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Methods of solution of singular integral equations

Aloknath Chakrabarti[1] and Subash Chandra Martha[2*]

*Correspondence: scmartha@iitrpr.ac.in
[2]Department of Mathematics, Indian Institute of Technology Ropar, Nangal Road, Rupnagar, Punjab, 140001, India
Full list of author information is available at the end of the article

Abstract

Purpose: This paper investigates a different method to evaluate different real improper integrals and also to obtain the solutions of various types of Cauchy-type singular integral equations of the first kind.

Methods: Methods using the analysis of functions of real variables only are reviewed and utilized for the above purpose. These methods clearly demonstrate that details of complex function theory which are normally employed in handling such integral equations for their solutions can be avoided altogether. Also, some approximate methods of solution of such integral equations are developed.

Results: The solutions of real singular integral equations over different intervals such as $(-1, 1)$; (a, b); $(0, a) \cup (b, c)$; $(-1, k) \cup (k, 1)$; $(-\infty, b)$; $(a, +\infty)$; $(-\infty, +\infty)$; infinite intervals with a gap are obtained by using the proposed methods.

Conclusion: The proposed methods are new and each has its own structure.

Keywords: Real singular integral equations, Cauchy-type kernels, Real variable method

MSC: 45E05

Background

Real singular integral equations involving Cauchy-type singularities arise (see [1-10]) in a natural way in handling a large class of mixed boundary value problems of mathematical physics, especially when two-dimensional problems are encountered. The integrals occurring in these integral equations are in fact improper and their evaluations in most cases can be rendered by using the theory of functions of complex variables involving the application of Cauchy's residue theorem. It is desirable, as is always felt, to avoid the use of complex function theory to evaluate real integrals because the details can be more involved analytically speaking than what is actually necessary for being able to use the final results in practical problems. It is with this idea in mind, in the present paper, that we have first reviewed the problems of evaluation of several real improper integrals (see [11]) by the help of the theory of functions of real variables only whilst the application of complex function theory is also demonstrated for these problems for comparison as well as for realizing the major differences of the analysis involved.

Thus, by developing the feeling that complex function theory can be avoided for problems involving improper real integrals, we have next taken up some known real singular integral equations involving Cauchy-type kernels and have presented the real variable method of solution of these equations.

The plan of this paper is as follows: In the 'Evaluation of real improper integrals' section, we consider some problems of real improper integrals and their solutions. In the 'Solution of a Cauchy-type singular integral equation of the first kind' section, a Cauchy-type singular integral equation of the first kind is considered for its solution in the intervals $(-1, 1)$ and (a, b). We consider a Cauchy-type singular integral equation of the first kind over two disjoint intervals, $(0, a) \cup (b, c)$ and $(-1, -k) \cup (k, 1)$ and its solution in the 'Solution of a Cauchy-type singular integral equation of the first kind over an interval with a gap' section. In the 'Verification of the solutions for homogeneous Cauchy-type singular integral equation of the first kind' section, we verify the solution of the homogeneous problem obtained in the previous section. We then we determine the approximate solution of Cauchy-type singular integral equations in the intervals (0,1), $(-1, -k) \cup (k, 1)$ and then $(0, a) \cup (b, c)$ in the 'Approximate solution of singular integral equations of the Cauchy type' section. Finally, in the 'Solutions of Cauchy-type singular integral equations over semi-infinite and infinite intervals' and 'Solution of Cauchy-type singular integral equations of the first kind over infinite intervals with a gap' sections, we derive the solutions of singular integral equations of the Cauchy type, involving semi-infinite as well as infinite intervals, as special limiting cases and show that the final results agree with the known ones.

Results and discussion

Evaluation of real improper integrals

In this section, we consider the problems of evaluation of certain special real improper integrals and their solutions by using the complex variable method as well as the real variable method.

Problem 1. Evaluate

$$I = \int_0^\infty \frac{\sin x}{x} dx. \tag{1}$$

Solution (using complex analysis):

Let $F(z) = \frac{e^{iz}}{z}$. Using Cauchy's residue theorem, we obtain

$$\int_\Gamma F(z)dz = 2\pi i \sum_\Gamma \text{Res}\, F(z) = 0, \tag{2}$$

where Γ is the closed contour consisting of the upper half of the large circle $|z| = R$ and the real axis from *-R* to *R* which avoids the origin, with a semicircular indentation of radius *r*.

Then, letting $R \to \infty$ and $r \to 0$, we get

$$\int_{-\infty}^\infty F(x)dx + i\int_\pi^0 d\theta = 0, \tag{3}$$

giving

$$\int_0^\infty \frac{\sin x}{x} dx = \pi/2. \tag{4}$$

Solution (without using complex analysis):

We write

$$\begin{aligned}\int_0^\infty \frac{\sin x}{x}dx &= \lim_{\varepsilon\to 0^+}\int_0^\infty e^{-\varepsilon x}\frac{\sin x}{x}dx = \lim_{\varepsilon\to 0^+}\int_0^\infty e^{-\varepsilon x}\left(\int_0^1 \cos(\alpha x)d\alpha\right)dx \\ &= \lim_{\varepsilon\to 0^+}\int_0^1 d\alpha\left(\int_0^\infty e^{-\varepsilon x}\cos(\alpha x)dx\right) = \lim_{\varepsilon\to 0^+}\int_0^1 d\alpha\left[\frac{\varepsilon}{\alpha^2+\varepsilon^2}\right] \\ &= \lim_{\varepsilon\to 0^+}\tan^{-1}\left(\frac{\alpha}{\varepsilon}\right)\Big|_{\alpha=0}^{1} = \frac{\pi}{2},\end{aligned} \tag{5}$$

which matches with Equation 4.

Problem 2. Evaluate

$$I = \int_0^\infty \frac{\sin^2 kx}{x^2}dx,\;\; k>0. \tag{6}$$

Solution (using complex analysis):

Using Cauchy's residue theorem, we obtain

$$\int_\Gamma F(z)dz = 2\pi i\sum_\Gamma \text{Res}\, F(z) = 0, \tag{7}$$

where $F(z) = \frac{1-e^{-2iz}}{z^2}$ and Γ is the same contour as was used for problem 1.

Again, letting $R\to\infty$ and $r\to 0$, we get

$$\int_{-\infty}^\infty F(x)dx + 2\int_\pi^0 d\theta = 0, \tag{8}$$

giving

$$\begin{aligned}&\int_{-\infty}^\infty \frac{1-\cos 2x + i\sin 2x}{x^2}dx = 2\pi \\ \Rightarrow\;\; &\int_0^\infty \frac{\sin^2 x}{x^2}dx = \pi/2.\end{aligned} \tag{9}$$

Thus, we find that

$$\int_0^\infty \frac{\sin^2 kx}{x^2}dx = \frac{\pi k}{2}. \tag{10}$$

Solution (without using complex analysis):

We write

$$\begin{aligned}\int_0^\infty \frac{\sin^2 kx}{x^2}dx &= \int_0^\infty \frac{1-\cos 2kx}{2x^2}dx = \frac{1}{2}\int_0^\infty \frac{dx}{x}\left(\int_0^{2k}\sin(\alpha x)d\alpha\right) \\ &= \frac{1}{2}\int_0^{2k} d\alpha\left(\int_0^\infty \frac{\sin(\alpha x)}{x}dx\right) = \frac{\pi k}{2}.\end{aligned} \tag{11}$$

which agrees with Equation 10.

Problem 3. Evaluate

$$I = \int_0^\infty \frac{x^2 dx}{x^4+6x^2+13}. \tag{12}$$

Solution (using complex analysis):

$$I = \frac{1}{2}J, \tag{13}$$

where

$$J = \int_0^\infty \frac{\sqrt{u}du}{u^2 + 6u + 13} \qquad \text{(by setting } x^2 = u). \tag{14}$$

We use Cauchy's residue theorem and get

$$\int_\Gamma F(z)dz = 2\pi i \sum_\Gamma \text{Res}F(z),$$

where $F(z) = \frac{\sqrt{z}}{z^2+6z+13}$ and Γ is a contour comprising of a circular indentation of radius r at the origin along with the two parts of the positive real axis, one lying above and the other lying below, as well as a large circle of radius R.

Then, letting $R \to \infty$ and $r \to 0$, we obtain

$$\int_0^\infty \frac{\sqrt{u}du}{u^2 + 6u + 13} = \pi i \left[\frac{\sqrt{z_1}}{z_1 - z_2} + \frac{\sqrt{z_2}}{z_2 - z_1} \right], \tag{15}$$

where

$$z_1 = \sqrt{13}e^{i(\pi - \tan^{-1}\frac{2}{3})}, z_2 = \sqrt{13}e^{i(\pi + \tan^{-1}\frac{2}{3})}.$$

We then find that

$$I = \frac{1}{2}J = \frac{\pi}{4}13^{1/4}\sin\left(\frac{\tan^{-1}\frac{2}{3}}{2}\right). \tag{16}$$

Solution (without using complex analysis):

We write

$$\begin{aligned} I &= \frac{1}{2}\int_0^\infty \frac{\sqrt{u}du}{u^2+6u+13} \qquad \text{(by setting } x^2 = u) \\ &= \frac{1}{2(u-\bar{u_0})} \lim_{R\to\infty}\int_0^R \left[\frac{\sqrt{u}}{(u-u_0)} - \frac{\sqrt{u}}{(u-\bar{u_0})}\right] du \qquad (17) \\ &= \frac{\pi}{2(u-\bar{u_0})}\left[\sqrt{-\bar{u_0}} - \sqrt{-u_0}\right] \\ &= \frac{\pi}{4}(13)^{1/4}\sin\left(\frac{1}{2}\tan^{-1}(2/3)\right), \qquad (18) \end{aligned}$$

where $u_0 = -3 + 2i$, $\bar{u_0} = -3 - 2i$.

Equation 18 agrees with Equation 16.

Problem 4. Evaluate

$$I = \int_0^\infty \frac{\log x}{(x+1)^2}dx. \tag{19}$$

Solution: We write

$$I = \int_0^\infty \frac{\log x}{(x+1)^2}dx = \left[-\frac{1}{2k}\frac{d}{dk}J(k)\right]_{k=1}, \tag{20}$$

where

$$J(k) = \int_0^\infty \frac{\log x}{(x^2+k^2)}dx. \tag{21}$$

Solution (using complex analysis):

Applying Cauchy's residue theorem, we first get

$$\int_\Gamma \frac{\log z}{(z^2+k^2)}dz = 2\pi i \times [\text{ Residue at } z = ik], \tag{22}$$

where Γ is the contour lying above the real axis, with a small semicircular indentation of radius h and a large semicircular arc of radius R, giving

$$\int_0^\infty \frac{\ln r dr}{r^2+k^2} = \frac{\pi}{2k}\ln k. \tag{23}$$

Then, we find that

$$I = \frac{-\pi}{4} \text{ (by using Equation 20).} \tag{24}$$

Solution (without using complex analysis):

We can write

$$\begin{aligned} J(k) &= \int_0^\infty \frac{\log x}{(x^2+k^2)}dx \\ &= \frac{1}{k}\int_0^{\pi/2} [\ln k + \ln(\tan\theta)]\, d\theta \qquad \text{(by setting } x = k\tan\theta) \\ &= \frac{\pi}{2k}\ln k. \end{aligned} \tag{25}$$

Then, using Equation 20, we get

$$I = \frac{-\pi}{4}, \tag{26}$$

which agrees with Equation 24.

Problem 5. Evaluate

$$I = \int_0^\infty \frac{\cos x}{x^2+a^2}dx, \ a > 0. \tag{27}$$

Solution (using complex analysis):

Cauchy's residue theorem gives

$$J = \int_\Gamma \frac{e^{-z}dz}{z^2-a^2} = 2\pi i \times [\text{ Residue at } z = a], \tag{28}$$

where Γ is the contour lying at the right side of the y-axis with a large semicircular arc of radius R, giving

$$J = 2\pi i \frac{e^{-a}}{2a} \text{ (taking limit as } R \to \infty).$$

But $J = 2iI$, which gives

$$I = \frac{\pi e^{-a}}{2a}. \tag{29}$$

Solution (without using complex analysis):

We write

$$\begin{aligned} I &= \int_0^\infty \frac{\cos x}{a}dx \left(\int_0^\infty e^{-at}\cos(tx)dt\right) \qquad (30) \\ &= \frac{1}{2a}\int_0^\infty e^{-at}\left\{\lim_{\varepsilon\to 0^+}\int_0^\infty e^{-\varepsilon x}[\cos(1-t)x + \cos(1+t)x]\,dx\right\}dt \\ &= \frac{1}{2a}\lim_{\varepsilon\to 0^+}\left[\varepsilon\int_0^\infty \frac{e^{-at}dt}{(1-t)^2+\varepsilon^2} + \varepsilon\int_0^\infty \frac{e^{-at}dt}{(1+t)^2+\varepsilon^2}\right] \\ &= \frac{1}{2a}\lim_{\varepsilon\to 0^+}\left[-e^{-a}\lim_{\varepsilon\to 0^+}\left\{-\frac{\pi}{2} - \tan^{-1}1/\varepsilon\right\} + e^{a}\lim_{\varepsilon\to 0^+}\left\{\frac{\pi}{2} - \tan^{-1}1/\varepsilon\right\}\right] \\ &= \frac{\pi e^{-a}}{2a}, \qquad (31) \end{aligned}$$

which matches with Equation 29.

A different approach:

We write

$$
\begin{aligned}
I &= \int_0^\infty \frac{xdx}{x^2+a^2}\left[-\int_0^1 \sin\alpha x d\alpha + \frac{1}{x}\right]\\
&= -\int_0^1 d\alpha\left[\int_0^\infty \frac{x\sin\alpha x}{x^2+a^2}dx\right] + \int_0^\infty \frac{dx}{x^2+a^2}\\
&= -\int_0^1 d\alpha\left[\int_0^\infty \sin\alpha x\left\{\int_0^\infty e^{-at}\sin(tx)dt\right\}dx\right] + \frac{\pi}{2a}\\
&= -\int_0^1 d\alpha\left[\int_0^\infty e^{-at}dt\left\{\lim_{\varepsilon\to 0}\int_0^\infty e^{-\varepsilon x}\sin(\alpha x)\sin(tx)dt\right\}\right] + \frac{\pi}{2a}\\
&= -\int_0^1 d\alpha\left[\int_0^\infty e^{-at}dt\left\{\lim_{\varepsilon\to 0}\left(\frac{1}{2}\int_0^\infty e^{-\varepsilon x}\left[\cos(\alpha-t)x - \cos(\alpha+t)x\right]dx\right)\right\}\right] + \frac{\pi}{2a}\\
&= -\int_0^1 \frac{d\alpha}{2}\left[\lim_{\varepsilon\to 0}\left\{\int_0^\infty \frac{e^{-at}.\varepsilon}{(\alpha-t)^2+\varepsilon^2}dt - \int_0^\infty \frac{e^{-at}.\varepsilon dt}{(\alpha+t)^2+\varepsilon^2}\right\}\right] + \frac{\pi}{2a}\\
&= -\frac{1}{2}\int_0^1 d\alpha\left\{e^{-a\alpha}\lim_{\varepsilon\to 0}\left(\frac{\pi}{2}+\tan^{-1}(\alpha/\varepsilon)\right) - e^{a\alpha}\lim_{\varepsilon\to 0}\left(\frac{\pi}{2}-\tan^{-1}(\alpha/\varepsilon)\right)\right\} + \frac{\pi}{2a}\\
&= \frac{\pi e^{-a}}{2a}, \qquad (32)
\end{aligned}
$$

which matches with Equations 29 and 31.

Problem 6. Evaluate

$$
I = \int_0^{\pi/2} \frac{dx}{a+\sin^2 x} \quad (a>0). \qquad (33)
$$

Solution (using complex analysis):

We write

$$
\begin{aligned}
I &= \int_0^{\pi/2} \frac{dx}{a+\sin^2 x} = 2\int_0^{\pi/2} \frac{dx}{2a+(1-\cos 2x)}\\
&= \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{(1+2a)-\cos\theta} \text{ (setting } \theta = 2x \text{ and noting that } \cos\theta \text{ is even)}\\
&= \frac{1}{i}\int_\Gamma \frac{dz}{2(1+2a)z-1-z^2} \text{ (putting } e^{i\theta}=z), \qquad (34)
\end{aligned}
$$

where Γ is the unit circle around the origin.

Then, applying Cauchy's residue theorem, we get

$$
I = \frac{\pi}{2\sqrt{a(1+a)}}. \qquad (35)
$$

Solution (without using complex analysis):

We can write Equation 33 as

$$
\begin{aligned}
I &= \int_0^{\pi/2} \frac{\sec^2 x dx}{a\sec^2 x + \sin^2 x\sec^2 x}\\
&= \int_0^{\pi/2} \frac{\sec^2 x dx}{a\sec^2 x + \tan^2 x}\\
&= \int_0^\infty \frac{dt}{a+(1+a)t^2} \text{ (setting } t=\tan x)\\
&= \left(\frac{1}{1+a}\right)\int_0^\infty \frac{dt}{\frac{a}{(1+a)}+t^2} = \frac{\pi}{2\sqrt{a(1+a)}}, \qquad (36)
\end{aligned}
$$

and this agrees with Equation 35.

Problem 7. Evaluate

$$I = \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx. \tag{37}$$

Solution:

We can write Equation 37 as

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \frac{(x^2+2)dx}{x^4+10x^2+9} dx \ \ \left(\text{since } \int_{-\infty}^{\infty} \frac{xdx}{x^4+10x^2+9} = 0\right) \\ &= \int_{-\infty}^{\infty} \frac{(x^2+2)dx}{(x^2+5)^2 - 4^2} = \int_{-\infty}^{\infty} \frac{(x^2+2)dx}{(x^2+9)(x^2+1)}. \end{aligned} \tag{38}$$

Solution (using complex analysis):

Applying Cauchy's residue theorem, involving the contour Γ comprising of the real axis and a large semicircular arc lying above the real axis, we obtain

$$I = \int_{\Gamma} \frac{(z^2+2)dz}{(z^2+9)(z^2+1)} = \frac{5}{12}\pi. \tag{39}$$

Solution (without using complex analysis):

We can write Equation 38 as

$$\begin{aligned} I &= 2\int_0^{\infty} \frac{x^2+2}{(x^2+9)(x^2+1)} dx \\ &= 2\int_0^{\infty} \left[\frac{1}{8(x^2+1)} + \frac{7}{8(x^2+9)}\right] dx \ \ \text{(by partial fraction)} \\ &= \frac{1}{4}\left[\tan^{-1} x + \frac{7}{3}\tan^{-1}\frac{x}{3}\right]_0^{\infty} \\ &= \frac{5}{12}\pi, \end{aligned} \tag{40}$$

and this matches with Equation 39.

Problem 8. Evaluate

$$I(x) = \int_{-1}^{1} \frac{(1+t)^{\alpha}(1-t)^{1-\alpha}}{t-x} dt, \ \ (0 < \alpha < 1), (-1 < x < 1). \tag{41}$$

Solution:

$I(x)$ can be written as

$$\begin{aligned} I(x) &= \int_{-1}^{1} \frac{(1+t)^{1/2}(1-t)^{1/2}(1+t)^{\alpha-\frac{1}{2}}(1-t)^{\frac{1}{2}-\alpha}}{t-x} dt \\ &= \int_{-1}^{1} \frac{(1-t^2)^{1/2}\psi(t)}{t-x} dt, \end{aligned} \tag{42}$$

where

$$\psi(t) = (1+t)^{\alpha-\frac{1}{2}}(1-t)^{\frac{1}{2}-\alpha}. \tag{43}$$

This $\psi(t)$ can be written as

$$\psi(t) = \sum_{n=0}^{\infty} c_n t^n, \tag{44}$$

where $c_0 = 1, c_1 = 2\beta, c_2 = 2\beta^2, c_3 = 2\left(\frac{2\beta^3+\beta}{3}\right), \ldots$ with $\beta = \alpha - \frac{1}{2}$.

From Equations 42 and 44, $I(x)$ can be written as

$$I(x) = \sum_{n=0}^{\infty} c_n I_n(x), \tag{45}$$

where

$$I_n(x) = \int_{-1}^{1} \frac{(1-t^2)^{1/2} t^n}{t-x} dt. \tag{46}$$

Evaluation of $I_n(x)$ ***(using complex analysis):***

$$I_n(x) = -\pi \, \mathrm{PP} \left\{ x^{n+1} \left(1 - \frac{1}{x^2}\right)^2 \right\}_{\text{for large } x} \quad \text{(see [2])}, \tag{47}$$

from which we obtain

$$I_0(x) = -\pi x, I_1(x) = -\pi \left(x^2 - \frac{1}{2}\right), I_2(x) = -\pi \left(x^3 - \frac{x}{2}\right),$$
$$I_3(x) = -\pi \left(x^4 - \frac{x^2}{2} - \frac{1}{8}\right), \ldots . \tag{48}$$

Evaluation of $I_n(x)$ ***(without using complex analysis):***
$I_n(x)$ can also be written as

$$I_n(x) = \begin{cases} 2x \int_0^1 \frac{(1-t^2)^{1/2} t^n}{t^2-x^2} dt, \text{when } n \text{ is even} \\ 2 \int_0^1 \frac{(1-t^2)^{1/2} t^{n+1}}{t^2-x^2} dt, \text{when } n \text{ is odd,} \end{cases} \tag{49}$$

$$\Rightarrow I_n(x) = \begin{cases} 2x \left[-S_n + (1-x^2)\{S_{n-2} + x^2 S_{n-4} + x^4 S_{n-6} + \ldots + x^{n-2} S_0\}\right], \text{when } n \text{ is even} \\ 2\left[-S_{n+1} + (1-x^2)\{S_{n-1} + x^2 S_{n-3} + x^4 S_{n-5} + \ldots + x^{n-1} S_0\}\right], \text{when } n \text{ is odd,} \end{cases} \tag{50}$$

where

$$S_n = \frac{1}{2} B\left(\frac{n+1}{2}, \frac{1}{2}\right). \tag{51}$$

The values of $I_n(x)$ can then be determined easily which are the same as the ones obtained in Equation 48.

Substituting the values of $I_n(x)(n = 0, 1, 2, \ldots)$ in Equation 45, the approximate value of $I(x)$ can be written as

$$I(x) = -\pi \left[x + 2\beta \left(x^2 - \frac{1}{2}\right) + 2\beta^2 \left(x^3 - \frac{x}{2}\right) + 2\left(\frac{2\beta^3 + \beta}{3}\right)\left(x^4 - \frac{x^2}{2} - \frac{1}{8}\right) + \ldots\right]. \tag{52}$$

The exact value of $I(x)$ is given by (see [2])

$$I(x) = \pi \left[(1 - 2\alpha - x)\csc(\pi\alpha) - (1+x)^{\alpha}(1-x)^{1-\alpha}\cot(\pi\alpha)\right], \quad (0 < \alpha < 1), (-1 < x < 1). \tag{53}$$

In Table 1, we find that the exact values and approximate values of $I(x)$ are nearly equal.

Table 1 Comparison between exact values and approximate values of *I*(*x*)

	$\alpha = 1/4, x = 0.5$	$\alpha = 1/4, x = 0$	$\alpha = 1/4, x = -0.5$
I_{exact}	−2.0673	−0.9202	0.8622
I_{approx}	−2.0249	−0.8590	1.0186

Evaluation of some real singular integrals by real variable method

Here, we consider some problems involving singular integrals and their solutions by real variable method. These results are useful to evaluate some other integral equations in the succeeding sections.

Problem 1. Evaluate

$$I = \int_0^1 \frac{\sqrt{t(1-t)}dt}{t-x}, \text{ for } x \notin (0,1), \text{ real.} \tag{54}$$

Solution: Setting $t = \sin^2\theta$ $[t \in (0,1) \Rightarrow \theta \in (0, \pi/2)]$, we get

$$\begin{aligned} I(x) &= \int_0^{\pi/2} \frac{2\sin^2\theta\cos^2\theta d\theta}{\sin^2\theta - x} \\ &= 2(1-x)\int_0^{\pi/2} \frac{\sin^2\theta d\theta}{\sin^2\theta - x} - 2\int_0^{\pi/2} \sin^2\theta d\theta \\ &= 2(1-x)\left[\int_0^{\pi/2} d\theta + x\int_0^{\pi/2} \frac{d\theta}{\sin^2\theta - x}\right] - \frac{\pi}{2} \\ &= \pi(1-x) + 2x(1-x)\int_0^{\pi/2} \frac{d(\tan\theta)}{\tan^2\theta(1-x) - x} - \frac{\pi}{2}. \end{aligned} \tag{55}$$

Case 1: When $x > 1$, we obtain

$$\begin{aligned} I(x) &= \pi(1-x) + 2x\int_0^{\pi/2} \frac{d(\tan\theta)}{\tan^2\theta + \frac{x}{x-1}} - \frac{\pi}{2} \\ &= \pi\left(\frac{1}{2} - x\right) + \pi\sqrt{x(x-1)}. \end{aligned} \tag{56}$$

Case 2: When $x < 0$, we obtain

$$\begin{aligned} I(x) &= \pi(1-x) + 2x\int_0^{\pi/2} \frac{d(\tan\theta)}{\tan^2\theta + \frac{(-x)}{1-x}} - \frac{\pi}{2} \\ &= \pi\left(\frac{1}{2} - x\right) + \pi\sqrt{x(x-1)}. \end{aligned} \tag{57}$$

So, we get

$$I(x) = \pi\left(\frac{1}{2} - x\right) + \pi\sqrt{x(x-1)}. \quad \blacksquare \tag{58}$$

Problem 2. Evaluate

$$I = \int_0^1 \frac{dt}{\sqrt{t(1-t)}\,(t-x)}, \text{ for } x \notin (0,1), \text{ real.} \tag{59}$$

Solution: Setting $t = \sin^2\theta$ $[t \in (0,1) \Rightarrow \theta \in (0, \pi/2)]$, we get

$$I(x) = 2\int_0^{\pi/2} \frac{d\theta}{\sin^2\theta - x} = 2\int_0^{\pi/2} \frac{d(\tan\theta)}{\tan^2\theta(1-x) - x}. \tag{60}$$

Case 1: When $x > 1$, we obtain

$$I(x) = \frac{2}{1-x}\int_0^{\pi/2} \frac{d(\tan\theta)}{\tan^2\theta + \frac{x}{x-1}} = \frac{-\pi}{\sqrt{x(x-1)}}. \tag{61}$$

Case 2: When $x < 0$, we obtain

$$I(x) = \frac{2}{1-x}\int_0^{\pi/2} \frac{d(\tan\theta)}{\tan^2\theta + \frac{(-x)}{1-x}} = \frac{-\pi}{\sqrt{x(x-1)}}. \tag{62}$$

So, we get

$$I(x) = \frac{-\pi}{\sqrt{x(x-1)}}. \qquad \blacksquare \qquad (63)$$

Problem 3. Evaluate

$$I(x) = \int_a^c \frac{\sqrt{(t-a)(c-t)}dt}{t-x}, \text{ for } x \notin (a,c), \text{ real.} \qquad (64)$$

Solution: Setting $t = a\cos^2\theta + c\sin^2\theta$ $[\,t \in (a,c) \Rightarrow \theta \in (0, \pi/2)]$, we get

$$\begin{aligned} I(x) &= \int_0^{\pi/2} \frac{2(c-a)^2 \sin^2\theta \cos^2\theta d\theta}{a\cos^2\theta + c\sin^2\theta - x} \\ &= 2(c-a)\int_0^{\pi/2} \frac{(c-x)\sin^2\theta - \sin^2\theta\left(a\cos^2\theta + c\sin^2\theta - x\right)}{a\cos^2\theta + c\sin^2\theta - x} d\theta \\ &= 2(c-a)(c-x)\int_0^{\pi/2} \frac{\sin^2\theta d\theta}{(a-x)+(c-a)\sin^2\theta} - (c-a)\frac{\pi}{2} \\ &= 2(c-x)\left[\frac{\pi}{2} + \frac{x-a}{c-a}\int_0^{\pi/2} \frac{d(\tan\theta)}{\left\{1 - \frac{x-a}{c-a}\right\}\tan^2\theta - \frac{x-a}{c-a}}\right] - (c-a)\frac{\pi}{2}. \qquad (65) \end{aligned}$$

Case 1: When $x > c$ (i.e., $x - c > 0$ and $x - a > 0$), we obtain

$$\begin{aligned} I(x) &= 2(c-x)\left[\frac{\pi}{2} + \frac{x-a}{c-x}\int_0^{\pi/2} \frac{d(\tan\theta)}{\tan^2\theta + \frac{x-a}{x-c}}\right] - (c-a)\frac{\pi}{2} \\ &= \pi\left[\frac{(c+a)}{2} - x\right] + \pi\sqrt{(x-a)(x-c)}. \qquad (66) \end{aligned}$$

Case 2: When $x < a$ (i.e., $a - x > 0, c - x > 0$), we obtain

$$\begin{aligned} I(x) &= 2(c-x)\left[\frac{\pi}{2} + \frac{x-a}{c-x}\int_0^{\pi/2} \frac{d(\tan\theta)}{\tan^2\theta + \frac{a-x}{c-x}}\right] - (c-a)\frac{\pi}{2} \\ &= \pi\left[\frac{(c+a)}{2} - x\right] - \pi\sqrt{(a-x)(c-x)}. \qquad (67) \end{aligned}$$

So, we obtain

$$I(x) = \pi\left[\frac{(c+a)}{2} - x\right] + \pi\sqrt{(x-a)(x-c)}\ \mathrm{sgn}(x-a). \qquad \blacksquare \qquad (68)$$

Problem 4. Evaluate

$$I(x) = \int_a^c \frac{dt}{\sqrt{(t-a)(c-t)}\,(t-x)}, \text{ for } x \notin (a,c), \text{ real.} \qquad (69)$$

Solution: Setting $t = a\cos^2\theta + c\sin^2\theta$ $[\,t \in (a,c) \Rightarrow \theta \in (0, \pi/2)]$, we get

$$\begin{aligned} I(x) &= 2\int_0^{\pi/2} \frac{d\theta}{a\cos^2\theta + c\sin^2\theta - x} \\ &= 2\int_0^{\pi/2} \frac{d\theta}{(a-x)\cos^2\theta + (c-x)\sin^2\theta}. \qquad (70) \end{aligned}$$

Case 1: When $x > c$ (i.e., $x - c > 0$ and $x - a > 0$), we get

$$I(x) = \frac{2}{c-x}\int_0^{\pi/2} \frac{d(\tan\theta)}{\tan^2\theta + \frac{x-a}{x-c}} = \frac{-\pi}{\sqrt{(x-a)(x-c)}}. \tag{71}$$

Case 2: When $x < a$ (i.e., $a - x > 0, c - x > 0$), we get

$$I(x) = \frac{2}{c-x}\int_0^{\pi/2} \frac{d(\tan\theta)}{\tan^2\theta + \frac{a-x}{c-x}} = \frac{\pi}{\sqrt{(c-x)(a-x)}}. \tag{72}$$

So, we obtain

$$I(x) = \frac{-\pi}{\sqrt{(x-a)(x-c)}}\,\mathrm{sgn}(x-a). \quad \blacksquare \tag{73}$$

Solution of a Cauchy-type singular integral equation of the first kind

Here, we consider a Cauchy-type singular integral equation of the first kind in the interval (-1,1) and obtain its solution by real variable method. Then, we have generalized the result for the interval (*a*,*b*).

Problem 1. Solve the singular integral equation of the first kind

$$\int_{-1}^{1} \frac{\phi(t)dt}{t-x} = f(x), \quad -1 < x < 1. \tag{74}$$

Solution: Set

$$t = 2u - 1, x = 2\xi - 1 \qquad [t \in (-1,1) \Rightarrow u \in (0,1),\ \ x \in (-1,1) \Rightarrow \xi \in (0,1)]. \tag{75}$$

Get: $t - x = 2(u - \xi)$.

Hence, the given integral equation (Equation 74) becomes

$$\int_0^1 \frac{\phi(2u-1)2du}{2(u-\xi)} = f(2\xi - 1),$$

$$\text{or} \int_0^1 \frac{\psi(u)du}{u-\xi} = g(\xi), \tag{76}$$

where

$$\psi(u) = \phi(2u-1), g(\xi) = f(2\xi - 1). \tag{77}$$

Now, we set

$$u = \cos^2\theta, \quad \xi = \cos^2\alpha. \tag{78}$$

Then, we get, from Equation 76,

$$2\int_0^{\pi/2} \frac{\psi(\cos^2\theta)\sin\theta\cos\theta d\theta}{\cos^2\theta - \cos^2\alpha} = g(\cos^2\alpha). \tag{79}$$

Now, we set (see [12])

$$2\psi(\cos^2\theta)\sin\theta\cos\theta = \frac{1}{2}a_0 + \sum_{r=1}^{\infty} a_{2r}\cos 2r\theta. \tag{80}$$

Then, Equations 79 and 80 give

$$g(\cos^2\alpha) = \frac{1}{2}a_0\int_0^{\pi/2}\frac{d\theta}{\cos^2\theta-\cos^2\alpha} + \sum_{r=1}^{\infty}a_{2r}\int_0^{\pi/2}\frac{\cos 2r\theta d\theta}{\cos^2\theta-\cos^2\alpha}. \tag{81}$$

Now, we have the following results:

Result 1:

$$\begin{aligned} I_0 \equiv \int_0^{\pi/2}\frac{d\theta}{\cos^2\theta-\cos^2\alpha} &= \int_0^{\pi/2}\frac{\sec^2\theta d\theta}{1-\cos^2\alpha(1+\tan^2\theta)} \\ &= \frac{-\sec^2\alpha}{2\tan\alpha}\left[\log\left|\frac{\tan\theta-\tan\alpha}{\tan\theta+\tan\alpha}\right|\right]_0^{\pi/2} \\ &= 0. \end{aligned} \tag{82}$$

Result 2: Let

$$I_r = \int_0^{\pi/2}\frac{\cos(2r\theta)d\theta}{\cos^2\theta-\cos^2\alpha}, \quad (r=1,2,3,\ldots). \tag{83}$$

Now,

$$\begin{aligned} I_1 &= \int_0^{\pi/2}\frac{\cos 2\theta d\theta}{\cos^2\theta-\cos^2\alpha} \\ &= \int_0^{\pi/2}\frac{2(\cos^2\theta-\cos^2\alpha)+(2\cos^2\alpha-1)}{\cos^2\theta-\cos^2\alpha}d\theta \\ &= 2\int_0^{\pi/2}d\theta + (2\cos^2\alpha-1)I_0 = \pi, \quad \text{by using Equation 82.} \end{aligned}$$

$$\text{So,}\quad I_1 \equiv \frac{\pi\sin 2\alpha}{\sin 2\alpha}. \tag{84}$$

Again, we get

$$\begin{aligned} I_2 &= \int_0^{\pi/2}\frac{\cos(4\theta)d\theta}{\cos^2\theta-\cos^2\alpha} \\ &= \int_0^{\pi/2}\frac{8(\cos^2\theta-1)\cos^2\theta}{\cos^2\theta-\cos^2\alpha}d\theta \quad (\text{by using } I_0=0) \\ &= 4\int_0^{\pi/2}(\cos 2\theta+1)d\theta - 4\sin^2\alpha\int_0^{\pi/2}\frac{\cos 2\theta+1}{\cos^2\theta-\cos^2\alpha}d\theta \\ &= 2\pi - 4\sin^2\alpha(I_1+I_0) = 2\pi\cos 2\alpha, \text{(see Equations 82 and 84).} \end{aligned} \tag{85}$$

So, we get

$$I_2 = \frac{2\pi\cos 2\alpha\sin 2\alpha}{\sin 2\alpha} = \frac{\pi\sin 4\alpha}{\sin 2\alpha} = \frac{\pi\sin 2r\alpha}{\sin 2\alpha}, \ (r=2). \tag{86}$$

Now, let us assume that

$$I_r = \frac{\pi\sin 2r\alpha}{\sin 2\alpha}, \ (r=1,2,\ldots). \tag{87}$$

We have

$$
\begin{aligned}
I_{r+1} &= \int_0^{\pi/2} \frac{\cos\,(2(r+1)\theta)d\theta}{\cos^2\theta - \cos^2\alpha} \\
&= \cos(2\alpha)I_r - \int_0^{\pi/2} \frac{\{\cos\,(r+1)\theta + \cos\,(r-1)\theta\}\{\cos\,(r-1)\theta - \cos\,(r+1)\theta\}}{\cos^2\theta - \cos^2\alpha} d\theta \\
&= \cos(2\alpha)\frac{\pi\,\sin(2r\alpha)}{\sin(2\alpha)} - \int_0^{\pi/2} \frac{\{\cos^2\,(r-1)\theta - \cos^2\,(r+1)\theta\}}{\cos^2\theta - \cos^2\alpha} d\theta \\
&= \pi\frac{\sin 2(r+1)\alpha}{\sin(2\alpha)} - \pi\,\cos(2r\alpha) + \frac{1}{2}\,[I_{r+1} - I_{r-1}] \\
\Rightarrow \frac{1}{2}I_{r+1} &= \pi\frac{\sin 2(r+1)\alpha}{\sin(2\alpha)} - \pi\,\cos(2r\alpha) - \frac{\pi}{2}\frac{\sin 2(r-1)\alpha}{\sin(2\alpha)}, \text{by Equation 87} \\
&= \frac{\pi}{2}\frac{\sin 2(r+1)\alpha}{\sin(2\alpha)}.
\end{aligned} \tag{88}
$$

We then get

$$
I_{r+1} = \pi\frac{\sin 2(r+1)\alpha}{\sin(2\alpha)}. \tag{89}
$$

Hence, the assumption in Equation 87 holds good for all $r = 1, 2, \ldots$, by induction.

Thus, by using Equation 87, we get, from Equation 81,

$$
g(\cos^2\alpha) = \pi\sum_{r=1}^{\infty} a_{2r}\frac{\sin 2r\alpha}{\sin(2\alpha)}. \tag{90}
$$

Hence, we obtain, by the Fourier series method,

$$
\pi a_{2r} = \frac{4}{\pi}\int_0^{\pi/2} g(\cos^2\alpha)\sin(2\alpha)\sin(2r\alpha)d\alpha. \tag{91}
$$

This, then, gives, from Equation 80,

$$
2\psi(\cos^2\theta)\sin\theta\cos\theta = \frac{1}{2}a_0 + \frac{4}{\pi^2}\int_0^{\pi/2} g(\cos^2\alpha)\sin(2\alpha)\left(\sum_{r=1}^{\infty}\sin(2r\alpha)\cos(2r\theta)\right)d\alpha, \tag{92}
$$

$$
\left[\text{Use: } \sum_{r=1}^{\infty}\sin(2r\alpha)\cos(2r\theta) = \lim_{R\to 1-0}\sum_{r=1}^{\infty} R^{2r}\sin(2r\alpha)\cos(2r\theta)\right].
$$

Now, we have the following result (using a limiting procedure of the type explained above):

$$
\begin{aligned}
\sum_{r=1}^{\infty}\sin(2r\alpha)\cos(2r\theta) &= \frac{1}{2}\sum_{r=1}^{\infty}[\sin 2r(\alpha+\theta) + \sin 2r(\alpha-\theta)] \\
&= \frac{1}{2}\text{Im}\left[\sum_{r=1}^{\infty}\left(e^{2ir(\alpha+\theta)} + e^{2ir(\alpha-\theta)}\right)\right] \\
&= \frac{1}{2}\text{Im}\left[\frac{1}{1-e^{2i(\alpha+\theta)}} + \frac{1}{1-e^{2i(\alpha-\theta)}}\right] \\
&= \frac{1}{4}\,[\cot(\alpha+\theta) + \cot(\alpha-\theta)] \\
&= \frac{\sin(2\alpha)}{2\{\cos 2\theta - \cos 2\alpha\}} = \frac{\sin 2\alpha}{4(\cos^2\theta - \cos^2\alpha)}.
\end{aligned} \tag{93}
$$

Thus, Equations 92 and 93 give

$$2\psi(\cos^2\theta)\sin\theta\cos\theta = \frac{1}{2}a_0 + \frac{4}{\pi^2}\int_0^{\pi/2}\frac{g(\cos^2\alpha)\sin^2\alpha\cos^2\alpha}{(\cos^2\theta-\cos^2\alpha)}d\alpha. \tag{94}$$

Hence, putting back (see Equation 78): $u = \cos^2\theta, \xi = \cos^2\alpha, -d\xi = 2\sin(\alpha)\cos(\alpha)d\alpha,\ \ \alpha\in(0,\pi/2)\Rightarrow \xi\in(1,0)$, we obtain, from Equation 94,

$$\psi(u) = \frac{c}{\sqrt{u(1-u)}} - \frac{1}{\pi^2}\frac{1}{\sqrt{u(1-u)}}\int_0^1\frac{g(\xi)\sqrt{\xi(1-\xi)}}{(\xi-u)}d\xi, \tag{95}$$

where $c = (1/4)a_0 =$ an arbitrary constant.

Note that Equation 95 is the well-known form of the solution of the integral equation (Equation 76), obtainable by using the theory of Riemann-Hilbert problems involving functions of a complex variable.

Finally, substituting $u = \frac{1}{2}(t+1),\ \xi = \frac{1}{2}(x+1), \psi(u) = \phi(t), g(\xi) = f(x), [\,u\in(0,1)\Rightarrow t\in(-1,1);\ \ \xi\in(0,1)\Rightarrow x\in(-1,1)]$, we obtain

$$\phi(t) = \frac{c_0}{\sqrt{1-t^2}} - \frac{1}{\pi^2}\frac{1}{\sqrt{1-t^2}}\int_{-1}^1\frac{f(x)\sqrt{1-x^2}}{x-t}dx, \tag{96}$$

which is the well-known form of the solution of the given integral equation (Equation 74) where $c_0 = 2c$ is an arbitrary constant.

Now, we consider the integral equation given by Equation 74 for its solution in the interval (a, b). So, we have the following problem to solve:

Problem 2. Solve the singular integral equation

$$\int_a^b\frac{\phi(t)dt}{t-x} = f(x),\ \ a<x<b. \tag{97}$$

Solution: Set

$$t = a+(b-a)u,\ x = a+(b-a)\xi \qquad [\,t\in(a,b)\Rightarrow u\in(0,1),\ \ x\in(a,b)\Rightarrow\xi\in(0,1)\,]\,. \tag{98}$$

Get: $\qquad t-x = (b-a)(u-\xi)$.

Hence, the given integral equation (Equation 97) becomes

$$\int_0^1\frac{\psi(u)du}{u-\xi} = g(\xi), \tag{99}$$

where

$$\psi(u) = \phi\{a+(b-a)u\},\ g(\xi) = f\{a+(b-a)\xi\}. \tag{100}$$

By the help of Equations 76 and 95, the solution of the integral equation (Equation 99) can be written as

$$\psi(u) = \frac{c}{\sqrt{u(1-u)}} - \frac{1}{\pi^2}\frac{1}{\sqrt{u(1-u)}}\int_0^1\frac{g(\xi)\sqrt{\xi(1-\xi)}}{(\xi-u)}d\xi, \tag{101}$$

where $c = (1/4)a_0 =$ an arbitrary constant.

Now, substituting, $u = \frac{t-a}{b-a},\ \xi = \frac{x-a}{b-a}, \psi(u) = \phi(t), g(\xi) = f(x), [\,u\in(0,1)\Rightarrow t\in(a,b);\ \ \xi\in(0,1)\Rightarrow x\in(a,b)]$, in Equation 101, we get

$$\phi(t) = \frac{c_0}{\sqrt{(t-a)(b-t)}} - \frac{1}{\pi^2}\frac{1}{\sqrt{(t-a)(b-t)}}\int_a^b\frac{f(x)\sqrt{(x-a)(b-x)}}{(x-t)}dx, \tag{102}$$

which is the well-known form of the solution of the given integral equation (Equation 74), where $c_0 = c(b-a)$ is an arbitrary constant.

Solution of a Cauchy-type singular integral equation of the first kind over an interval with a gap

Here, we consider a Cauchy-type singular integral equation of the first kind over two disjoint intervals $(0, a) \cup (b, c)$ and obtain its solution by real variable method. We also find, here, its solution, applicable to the intervals $(-1, -k) \cup (k, 1)$ as considered by Tricomi [13].

Problem: Solve the singular integral equation of the first kind, involving a finite interval with a gap, as given by

$$\int_0^a \frac{\phi(t)dt}{t-x} + \int_b^c \frac{\phi(t)dt}{t-x} = f(x), \;\; x \in (0, a) \cup (b, c). \tag{103}$$

Solution: We shall use the known solution of the following singular integral equation:

$$\int_0^1 \frac{p(t)dt}{t-x} = q(x), \;\; 0 < x < 1, \tag{104}$$

which is given by

$$p(x) = \frac{c_0}{\sqrt{x(1-x)}} - \frac{1}{\pi^2\sqrt{x(1-x)}} \int_0^1 \frac{\sqrt{t(1-t)}q(t)dt}{t-x}, \;\; 0 < x < 1, \tag{105}$$

where c_0 is an arbitrary constant.

Setting

$$cx = \xi, ct = \tau, p(t) = \psi(\tau), q(x) = g(\xi), \tag{106}$$

we obtain, from Equations 104 and 106, that the solution of the singular integral equation

$$\int_0^c \frac{\psi(\tau)d\tau}{\tau-\xi} = g(\xi), \;\; 0 < \xi < c, \tag{107}$$

is given by

$$\psi(\xi) = \frac{D_0}{\sqrt{\xi(c-\xi)}} - \frac{1}{\pi^2\sqrt{\xi(c-\xi)}} \int_0^c \frac{\sqrt{\tau(c-\tau)}g(\tau)d\tau}{\tau-\xi}, \tag{108}$$

where D_0 is an arbitrary constant.

Now, we shall first solve the homogeneous integral equation (Equation 103) as follows:

Consider the homogeneous integral equation (Equation 103), as given by

$$\int_0^a \frac{\phi_0(t)dt}{t-x} + \int_b^c \frac{\phi_0(t)dt}{t-x} = 0, \;\; x \in (0, a) \cup (b, c). \tag{109}$$

Let us assume that there exist two functions: $\psi_0(t)$ and $f_0(x)$, such that

$$\psi_0(t) = \begin{cases} \phi_0(t), & \text{for } t \in (0, a) \cup (b, c), \\ 0, & \text{for } t \in (a, b), \end{cases} \tag{110}$$

and

$$f_0(x) = \begin{cases} 0, & \text{for } x \in (0, a) \cup (b, c), \\ \hat{f}_0(x), & \text{for } x \in (a, b), \end{cases} \tag{111}$$

where $\hat{f}_0(x)$ is an unknown function.

Then, using Equations 109, 110, and 111, we obtain an integral equation which is given by

$$\int_0^c \frac{\psi_0(t)dt}{t-x} = f_0(x), \text{ for } x \in (0,c), \tag{112}$$

which possesses the solution (see Equations 107 and 108)

$$\psi_0(x) = \frac{E_0}{\sqrt{x(c-x)}} - \frac{1}{\pi^2\sqrt{x(c-x)}} \int_a^b \frac{\sqrt{t(c-t)}\hat{f}_0(t)dt}{t-x}, \ \ 0<x<c, \tag{113}$$

where E_0 is an arbitrary constant.

Now, $\psi_0(x) = 0$, for $x \in (a,b)$ (see Equation 110), gives an integral equation for the unknown function $\hat{f}_0(x)$, as given by (see Equation 113)

$$\int_a^b \frac{\hat{f}_0(t)\sqrt{t(c-t)}dt}{t-x} = \pi^2 E_0, \quad x \in (a,b), \tag{114}$$

with its solution, as given by the equation (see Equations 105 and 108)

$$\begin{aligned}
\sqrt{x(c-x)}\hat{f}_0(x) &= \frac{F_0}{\sqrt{(b-x)(x-a)}} - \frac{1}{\pi^2\sqrt{(b-x)(x-a)}} \\
&\quad \int_a^b \frac{\sqrt{(b-t)(t-a)}}{t-x}(\pi^2 E_0)dt, \ \ a<x<b \\
&= \frac{F_0}{\sqrt{(b-x)(x-a)}} - \frac{E_0}{\sqrt{(b-x)(x-a)}} \\
&\quad \times \left[-\pi \,\mathrm{PP}\left\{x\left(1-\frac{a}{x}\right)^{1/2}\left(1-\frac{b}{x}\right)^{1/2}\right\}\right]_{(x \text{ large})} \quad \text{(see [2])} \\
&= \frac{G_0 + H_0 x}{\sqrt{(b-x)(x-a)}}, \ \ x \in (a,b),
\end{aligned} \tag{115}$$

where $G_0 = F_0 - \frac{\pi}{2}E_0(a+b), H_0 = \pi E_0$.

Thus, using Equations 113 and 115, as well as Equation 110, we obtain the solution of the homogeneous equation (Equation 109), as given by

$$\phi_0(x) = \frac{E_0}{\sqrt{x(c-x)}} - \frac{1}{\pi^2\sqrt{x(c-x)}} \int_a^b \frac{G_0 + H_0 t}{\sqrt{(b-t)(t-a)}(t-x)}dt, \ \ x \in (0,a)\cup(b,c). \tag{116}$$

Now, we can evaluate the integral in Equation 116 for $x \notin (a,b)$ (see [2]) and obtain

$$\int_a^b \frac{(G_0+H_0t)dt}{\sqrt{(b-t)(t-a)}(t-x)}dt = \begin{cases} \pi\left[\frac{-(G_0+H_0x)}{\sqrt{(x-b)(x-a)}} + H_0\right], & \text{for } x>b \\ \pi\left[\frac{(G_0+H_0x)}{\sqrt{(x-b)(x-a)}} + H_0\right], & \text{for } x<a. \end{cases} \tag{117}$$

Hence, by using Equations 116 and 117, we obtain

$$\phi_0(x) = \begin{cases} \frac{E_0}{\sqrt{x(c-x)}} - \frac{1}{\pi\sqrt{x(c-x)}}\left[\frac{-(G_0+H_0x)}{\sqrt{(x-b)(x-a)}} + H_0\right], & \text{for } x>b \\ \frac{E_0}{\sqrt{x(c-x)}} - \frac{1}{\pi\sqrt{x(c-x)}}\left[\frac{(G_0+H_0x)}{\sqrt{(x-b)(x-a)}} + H_0\right], & \text{for } x<a. \end{cases} \tag{118}$$

$$\Rightarrow \phi_0(x) = \frac{(\widehat{G_0} + \widehat{H_0}x)\ \mathrm{Sgn}\ (x-a)}{\sqrt{x(c-x)}\sqrt{(x-b)(x-a)}}, \ \ \text{for } x \in (0,a)\cup(b,c), \tag{119}$$

which is the solution of the homogeneous equation (Equation 118), where $\widehat{G_0} = \frac{G_0}{\pi}, \widehat{H_0} = \frac{H_0}{\pi}$, are two arbitrary constants.

Now, we solve the inhomogeneous equation (Equation 103) (for a particular solution): We define

$$\psi(x) = \begin{cases} \phi_(x), & \text{for } x \in (0,a) \cup (b,c), \\ 0, & \text{for } x \in (a,b), \end{cases} \tag{120}$$

and

$$h(x) = \begin{cases} f(x), & \text{for } x \in (0,a) \cup (b,c), \\ \hat{f}(x), & \text{for } x \in (a,b), \end{cases} \tag{121}$$

where $\hat{f}(x)$ is an unknown function.

Then, Equation 103 can be expressed as

$$\int_0^c \frac{\psi(t)dt}{t-x} = h(x), \text{ for } x \in (0,c), \tag{122}$$

with the particular solution

$$\psi(x) = \frac{-1}{\pi^2\sqrt{x(c-x)}} \int_0^c \frac{h(t)\sqrt{t(c-t)}}{t-x} dt. \tag{123}$$

Then, $\psi(x) = 0$, for $x \in (a,b)$, gives the integral equation (see Equations 120, 121, and 123):

$$\int_a^b \frac{\hat{f}(t)\sqrt{t(c-t)}dt}{t-x} = -\int_{(0,a)\cup(b,c)} \frac{f(t)\sqrt{t(c-t)}dt}{t-x}, \text{ for } x \in (a,b). \tag{124}$$

Now the singular integral equation (Equation 124) possesses the particular solution as given by

$$\hat{f}(x)\sqrt{x(c-x)} = \frac{1}{\pi^2\sqrt{(b-x)(x-a)}} \int_a^b \frac{\sqrt{(b-\tau)(\tau-a)}}{\tau - x} \times \left(\int_{(0,a)\cup(b,c)} \frac{f(t)\sqrt{t(c-t)}dt}{t-\tau} \right) d\tau, \text{ (for } x \in (a,b))$$

$$= \frac{1}{\pi^2\sqrt{(b-x)(x-a)}} \left[\int_{(0,a)\cup(b,c)} \frac{f(t)\sqrt{t(c-t)}dt}{t-x} \int_a^b \sqrt{(b-\tau)(\tau-a)} \times \left\{ \frac{1}{-(\tau-t)} + \frac{1}{\tau-x} \right\} d\tau \right], \text{ (for } x \in (a,b))$$

$$= \frac{1}{\pi^2\sqrt{(b-x)(x-a)}} \left[\int_{(0,a)\cup(b,c)} \frac{f(t)\sqrt{t(c-t)}dt}{t-x} \left\{ -\pi\left(\frac{b+a}{2} - t\right) - \pi\sqrt{(t-b)(t-a)} \text{ Sgn } (t-a) + \pi\left(\frac{b+a}{2} - x\right) \right\} \right] \text{ (for } x \in (a,b))$$

(see Equations 64, 68, and [2])

$$= \frac{1}{\pi\sqrt{(b-x)(x-a)}} \left[\left(\int_0^a + \int_b^c \right) f(t)\sqrt{t(c-t)}dt + \left(\int_0^a - \int_b^c \right) \frac{f(t)\sqrt{t(c-t)(t-b)(t-a)}}{t-x} dt \right] \text{(for } x \in (a,b)). \tag{125}$$

Using Equation 121, Equation 123 can be written as

$$\psi(x) = \frac{-1}{\pi^2\sqrt{x(c-x)}}\left[\left(\int_0^a + \int_b^c\right)\frac{f(t)\sqrt{t(c-t)}}{t-x}dt + \int_a^b \frac{\hat{f}(\tau)\sqrt{\tau(c-\tau)}}{\tau - x}d\tau\right] \tag{126}$$

$$= \frac{-1}{\pi^2\sqrt{x(c-x)}}\left(\int_0^a + \int_b^c\right)\frac{f(t)\sqrt{t(c-t)}}{t-x}dt - \frac{-1}{\pi^3\sqrt{x(c-x)}}$$

$$\int_a^b \frac{d\tau}{\sqrt{(b-\tau)(\tau-a)}(\tau-x)}\left[\left(\int_0^a + \int_b^c\right)f(t)\sqrt{t(c-t)}dt + \left(\int_0^a - \int_b^c\right)\right.$$

$$\left.\times\frac{f(t)\sqrt{t(c-t)(t-a)(t-b)}dt}{t-\tau}\right],$$

$x \in (0,a)\cup(b,c)$ (by substitution of Equation 125),

$$= \frac{-1}{\pi^2\sqrt{x(c-x)}}\left(\int_0^a + \int_b^c\right)\frac{f(t)\sqrt{t(c-t)}}{t-x}dt - \frac{M_0}{\pi^3\sqrt{x(c-x)}}\int_a^b \frac{d\tau}{\sqrt{(b-\tau)(\tau-a)}(\tau-x)}$$

$$- \frac{-1}{\pi^3\sqrt{x(c-x)}}\left[\left(\int_0^a - \int_b^c\right)f(t)\sqrt{t(c-t)(t-a)(t-b)}dt\times\right.$$

$$\left.\int_a^b \frac{d\tau}{\sqrt{(b-\tau)(\tau-a)}\ (t-x)}\left\{\frac{1}{-(\tau-t)} + \frac{1}{(\tau-x)}\right\}\right],\ x \in (0,a)\cup(b,c),$$

where $M_0 = \left(\int_0^a + \int_b^c\right)f(t)\sqrt{t(c-t)}dt$

$$= \frac{-1}{\pi^2\sqrt{x(c-x)}}\left(\int_0^a + \int_b^c\right)\frac{f(t)\sqrt{t(c-t)}}{t-x}dt + \frac{M_0\ \mathrm{Sgn}\ (x-a)}{\pi^2\sqrt{x(c-x)(x-a)(x-b)}}$$

$$- \frac{1}{\pi^3\sqrt{x(c-x)}}\left[\left(\int_0^a - \int_b^c\right)\frac{f(t)\sqrt{t(c-t)(t-a)(t-b)}dt}{t-x}\right.$$

$$\left.\times\left\{\frac{\pi\ \mathrm{Sgn}\ (t-a)}{\sqrt{(t-b)(t-a)}} - \frac{\pi\ \mathrm{Sgn}\ (x-a)}{\sqrt{(x-b)(x-a)}}\right\}\right],\ x \in (0,a)\cup(b,c),$$

(see Equations 69 and 73)

$$= \frac{-1}{\pi^2\sqrt{x(c-x)}}\left(\int_0^a + \int_b^c\right)\frac{f(t)\sqrt{t(c-t)}}{t-x}dt + \frac{M_0\ \mathrm{Sgn}\ (x-a)}{\pi^2\sqrt{x(c-x)(x-a)(x-b)}}$$

$$+ \frac{1}{\pi^2\sqrt{x(c-x)}}\left(\int_0^a + \int_b^c\right)\frac{f(t)\sqrt{t(c-t)}dt}{t-x} + \frac{1}{\pi^2\sqrt{x(c-x)}}$$

$$\times\left[\left(\int_0^a - \int_b^c\right)\frac{f(t)\sqrt{t(c-t)(t-a)(t-b)}dt}{\sqrt{(x-b)(x-a)}\ \ (t-x)}\mathrm{Sgn}\ (x-a)\right],\ x \in (0,a)\cup(b,c),$$

$$= \frac{M_0\ \mathrm{Sgn}\ (x-a)}{\pi^2\sqrt{x(c-x)(x-a)(x-b)}} - \frac{1}{\pi^2\sqrt{x(c-x)}}$$

$$\times\left[\left(\int_0^a + \int_b^c\right)\frac{f(t)\sqrt{t(c-t)(t-a)(t-b)}}{\sqrt{(x-b)(x-a)}\ \ (t-x)}\ \frac{\mathrm{Sgn}(t-a)}{\mathrm{Sgn}\ (x-a)}dt\right],\ x \in (0,a)\cup(b,c). \tag{127}$$

From the Equations 119 and 127, the general solution of the singular integral equation (Equation 103) can be obtained as

$$\Rightarrow \phi(x) = \frac{(N_0 + \widehat{H_0}x)\ \mathrm{Sgn}\ (x-a)}{\sqrt{x(c-x)}\sqrt{(x-b)(x-a)}} - \frac{1}{\pi^2\sqrt{x(c-x)}}\left[\left(\int_0^a + \int_b^c\right) \frac{f(t)\sqrt{t(c-t)(t-a)(t-b)}}{\sqrt{(x-b)(x-a)}\ (t-x)}\ \frac{\mathrm{Sgn}(t-a)}{\mathrm{Sgn}\ (x-a)}dt\right],\ \text{for } x \in (0,a)\cup(b,c), \tag{128}$$

where $N_0 = \widehat{G_0} + \frac{M_0}{\pi^2}$ (note that N_0 and $\widehat{H_0}$ are two arbitrary constants).

Now, we set

$$t = \frac{a(\tau+1)}{1-k},\ x = \frac{a(\xi+1)}{1-k},\ b = \frac{a(1+k)}{1-k},\ c = \frac{2a}{1-k},\ \phi(x) = \psi(\xi),\ f(t) = g(\tau) \tag{129}$$

and get (note that the integral equation (Equation 103) transforms to a new equation (see [13]))

$$\psi(\xi) = \frac{(A + B\xi)\ \mathrm{Sgn}\ (\xi)}{\sqrt{(1-\xi^2)(\xi^2-k^2)}} - \frac{1}{\pi^2}\left[\left(\int_{-1}^{-k} + \int_k^1\right) \frac{g(\tau)\sqrt{(1-\tau^2)(\tau^2-k^2)}}{\sqrt{(1-\xi^2)(\xi^2-k^2)}\ (\tau-\xi)}\ \frac{\mathrm{Sgn}(\tau)}{\mathrm{Sgn}\ (\xi)}d\tau\right], \tag{130}$$

which exactly matches with the result obtained by Tricomi [13], where $A = \frac{N_0(1-k)^2 + \widehat{H_0}a(1-k)}{a^2}$, $B = \frac{H_0}{a}(1-k)$ are two arbitrary constants.

Verification of the solutions for homogeneous Cauchy-type singular integral equation of the first kind

In this section, we verify the solutions obtained in the 'Solution of Cauchy-type singular integral equations of the first kind over an interval with a gap' section for the homogeneous equation involving a Cauchy-type singular integral associated with the disjoint intervals $(0,a)\cup(b,c)$ and $(-1,-k)\cup(k,1)$.

Problem 1. Prove that

$$\phi(t) = \frac{(c_1 + c_2 t)\ \mathrm{Sgn}(t-a)}{\sqrt{t(t-a)(t-b)(c-t)}} \tag{131}$$

is a solution of the homogeneous integral equation

$$\int_0^a \frac{\phi(t)dt}{t-x} + \int_b^c \frac{\phi(t)dt}{t-x} = 0,\ \ x \in (0,a)\cup(b,c). \tag{132}$$

Solution: Substituting Equation 131 in the left-hand side of Equation 132, we get

$$I = -\int_0^a \frac{(c_1 + c_2 t)}{\sqrt{t(a-t)(b-t)(c-t)}}\frac{dt}{t-x} + \int_b^c \frac{(c_1 + c_2 t)}{\sqrt{t(a-t)(b-t)(c-t)}}\frac{dt}{t-x}. \tag{133}$$

For evaluation of the integrals, we will consider two cases: **Case I**: $x \in (0,a)$ and **Case II**: $x \in (b,c)$.

For $t \in (0,a)\cup(b,c)$, take

$$x = \begin{cases} \frac{a}{1-k}\left(1 - \sqrt{k^2\cos^2\phi + \sin^2\phi}\right) & \text{for the case } x \in (0,a), \\ \frac{a}{1-k}\left(1 + \sqrt{k^2\cos^2\phi + \sin^2\phi}\right) & \text{for the case } x \in (b,c), \end{cases}$$

$$b = \frac{(1+k)a}{1-k}, \text{ and } c = \frac{2a}{1-k}.$$

For $x \in (0,a) \cup (b,c)$, take

$$t = \begin{cases} \frac{a}{1-k}\left(1 - \sqrt{k^2\cos^2\theta + \sin^2\theta}\right) & \text{for } t \in (0,a), \\ \frac{a}{1-k}\left(1 + \sqrt{k^2\cos^2\theta + \sin^2\theta}\right) & \text{for } t \in (b,c). \end{cases}$$

Case I: When $x \in (0,a)$

$$I = \frac{(1-k)^2}{a^2}\int_0^{\pi/2} \frac{1}{\sqrt{k^2\cos^2\theta+\sin^2\theta}}\left[\frac{c_1 + \frac{c_2 a}{1-k}\left(1-\sqrt{k^2\cos^2\theta+\sin^2\theta}\right)}{\sqrt{k^2\cos^2\theta+\sin^2\theta}-\sqrt{k^2\cos^2\phi+\sin^2\phi}}\right.$$

$$\left.+\frac{c_1 + \frac{c_2 a}{1-k}\left(1+\sqrt{k^2\cos^2\theta+\sin^2\theta}\right)}{\sqrt{k^2\cos^2\theta+\sin^2\theta}+\sqrt{k^2\cos^2\phi+\sin^2\phi}}\right]d\theta.$$

$$= \frac{2(1-k)^2\left\{c_1 + \frac{c_2 a}{1-k}\left(1-\sqrt{k^2\cos^2\phi+\sin^2\phi}\right)\right\}}{a^2}$$

$$\int_0^{\pi/2}\frac{d\theta}{(k^2\cos^2\theta+\sin^2\theta)-(k^2\cos^2\phi+\sin^2\phi)}$$

$$= \frac{2(1-k)^2\left\{c_1 + \frac{c_2 a}{1-k}\left(1-\sqrt{k^2\cos^2\phi+\sin^2\phi}\right)\right\}}{a^2\{1-(k^2\cos^2\phi+\sin^2\phi)\}}$$

$$\int_0^{\pi/2}\frac{d(\tan\theta)}{\tan^2\theta - \frac{(k^2\cos^2\phi+\sin^2\phi)-k^2}{1-(k^2\cos^2\phi+\sin^2\phi)}} = 0. \tag{134}$$

Case II: When $x \in (b,c)$

$$I = \frac{(1-k)^2}{a^2}\int_0^{\pi/2} \frac{1}{\sqrt{k^2\cos^2\theta+\sin^2\theta}}\left[\frac{c_1 + \frac{c_2 a}{1-k}\left(1-\sqrt{k^2\cos^2\theta+\sin^2\theta}\right)}{\sqrt{k^2\cos^2\theta+\sin^2\theta}+\sqrt{k^2\cos^2\phi+\sin^2\phi}}\right.$$

$$\left.+\frac{c_1 + \frac{c_2 a}{1-k}\left(1+\sqrt{k^2\cos^2\theta+\sin^2\theta}\right)}{\sqrt{k^2\cos^2\theta+\sin^2\theta}-\sqrt{k^2\cos^2\phi+\sin^2\phi}}\right]d\theta.$$

$$= \frac{2(1-k)^2\left\{c_1 + \frac{c_2 a}{1-k}\left(1-\sqrt{k^2\cos^2\phi+\sin^2\phi}\right)\right\}}{a^2}$$

$$\int_0^{\pi/2}\frac{d\theta}{(k^2\cos^2\theta+\sin^2\theta)-(k^2\cos^2\phi+\sin^2\phi)} = 0. \tag{135}$$

Hence, the problem gets solved. ■

Problem 2. Prove that

$$\phi(\tau) = \frac{(A+B\tau)\,\mathrm{Sgn}(\tau)}{\sqrt{(1-\tau^2)(\tau^2-k^2)}} \tag{136}$$

is a solution of the homogeneous integral equation

$$\int_{-1}^{-k}\frac{\phi(\tau)d\tau}{\tau-\xi} + \int_k^1 \frac{\phi(\tau)d\tau}{\tau-\xi} = 0, \;\; \xi \in (-1,-k)\cup(k,1). \tag{137}$$

Solution: Substituting Equation 136 in the left-hand side of equation 137, we get

$$\begin{aligned} I &= -\int_{-1}^{-k} \frac{(A+B\tau)}{\sqrt{(1-\tau^2)(\tau^2-k^2)}} \frac{d\tau}{\tau-\xi} + \int_k^1 \frac{(A+B\tau)}{\sqrt{(1-\tau^2)(\tau^2-k^2)}} \frac{d\tau}{\tau-\xi} \\ &\qquad \text{for } \xi \in (-1,-k) \cup (k,1), \\ &= \int_k^1 \frac{1}{\sqrt{(1-\tau^2)(\tau^2-k^2)}} \left[\frac{(A-B\tau)}{\tau+\xi} + \frac{(A+B\tau)}{\tau-\xi} \right] d\tau \\ &= (A+B\xi) \int_k^1 \frac{2\tau d\tau}{\sqrt{(1-\tau^2)(\tau^2-k^2)}(\tau^2-\xi^2)}. \end{aligned}$$

Putting $\tau^2 = u$ and $\xi^2 = v$, we get

$$I = (A+B\sqrt{v}) \int_{k^2}^1 \frac{du}{\sqrt{(1-u)(u-k^2)}(u-v)}. \tag{138}$$

Now, substituting $u = k^2 \cos^2\theta + \sin^2\theta$, we get

$$\begin{aligned} I &= 2(A+B\sqrt{v}) \int_0^{\pi/2} \frac{d\theta}{k^2\cos^2\theta + \sin^2\theta - v} \\ &= \frac{2(A+B\sqrt{v})}{1-v} \int_0^{\pi/2} \frac{d(\tan\theta)}{\tan^2\theta + \frac{k^2-v}{1-v}} = 0. \end{aligned} \tag{139}$$

This solves the problem. ■

Approximate solution of singular integral equations of the Cauchy type

Here, we find an approximate solution of singular integral equations of the Cauchy type, involving the intervals (0,1), $(-1,-k) \cup (k,1)$, and $(0,a) \cup (b,c)$.

Problem 1. Solve

$$\int_0^1 \frac{\phi(t)dt}{t-x} = f(x), \; x \in (0,1). \tag{140}$$

Solution: Let

$$\phi(t) = \frac{\sum\limits_{n=0}^{\infty} a_n t^n}{\sqrt{t(1-t)}}, \; f(x) = \sum_{n=0}^{\infty} f_n x^n, \tag{141}$$

where $a_n (n = 0,1,2,\ldots)$ and $f_n (n = 0,1,2,\ldots)$ are real constants.

Substituting Equation 141 in Equation 140, we get

$$\begin{aligned} &\sum_{n=0}^{\infty} a_n \left(\int_0^1 \frac{t^n}{\sqrt{t(1-t)}} \frac{dt}{t-x} \right) = \sum_{n=0}^{\infty} f_n x^n, \\ \Rightarrow &\sum_{n=0}^{\infty} a_n \pi \ \text{PP} \left\{ \frac{x^n}{\sqrt{x(x-1)}} \right\}_{x \text{ large}} = \sum_{n=0}^{\infty} f_n x^n \text{ (see [2])}. \end{aligned} \tag{142}$$

For n=0,1,2,3, we obtain

$$\pi \left[a_1 + a_2(x+\frac{1}{2}) + a_3(x^2+\frac{1}{2}x+\frac{3}{8}) \right] = f_0 + f_1 x + f_2 x^2 + f_3 x^3. \tag{143}$$

Equating the coefficients of x^n (n=0,1,2,3) from both sides, we get

$$x^0 \ : \ a_1 + \frac{a_2}{2} + \frac{3a_3}{8} = \frac{f_0}{\pi}, \tag{144a}$$

$$x^1 \ : \ a_2 + \frac{a_3}{2} = \frac{f_1}{\pi}, \tag{144b}$$

$$x^2 \ : \ a_3 = \frac{f_2}{\pi}, \tag{144c}$$

$$x^3 \ : \ f_3 = 0. \tag{144d}$$

Example: Let $f(x) = 1$ (i.e., $f_0 = 1, f_1 = f_2 = 0$).

From Equations 144a to 144d, we obtain

$$a_3 = 0, \ \ a_2 = 0, \ \text{ and } \ a_1 = \frac{f_0}{\pi} = \frac{1}{\pi}. \tag{145}$$

Hence,

$$\phi(t) = \frac{a_0 + \frac{1}{\pi}t}{\sqrt{t(1-t)}}, \tag{146}$$

with a_0 as an arbitrary constant.

A different approach

From Equations 97 and 102, the solution of Equation 140 can be written as

$$\phi(t) = \frac{c_0}{\sqrt{t(1-t)}} - \frac{1}{\pi^2\sqrt{t(1-t)}}\int_0^1 \frac{\sqrt{x(1-x)}f(x)dx}{x-t}, \ \ 0 < t < 1. \tag{147}$$

For $f(x) = 1$,

$$\begin{aligned}\int_0^1 \frac{\sqrt{x(1-x)}}{x-t}dx &= -\pi \,\mathrm{PP}\left\{\sqrt{t(t-1)}\right\}_{t \text{ large}} \ (\text{see [2]}) \\ &= -\pi\left(t - \frac{1}{2}\right).\end{aligned} \tag{148}$$

Substituting Equation 148 in Equation 147, we get

$$\begin{aligned}\phi(t) &= \frac{1}{\sqrt{t(1-t)}}\left[c_0 + \frac{1}{\pi}(t - \frac{1}{2})\right] \\ &= \frac{A_0 + \frac{1}{\pi}t}{\sqrt{t(1-t)}},\end{aligned} \tag{149}$$

which matches exactly with Equation 146, where $A_0 = c_0 - \frac{1}{2\pi}$ is an arbitrary constant.

Problem 2. Find an approximate solution of the integral equation

$$\int_{-1}^{-k} \frac{\phi(\tau)d\tau}{\tau - \xi} + \int_k^1 \frac{\phi(\tau)d\tau}{\tau - \xi} = g(\xi), \ \ \xi \in (-1, -k) \cup (k, 1). \tag{150}$$

Solution: Assume

$$\phi(\tau) = \frac{\sum\limits_{n=0}^{\infty} a_n \tau^n}{\sqrt{(1-\tau^2)(\tau^2 - k^2)}} \,\mathrm{Sgn}(\tau) \tag{151}$$

is a solution of the integral equation (Equation 150) and also assume that

$$g(\xi) = \sum_{n=0}^{\infty} g_n \xi^n, \tag{152}$$

where $a_n (n = 0, 1, 2, \ldots)$ and $g_n (n = 0, 1, 2, \ldots)$ are real constants.

Substituting Equations 151 and 152 in Equation 150, we get

$$\int_k^1 \frac{\sum_{n=0}^{\infty} a_n (-1)^n \tau^n}{\sqrt{(1-\tau^2)(\tau^2-k^2)}} \frac{d\tau}{\tau+\xi} + \int_k^1 \frac{\sum_{n=0}^{\infty} a_n \tau^n}{\sqrt{(1-\tau^2)(\tau^2-k^2)}} \frac{d\tau}{\tau-\xi} = \sum_{n=0}^{\infty} g_n \xi^n$$

$$\Rightarrow \quad \int_k^1 \frac{1}{\sqrt{(1-\tau^2)(\tau^2-k^2)}} \frac{2\tau}{(\tau^2-\xi^2)} \left[\sum_{k=0}^{\infty} a_{2k} \tau^{2k} + \xi \sum_{l=0}^{\infty} a_{2l+1} \tau^{2l} \right] d\tau = \sum_{n=0}^{\infty} g_n \xi^n. \tag{153}$$

Now, consider the left side (LS) of Equation 153 and take k, l=0,1 and $\tau^2 = u, \xi^2 = v$, and obtain (see [2])

$$\begin{aligned} \text{LS} &= \int_k^1 \frac{1}{\sqrt{(1-u)(u-k^2)}(u-v)} (a_0 + a_1 v^{1/2} + a_2 u + a_3 u v^{1/2}) du \\ &= \pi \left[(a_0 + a_1 v^{1/2}) \text{ PP} \left\{ \frac{1}{\sqrt{(v-1)(v-k^2)}} \right\}_{v \text{ large}} \right. \\ &\quad \left. + (a_2 + a_3 v^{1/2}) \text{ PP} \left\{ \frac{v}{\sqrt{(v-1)(v-k^2)}} \right\}_{v \text{ large}} \right] \quad \text{(see [2])} \\ &= \pi \left[a_2 + a_3 v^{1/2} \right] = \pi \left[a_2 + a_3 \xi \right]. \end{aligned} \tag{154}$$

Then, from Equations 153 and 154, we get

$$\pi [a_2 + a_3 \xi] = g_0 + g_1 \xi + g_2 \xi^2 + g_3 \xi^3. \tag{155}$$

Equating the coefficients of $\xi^n (n = 0, 1, 2, 3)$, we obtain

$$\xi^0 \;:\; a_2 = \frac{g_0}{\pi} \tag{156a}$$

$$\xi^1 \;:\; a_3 = \frac{g_1}{\pi} \tag{156b}$$

$$\xi^2 \;:\; g_2 = 0; \quad \xi^3 : \quad g_3 = 0. \tag{156c}$$

In particular, taking $g(\xi) = 1$ (i.e., $g_0 = 1, g_1 = g_2 = g_3 = 0$), we get

$$a_2 = \frac{1}{\pi}, \quad a_3 = 0.$$

Hence,

$$\phi(\tau) = \frac{a_0 + a_1 \tau + \frac{1}{\pi} \tau^2}{\sqrt{(1-\tau^2)(\tau^2-k^2)}} \text{Sgn}(\tau), \tag{157}$$

is the special case g=1, where a_0 and a_1 are two arbitrary constants.

Matching with the closed form solution (Equation 130): Using Equation 130, the solution of Equation 150 can be written as

$$\phi(\tau) = \frac{(A + B\tau) \text{ Sgn } (\tau)}{\sqrt{(1-\tau^2)(\tau^2-k^2)}} - \frac{\text{Sgn } (\tau)}{\pi^2 \sqrt{(1-\tau^2)(\tau^2-k^2)}} \times Q, \tag{158}$$

where

$$Q = \left[\left(\int_{-1}^{-k} + \int_{k}^{1} \right) \frac{g(\xi)\sqrt{(1-\xi^2)(\xi^2-k^2)}}{(\xi-\tau)} \operatorname{Sgn}(\xi) d\xi \right]. \tag{159}$$

Now Q can be written as

$$Q = \int_k^1 \sqrt{(1-\xi^2)(\xi^2-k^2)} \left[\frac{g(-\xi)}{\xi+\tau} + \frac{g(\xi)}{\xi-\tau} \right] d\xi.$$

In particular, taking $g(\xi) = 1$ $(g(-\xi) = 1)$

$$\begin{aligned} Q &= \int_k^1 \sqrt{(1-\xi^2)(\xi^2-k^2)} \frac{2\xi d\xi}{\xi^2-\tau^2} \\ &= \int_{k^2}^1 \sqrt{(1-u)(u-k^2)} \frac{du}{u-v} \text{ (by substituting } u=\xi^2, v=\tau^2) \\ &= -\pi \text{ PP} \left\{ \sqrt{(v-1)(v-k^2)} \right\}_{v \text{ large}} \text{ (see [2])} \\ &= -\pi \left(v - \frac{1+k^2}{2} \right) = -\pi \left(\tau^2 - \frac{1+k^2}{2} \right). \end{aligned} \tag{160}$$

From Equations 158 and 160, we then get

$$\begin{aligned} \phi(\tau) &= \frac{\operatorname{Sgn}(\tau)}{\sqrt{(1-\tau^2)(\tau^2-k^2)}} \left[(A+B\tau) + \frac{1}{\pi} \left(\tau^2 - \frac{1+k^2}{2} \right) \right] \\ &= \frac{\left(A_1 + B\tau + \frac{1}{\pi}\tau^2 \right) \operatorname{Sgn}(\tau)}{\sqrt{(1-\tau^2)(\tau^2-k^2)}}, \end{aligned} \tag{161}$$

which exactly matches with Equation 157, where $A_1 = A - \frac{1+k^2}{2\pi}$.

Note that A_1 and B are two arbitrary constants.

We can also find an approximate solution for the non-homogeneous integral equation (Equation 150) involving the disjoint intervals $(0, a) \cup (b, c)$: Substituting $\tau = \frac{1}{a} t(1-k) - 1$, $b = \frac{a(1+k)}{1-k}$, $c = \frac{2a}{1-k}$, and $\phi(\tau) = \phi(t)$ in Equation 157, we get

$$\begin{aligned} \phi(t) &= \frac{a^2(a_0 - a_1 + 1) + (aa_1 - 2a)(1-k)t + (1-k)^2 t^2}{(1-k)^2 \sqrt{t(t-a)(t-b)(c-t)}} \\ &= \frac{B_1 + B_2 t + t^2}{\sqrt{t(t-a)(t-b)(c-t)}}, \end{aligned} \tag{162}$$

where $B_1 = \frac{a^2}{(1-k)^2}(a_0 - a_1 + 1)$ and $B_2 = \frac{aa_1 - 2a}{1-k}$ are two arbitrary constants.

Solutions of Cauchy-type singular integral equations over semi-infinite and infinite intervals

Here, we consider a Cauchy-type singular integral equation of the first kind in the intervals $(-\infty, b)$, (a, ∞), and $(-\infty, \infty)$ and obtain the solutions by a limiting process applied to the solutions of similar equations associated with finite intervals.

Problem 1. Solve the singular integral equation

$$\int_{-\infty}^{b} \frac{\phi(t)dt}{t-x} = f(x), \quad x \in (-\infty, b). \tag{163}$$

Solution: We know that the general solution of the singular integral equation

$$\int_a^b \frac{\phi(t)dt}{t-x} = f(x), \quad x \in (a,b), \tag{164}$$

is given by

$$\phi(x) = \frac{c_0}{\sqrt{(x-a)(b-x)}} - \frac{1}{\pi^2\sqrt{(x-a)(b-x)}} \int_a^b \frac{f(t)\sqrt{(t-a)(b-t)}}{(t-x)} dt, \tag{165}$$

where c_0 is an arbitrary constant.

Taking limit as $a \to -\infty$, keeping x fixed, we get

$$\phi(x) = \frac{c_1}{\sqrt{b-x}} - \frac{1}{\pi^2} \int_{-\infty}^b \sqrt{\frac{b-t}{b-x}} \frac{f(t)dt}{(t-x)}, \tag{166}$$

which is the general solution of the integral equation (Equation 163), where c_1 is an arbitrary constant as given by $c_1 = \lim_{a\to-\infty} \frac{c_0}{\sqrt{(x-a)}}$.

Problem 2. Solve the singular integral equation

$$\int_a^\infty \frac{\phi(t)dt}{t-x} = f(x), \quad x \in (a,\infty). \tag{167}$$

Solution: Taking limit as $b \to \infty$, keeping x fixed, in Equation 165, we get

$$\phi(x) = \frac{c_2}{\sqrt{x-a}} - \frac{1}{\pi^2} \int_a^\infty \sqrt{\frac{t-a}{x-a}} \frac{f(t)dt}{(t-x)}, \tag{168}$$

which is the general solution of the integral equation (Equation 167), where c_2 is an arbitrary constant as given by $c_2 = \lim\limits_{b\to\infty} \frac{c_0}{\sqrt{b-x}}$.

Problem 3. Solve the singular integral equation

$$\int_{-\infty}^\infty \frac{\phi(t)dt}{t-x} = f(x), \quad x \in (-\infty,\infty). \tag{169}$$

Solution: We know that the general solution of the singular integral equation

$$\int_{-a}^a \frac{\phi(t)dt}{t-x} = f(x), \quad x \in (-a,a), \tag{170}$$

is given by

$$\phi(x) = \frac{c_3}{\sqrt{a^2-x^2}} - \frac{1}{\pi^2\sqrt{a^2-x^2}} \int_{-a}^a \frac{f(t)\sqrt{a^2-t^2}}{t-x} dt, \tag{171}$$

where c_3 is an arbitrary constant.

Taking limit as $a \to \infty$, keeping x fixed, we get

$$\phi(x) = c_4 - \frac{1}{\pi^2} \int_{-\infty}^\infty \frac{f(t)dt}{t-x}, \tag{172}$$

where $c_4 = \lim_{a\to\infty} \frac{c_3}{\sqrt{a^2-x^2}}$ is an arbitrary constant.

Now, for consistency, we must have $c_4 = 0$, since $\phi(x)$ must tend to 0 as $x \to \infty$. Hence, from Equation 172, we get

$$\phi(x) = -\frac{1}{\pi^2} \int_{-\infty}^\infty \frac{f(t)dt}{t-x}, \tag{173}$$

which is the general solution of the integral equation (Equation 169). Equations 169 and 173 exactly match with the results obtained in [14], known as Hilbert's pair of formulae.

Solution of Cauchy-type singular integral equations of the first kind over infinite intervals with a gap

Here, we consider a Cauchy-type singular integral equation of the first kind over two disjoint intervals $(-\infty, b_1) \cup (c_1, d_1)$; $(a_1, b_1) \cup (c_1, \infty)$; and $(-\infty, -k) \cup (k, \infty)$ and obtain its solution by limiting process applied to the known solution for equations involving two disjoint finite intervals.

Problem 1. Solve the singular integral equation of the first kind, involving a semi-infinite interval with a gap, as given by

$$\int_{-\infty}^{b_1} \frac{\psi(t)dt}{t-x} + \int_{c_1}^{d_1} \frac{\psi(t)dt}{t-x} = g(x), \quad x \in (-\infty, b_1) \cup (c_1, d_1). \tag{174}$$

Solution: We know that the solution of the integral equation

$$\int_0^a \frac{\phi(\tau)d\tau}{\tau-\xi} + \int_b^c \frac{\phi(\tau)d\tau}{\tau-\xi} = f(\xi), \quad \xi \in (0,a) \cup (b,c). \tag{175}$$

is given by

$$\begin{aligned} \phi(\xi) \quad = \quad & \frac{(N_0 + H_0\xi)\,\text{Sgn}\,(\xi-a)}{\sqrt{\xi(c-\xi)}\sqrt{(\xi-b)(\xi-a)}} - \frac{1}{\pi^2\sqrt{\xi(c-\xi)}}\left[\left(\int_0^a + \int_b^c\right)\right. \\ & \left.\frac{f(\tau)\sqrt{\tau(c-\tau)(\tau-a)(\tau-b)}}{\sqrt{(\xi-b)(\xi-a)}\ (\tau-\xi)} \frac{\text{Sgn}(\tau-a)}{\text{Sgn}\,(\xi-a)} d\tau\right], \text{ for } \xi \in (0,a) \cup (b,c), \end{aligned} \tag{176}$$

where N_0 and H_0 are two arbitrary constants.

Now, we put

$$\tau = \frac{a(t-a_1)}{b_1-a_1}, \ \ \xi = \frac{a(x-a_1)}{b_1-a_1}, \ b = \frac{a(c_1-a_1)}{b_1-a_1}, \ \ c = \frac{a(d_1-a_1)}{b_1-a_1}, \ \phi(\xi) = \psi(x), \ f(\tau) = g(t) \tag{177}$$

in Equations 175 and 176 and find that the solution of the integral equation

$$\int_{a_1}^{b_1} \frac{\psi(t)dt}{t-x} + \int_{c_1}^{d_1} \frac{\psi(t)dt}{t-x} = g(x), \quad x \in (a_1, b_1) \cup (c_1, d_1) \tag{178}$$

is given by

$$\begin{aligned} \psi(x) \quad = \quad & \frac{(A + Bx)\,\text{Sgn}\,(x-b_1)}{\sqrt{(x-a_1)(d_1-x)(x-b_1)(x-c_1)}} - \frac{1}{\pi^2}\left[\left(\int_{a_1}^{b_1} + \int_{c_1}^{d_1}\right)\right. \\ & \left.\frac{\sqrt{(t-a_1)(d_1-t)(t-b_1)(t-c_1)}}{\sqrt{(x-a_1)(d_1-x)(x-b_1)(x-c_1)}} \frac{\text{Sgn}(t-b_1)}{\text{Sgn}\,(x-b_1)} \frac{g(t)dt}{(t-x)}\right], \\ & \text{for } x \in (a_1, b_1) \cup (c_1, d_1), \end{aligned} \tag{179}$$

where $A = \left[\frac{N_0(b_1-a_1)}{a^2} - \frac{H_0 a_1}{a}\right](b_1 - a_1)$ and $B = \frac{H_0(b_1-a_1)}{a}$ are arbitrary constants.

Then, taking limit as $a_1 \to -\infty$, keeping x fixed, we get

$$\begin{aligned} \psi(x) \quad = \quad & \frac{A_1\,\text{Sgn}\,(x-b_1)}{\sqrt{(d_1-x)(x-b_1)(x-c_1)}} - \frac{1}{\pi^2}\left[\left(\int_{-\infty}^{b_1} + \int_{c_1}^{d_1}\right)\right. \\ & \left.\frac{\sqrt{(d_1-t)(t-b_1)(t-c_1)}}{\sqrt{(d_1-x)(x-b_1)(x-c_1)}} \frac{\text{Sgn}(t-b_1)}{\text{Sgn}\,(x-b_1)} \frac{g(t)dt}{(t-x)}\right], \\ & \text{for } x \in (-\infty, b_1) \cup (c_1, d_1), \end{aligned} \tag{180}$$

which is the general solution of the integral equation (Equation 174), where $A_1 = \lim_{a_1 \to -\infty} \frac{A}{\sqrt{x-a_1}}$ is an arbitrary constant.

Problem 2. Solve the singular integral equation of the first kind, involving a semi-infinite interval with a gap, as given by

$$\int_{a_1}^{b_1} \frac{\psi(t)dt}{t-x} + \int_{c_1}^{\infty} \frac{\psi(t)dt}{t-x} = g(x), \;\; x \in (a_1, b_1) \cup (c_1, \infty). \tag{181}$$

Solution: Taking limit as $d_1 \to \infty$, keeping x fixed, in Equation 179, we obtain

$$\begin{aligned}\psi(x) &= \frac{A_2 \,\mathrm{Sgn}\,(x-b_1)}{\sqrt{(x-a_1)(x-b_1)(x-c_1)}} - \frac{1}{\pi^2}\left[\left(\int_{a_1}^{b_1} + \int_{c_1}^{\infty}\right)\right. \\ &\left.\frac{\sqrt{(t-a_1)(t-b_1)(t-c_1)}}{\sqrt{(x-a_1)(x-b_1)(x-c_1)}}\,\frac{\mathrm{Sgn}(t-b_1)}{\mathrm{Sgn}\,(x-b_1)}\,\frac{g(t)dt}{(t-x)}\right], \\ &\text{for } x \in (a_1, b_1) \cup (c_1, \infty),\end{aligned} \tag{182}$$

which is the general solution of the integral equation (Equation 181), where $A_2 = \lim_{d_1 \to \infty} \frac{A}{\sqrt{d_1 - x}}$ is an arbitrary constant.

Problem 3. Solve the singular integral equation of the first kind, involving an infinite interval with a gap, as given by

$$\int_{-\infty}^{-k} \frac{\psi(t)dt}{t-x} + \int_{k}^{\infty} \frac{\psi(t)dt}{t-x} = g(x), \;\; x \in (-\infty, -k) \cup (k, \infty). \tag{183}$$

Solution: Taking limit as $a_1 \to -\infty$, $d_1 \to \infty$, $b_1 \to -k$, and $c_1 \to k$, keeping x fixed, in Equation 179, we obtain

$$\begin{aligned}\psi(x) = \frac{A_3 \,\mathrm{Sgn}\,(x+k)}{\sqrt{x^2-k^2}} - \frac{1}{\pi^2}\left[\left(\int_{-\infty}^{-k} + \int_{k}^{\infty}\right)\sqrt{\frac{t^2-k^2}{x^2-k^2}}\,\frac{g(t)}{(t-x)}\,\frac{\mathrm{Sgn}(t+k)}{\mathrm{Sgn}\,(x+k)}dt\right], \\ \text{for } x \in (-\infty, -k) \cup (k, \infty),\end{aligned} \tag{184}$$

which is the required solution of the integral equation (183), where $A_3 = \lim_{\substack{a_1 \to -\infty \\ d_1 \to +\infty}} \frac{A}{\sqrt{(x-a_1)(d_1-x)}}$ is an arbitrary constant.

Conclusions

Methods involving evaluation of real improper integrals by the use of the ideas of the theory of functions of real variables only are highlighted, and solutions of certain real singular integral equations of the Cauchy type have been re-derived. Particular examples have been worked out in detail, and the final results have been compared with the known ones.

Methods

The principal methods used in the present work involve application of the theory of functions of real variables only to analyze and solve singular integral equations involving real valued functions everywhere. As a result, the presently developed methods are free from other complicated methods used earlier by various workers to determine the solutions of such integral equations.

Endnotes

Methods of solution of singular integral equations involve, generally speaking, details of complex function theory needing to analyze new types of boundary value problems of the Riemann Hilbert type. In this paper, we have demonstrated, in a systematic manner, that if our concern is to determine solutions of singular integral equations in which the unknown function, the kernel, as well as the forcing term are all functions of real variables, then methods based on the theory of functions of real variables only help in finding out the all wanted real valued solution function. The methods developed in this paper replaces the detailed use of complex function theory and the final forms of the solutions are expected to be useful for direct application to practical problems.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AC and SCM contributed equally to this work. Both authors read and approved the final manuscript.

Authors' information
AC received his M.Sc. and Ph.D. in Applied Mathematics from Calcutta University, Kolkata, India and retired as a professor in the Department of Mathematics, Indian Institute of Science, Bangalore, India. He is currently a NASI-Senior Scientists Platinum Jubilee Fellow in the same institute. His areas of research are diffraction theory, elasticity, fluid mechanics, integral equations, and boundary value problems. SCM received his M.Sc. from Sambalpur University, Odisha, India and Ph.D. from Indian Institute of Technology Guwahati, Guwahati, India in Mathematics. He is an assistant professor in the Department of Mathematics, Indian Institute of Technology Ropar, India. His main areas of interest are mathematical modeling of water wave phenomena, scattering by very large floating structures (VLFS), and integral equation.

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Author details
[1]Department of Mathematics, Indian Institute of Science, Bangalore, 560012, India. [2]Department of Mathematics, Indian Institute of Technology Ropar, Nangal Road, Rupnagar, Punjab, 140001, India.

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