\log\frac{1}{r}) r dr < \infty. \tag{21}$$

If

$$\lim_{|a|\to 1^-} \int_{\Delta} \frac{|\phi'(z)|^p |g(z)|^p}{(1-|\phi(z)|^2)^{2p}} (1-|z|^2)^q \frac{K_2(g(z,a))}{\omega(1-|z|)} dA(z) = 0, \tag{22}$$

then $C_\phi^g : Q_{K_1,\omega}(p,q) \to Q_{K_2,\omega,0}(p,q)$ is compact. Conversely assume that $C_\phi^g : Q_{K_1,\omega}(p,q) \to Q_{K_2,\omega,0}(p,q)$ is compact, (22) holds.

Proof. The proof is very similar as the proof of Theorem 2.2. ☐

Composition operators $C_\phi^g : \mathcal{B}_\omega \to Q_{K,\omega}(p,q)$

Using Riesz Factorization theorem and Vitali's convergence theorem, Shapiro and Taylor showed in [29] that, C_ϕ is compact on H^p, for some $0 < p < \infty$ if and only if C_ϕ is compact on H^2. Moreover, Shapiro solved the compactness problem for composition operators on H^p using the Nevanlinna counting function

$$N_{\phi(w)} = \sum_{\phi(z)=w} -\log|w| \qquad \text{(see [1])}$$

The counting function for the Besov space B_p is

$$N_p(w,\phi) = \sum_{\phi(z)=w} \left(|\phi'(z)|(1-|z|^2)\right)^{p-2} \quad \text{for } w\in\Delta,\ p>1 \qquad \text{(see [3])}.$$

In [6], Li and Wulan gave a modification of the Nevanlinna type counting function on $F(p,q,s)$ spaces as follows:

$$N_{p,q,s,\phi}(w) = \sum_{\phi(z)=w} |\phi'(z)|^{p-2}(1-|z|^2)^q g^s(z,a)$$

for $w \in \phi(\Delta)$, $2 \le p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$.

Now, we introduce the following definition:

Definition 4.1. *The counting function for the $Q_K(p,q)$ spaces is*

$$N_{K,p,q,\phi}(w) = \sum_{\phi(z)=w} |\phi'(z)|^{p-2}(1-|z|^2)^q K(g(z,a)),$$

for $w \in \phi(\Delta)$, $2 \le p < \infty$, $-2 < q < \infty$ and $K : [0,\infty) \to [0,\infty)$. In this section, we characterize the boundedness and compactness of the generalized composition operator from weighted Bloch type spaces to $Q_{K,\omega}(p,q)$ spaces.

Theorem 4.1. *Let $\omega : (0,1] \to (0,\infty)$, $g(z) \in H(\Delta)$ and ϕ be an analytic self-map of Δ, $0 < p < \infty$, $-2 < q < \infty$ and $K \in (0,\infty)$. Then $C_\phi^g : \mathcal{B}_\omega \to Q_{K,\omega}(p,q)$ is compact if and only if*

$$\lim_{|a|\to 1^-} \|C_\phi^g \varphi_a\|_{K,\omega,p,q} = 0. \tag{23}$$

Proof. Assume that C_ϕ^g is compact from $\mathcal{B}_\omega$ to $Q_{K,\omega}(p,q)$. Since $\{\varphi_a : a \in \Delta\}$ is a bounded set in $\mathcal{B}_\omega$ and $\varphi_a - a \to 0$ uniformly on compact sets as $|a| \to 1$, the compactness of C_ϕ^g yields that $\|C_\phi^g \varphi_a\|_{K,\omega,p,q} \to 0$ as $|a| \to 1$.

Conversely, let $\{f_n\} \in \mathcal{B}_\omega$ be a bounded sequence. Since $f_n \in \mathcal{B}_\omega$, for $z \in \Delta$

$$|f_n(z)| \le C\|f_n\|_{\mathcal{B}_\omega^{\frac{q+2}{p}}}.$$

Hence $\{f_n\}$ is a normal family. Thus, there is a subsequence $\{f_{n_k}\}$ which converges to f analytic on Δ and both $f_{n_k} \to f$ and $f'_{n_k} \to f'$ uniformly on compact subsets of Δ. It is easy to know that $f \in \Delta$. We choose $h = C_\phi^g f$, we remark that C_ϕ^g is compact by showing

$$\|C_\phi^g f_{n_k} - C_\phi^g f\|_{K,\omega,p,q} \to 0 \quad \text{as} \quad k \to \infty.$$

Write

$$\begin{aligned}\|C_\phi^g \varphi_a\|_{K,\omega,p,q}^p &= \sup_{a\in\Delta} \int_\Delta \frac{(1-|a|^2)^p}{|1-\bar{a}\phi(z)|^{2p}} |\phi'(z)|^2 |\phi'(z)|^{p-2} \\ &\quad \times |g(z)|^p (1-|z|^2)^q \frac{K(g(z,a))}{\omega(1-|z|)} dA(z) \\ &= \sup_{a\in\Delta} \int_\Delta \frac{(1-|a|^2)^p}{|1-\bar{a}w|^{2p}} \frac{|g(z)|^p}{\omega(1-|z|)} \\ &\quad \times N_{K,p,q,\phi}(z_1) dA(z_1).\end{aligned}$$

Here

$$N_{K,p,q,\phi}(z_1) = \sum_{\phi(z)=z_1} \{|\phi'(z)|^{p-2}(1-|z|^2)^q K(g(z,a))\}, \times z_1 \in \phi(\Delta). \tag{24}$$

is a counting function. Thus (24) is equivalent to

$$\lim_{|a|\to 1^-} \sup_{a\in\Delta} \int_\Delta \frac{(1-|a|^2)^p}{|1-\bar{a}z_1|^{2p}} \frac{|g(a)|^p}{\omega(1-|z|)} \times N_{K,p,q,\phi}(z_1) dA(z_1) = 0.$$

For any $\epsilon > 0$ there exists a δ, $0 < \delta < 1$, such that for $0 < h < \delta$ and all $a \in \Delta$

$$\int_\Delta N_{K,p,q,\phi}(z_1) dA(z_1) < \epsilon K\left(\frac{1-|z_1|}{h}\right) \qquad \text{(see [30])},$$

where $S(h,\theta)$ is a Carleson box. The mean value property for analytic functions f'_{n_k} and f' yields

$$f'_{n_k}(z_1) - f'(z_1) \le \frac{4}{\pi(1-|z_1|)^2} \int_{|z_1-z|<\frac{1-|z|}{2}} \times (f'_{n_k}(z) - f'(z)) dA(z).$$

Then by Jensen's inequality (see [31]), we have

$$|f'_n(z_1) - f'(z_1)|^p \le \frac{4}{\pi(1-|z_1|)^2} \int_{|z_1-z|<\frac{1-|z|}{2}} \times |f'_{n_k}(z) - f'(z)|^p dA(z).$$

If $|z_1 - z| < \frac{1-|z|}{2}$, then $z_1 \in S(2(1-|z|),\theta)$ and $\frac{1}{(1-|z_1|)^2} \le \frac{C}{(1-|z|)^2}$. By Fubini's theorem

$$\begin{aligned}&\sup_{a\in\Delta} \int_\Delta |f'_{n_k}(z_1) - f'(z_1)|^p \frac{|g(z_1)|^p}{\omega(1-|z_1|)} N_{K,p,q,\phi}(z_1) dA(z_1) \\ &\le \sup_{a\in\Delta} \int_\Delta \frac{4}{\pi(1-|z_1|)^2} \int_{|z_1-z|<\frac{1-|z|}{2}} |f'_{n_k}(z_1) \\ &\quad - f'(z_1)|^p |g(z)|^p dA(z) N_{K,p,q,\phi}(z_1) dA(z_1) \\ &\le C \sup_{a\in\Delta} \int_\Delta \frac{|f'_{n_k}(z) - f'(z)|^p |g(z)|^p}{(1-|z|)^2 \omega(1-|z|)} \int_{S(2(1-|z|),\theta)} \\ &\quad \times N_{K,p,q,\phi}(z_1) dA(z_1) dA(z) \\ &= C \sup_{a\in\Delta} \int_{|z|>1-\frac{\delta}{2}} \frac{|f'_{n_k}(z) - f'(z)|^p |g(z)|^p}{(1-|z|)^2 \omega(1-|z|)} \\ &\quad \times \int_{S(2(1-|z|),\theta)} N_{K,p,q,\phi}(z_1) dA(z_1) dA(z) \\ &\quad + C \sup_{a\in\Delta} \int_{|z|\le 1-\frac{\delta}{2}} \frac{|f'_{n_k}(z) - f'(z)|^p |g(z)|^p}{(1-|z|)^2 \omega(1-|z|)} \\ &\quad \times \int_{S(2(1-|z|),\theta)} N_{K,p,q,\phi}(z_1) dA(z_1) dA(z).\end{aligned}$$

For one hand, since f_{n_k}, $f \in \mathcal{B}_\omega$ and $0 < p < \infty$, we have

$$\begin{aligned}
&\sup_{a\in\Delta}\int_{|z|>1-\frac{\delta}{2}} \frac{|f'_{n_k}(z)-f'(z)|^p|g(z)|^p}{(1-|z|)^2\omega(1-|z|)}\\
&\int_{S(2(1-|z|),\theta)} N_{K,p,q,\phi}(z_1)dA(z_1)dA(z)\\
&\le \epsilon K\left(\frac{1-|z_1|}{h}\right)\int_{|z|>1-\frac{\delta}{2}}|f'_{n_k}(z)-f'(z)|^p\frac{|g(z)|^p}{\omega(1-|z|)}\\
&\times(1-|z|)^{p-2}dA(z)\\
&\le C\epsilon\|f_{n_k}-f\|^{p-2}_{\mathcal{B}_\omega}\int_{|z|>1-\frac{\delta}{2}}|f'_{n_k}(z)-f'(z)|^p|g(z)|^p dA(z)\\
&\le C\epsilon\|f_{n_k}-f\|^{p-2}_{\mathcal{B}_\omega}\|f_{n_k}-f\|^2_{\mathcal{D}}|g(z)|^p\\
&\le \epsilon C'\|f_{n_k}-f\|^{p-2}_{\mathcal{B}_\omega}|g(z)|^p
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\sup_{a\in\Delta}\int_{|z|\le 1-\frac{\delta}{2}} \frac{|f'_{n_k}(z)-f'(z)|^p|g(z)|^p}{(1-|z|)^2\omega(1-|z|)}\\
&\int_{S(2(1-|z|),\theta)} N_{K,p,q,\phi}(z_1)dA(z_1)dA(z)\\
&\le C\left(\sup_{a\in\Delta}\int_\Delta N_{K,p,q,\phi}(z_1)dA(z_1)\right)\int_{|z|\le 1-\frac{\delta}{2}}\\
&\times|f'_{n_k}(z)-f'(z)|^p\frac{|g(z)|^p}{\omega(1-|z|)}dA(z)\le C\epsilon
\end{aligned}$$

for n large enough since $f'_{n_k}(z)-f'(z)\to 0$ uniformly on $\{z\in\Delta:|z|\le 1-\frac{\delta}{2}\}$.

Therefore, for sufficiently large k, the above discussion gives

$$\begin{aligned}
\|C^g_\phi f_{n_k}-C^g_\phi f\|_{K,\omega,p,q} &= \sup_{a\in\Delta}\int_\Delta\left|(f_{n_k}\circ\phi)'(z)g(z)\right.\\
&\left.-(f\circ\phi)'(z)g(z)\right|^p\left(1-|z|^2\right)^q\\
&\times\frac{K(g(z,a))}{\omega(1-|z|)}dA(z)\\
&=\sup_{a\in\Delta}\int_\Delta\left|f'_{n_k}(z_1)-f'(z_1)\right|^p\\
&\times\frac{|g(z_1)|^p}{\omega(1-|z_1|)}N_{K,p,q,\phi}\\
&\times(z_1)dA(z_1)<C\epsilon.
\end{aligned}$$

It follows that C^g_ϕ is a compact operator. □

Corollary 4.1. *Let $g(z)\in H(\Delta)$ and ϕ be an analytic self-map of Δ, $0<p<\infty$, $-2<q<\infty$ and $K\in(0,\infty)$. Then $C^g_\phi:\mathcal{B}\to Q_K(p,q)$ is compact if and only if*

$$\lim_{|a|\to 1^-}\|C^g_\phi\varphi_a\|_{K,p,q}=0.$$

Corollary 4.2. *Let $g(z)\in H(\Delta)$ and ϕ be an analytic self-map of Δ, $0<p<\infty$, $-2<q<\infty$ and $0<s<\infty$. Then $C^g_\phi:\mathcal{B}_\omega\to F_\omega(p,q,s)$ is compact if and only if*

$$\lim_{|a|\to 1^-}\|C^g_\phi\varphi_a\|_{F_\omega(p,q,s)}=0.$$

Conclusions

The boundedness and compactness of generalized composition operators on $Q_{K,\omega}(p,q)$-type spaces and the weighted Bloch space B_ω are investigated in the unit disc. Moreover, we characterized boundedness and compactness of generalized composition operators from $Q_{K_1,\omega}(p,q)$ into $Q_{K_2,\omega}(p,q)$. Compactness criteria is also provided for generalized composition operators on the space $Q_{K,\omega,0}(p,q)$ using mild conditions.

Competing interests
The authors of this paper declare that they have no competing interests.

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Each author contributed equally in the development of this paper. All authors read and approved the final version of this paper.

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