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Iterative reproducing kernel method for a beam equation with third-order nonlinear boundary conditions

Fazhan Geng[1]

Abstract

Purpose: This paper investigates an analytical approximate solution of a fourth-order differential equation with nonlinear boundary conditions modeling beams on elastic foundations using iterative reproducing kernel method.

Methods: The solution obtained using the method takes the form of a convergent series with easily computable components. However, the reproducing kernel method can not be used directly to solve the problems since there is no method of obtaining a reproducing kernel satisfying nonlinear boundary conditions. The aim of this paper is to fill this gap.

Results: Several illustrative examples are given to demonstrate the effectiveness of the present method.

Conclusions: Results obtained using the scheme presented here show that the numerical scheme is very effective and convenient for the beam equation with third-order nonlinear boundary conditions.

Keywords: Iterative reproducing kernel method; Beam equation; Fourth-order boundary value problem; Nonlinear boundary conditions

Background

This paper discusses the analytical approximate solution for fourth-order equations with nonlinear boundary conditions involving third-order derivatives which appears in the study of deformations of elastic beams on elastic bearings:

$$\begin{cases} u^{iv}(x) = h(x, u(x)),\ 0 \le x \le 1, \\ u(0) = 0,\ u'(0) = 0,\ u''(1) = 0,\ u'''(1) = g(u(1)), \end{cases} \tag{1.1}$$

where $h \in C([0, 1] \times \mathbb{R})$ and $g \in C(\mathbb{R})$.

Existence and multiplicity results for this kind of problem were studied recently by Grossinho and coworker [1-3]. However, it is very difficult to obtain its numerical solution due to the appearance of third-order nonlinear boundary conditions. Recently, Ma and Silva [4] proposed an iterative method for solving Equation 1.1.

In this paper, we will apply the iterative reproducing kernel method (IRKM) presented by Geng and Cui [5,6] to the beam equation (Equation 1.1).

Reproducing kernel theory has important application in numerical analysis, differential equation, probability and statistics, and so on [5-17]. Recently, using the RKM, the authors discussed two-point boundary value problems and periodic boundary value problems. For fourth-order equations with nonlinear boundary conditions, however, it can not be applied directly since there is no method of obtaining a reproducing kernel satisfying nonlinear boundary conditions. The aim of this paper is to fill this gap. We will show how IRKM can be used to solve Equation 1.1.

The rest of the paper is organized as follows: An equivalent equation is obtained in the next section. The IRKM is applied to the equivalent equation in the 'IRKM for Equation 2.1' section. The numerical examples are presented in the 'Numerical experiments' section. The 'Conclusions' section ends this paper with a brief conclusion.

Correspondence: gengfazhan@sina.com
[1] Department of Mathematics, Changshu Institute of Technology , Cangshu, Jiangsu , 215500, People's Republic of China

Table 1 Maximum absolute errors in Example 4.1

n	E^N (method in [4]; N=11)	E^N (method in [4]; N=21)	E^N (PM; N=11)	E^N (PM; N=21)
1	1.3243×10^{-2}	1.5197×10^{-2}	1.1852×10^{-1}	1.2798×10^{-1}
2	3.6212×10^{-2}	1.1493×10^{-2}	2.7107×10^{-2}	1.8765×10^{-2}
3	3.3284×10^{-2}	7.9974×10^{-3}	8.9798×10^{-3}	6.7401×10^{-4}
4	3.3653×10^{-2}	8.4501×10^{-3}	1.1391×10^{-2}	3.0623×10^{-3}
5	3.3607×10^{-2}	8.3914×10^{-3}	1.1107×10^{-2}	2.7495×10^{-3}

PM, proposed method.

Results and discussion

Numerical experiments

In this section, two numerical examples are studied to demonstrate the accuracy of the present method. The examples are computed using Mathematica 5.0. Results obtained by the present method are compared with those by the method in [4] and show that the present method is effective for the beam equation (Equation 1.1).

Example 4.1

We consider the problem (Equation 1.1) with

$$h(x,u) = \frac{24}{61}\left(183x^2 - 116x - 2\right),$$
$$g(u) = \frac{24\sin u}{61\sin(48/61)}.$$

The exact solution is given by $u(x) = \frac{x^6}{5} - \frac{116x^5}{305} - \frac{2x^4}{61} + x$. Using the present method, choosing initial approximation $u_0(x) = 0$ and taking $n = 1,2,3,4,5$; $N = 11, 21$; and $x_i = \frac{i-1}{N-1}$, where $i = 1,2,\cdots,N$, the maximum absolute errors $E^N = \sup\limits_{0\le x\le 1} \mid u_n^N(x) - u(x) \mid$ between the approximate solution and the exact solution are given in Table 1.

Example 4.2

We consider the problem (Equation 1.1) with

$$h(x,u) = u^2 - x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48,\ g(u) = 12u.$$

The exact solution is given by $u(x) = x^5 - 2x^4 + 2x^2$. Using the present method, choosing initial approximation $u_0(x) = 0$ and taking $n = 1,2,3,4,5$; $N = 11, 21$; and $x_i = \frac{i-1}{N-1}$, where $i = 1,2,\cdots,N$, the maximum absolute errors $E^N = \sup\limits_{0\le x\le 1} \mid u_n^N(x) - u(x) \mid$ between the approximate solution and the exact solution are given in Table 2.

Conclusions

In this paper, we apply IRKM to fourth-order boundary value problems with nonlinear boundary conditions arising in the study of deformations of elastic beams on elastic bearings and obtain approximate solutions with a high degree of accuracy. Results of numerical experiments show that IRKM is an accurate and reliable analytical technique for this class of fourth-order boundary value problems with a third-order nonlinear boundary condition.

Methods

The equivalent equation of 1.1

Equation 1.1 can not be solved directly using IRKM since it is impossible to obtain a reproducing kernel satisfying nonlinear boundary conditions of Equation 1.1. So, we will make great efforts to convert Equation 1.1 into an equivalent equation, which is easily solved using IRKM.

Integrating both sides of Equation 1.1 from 1 to x and substituting $u'''(1) = g(u(1))$ leads to:

$$\begin{cases} u'''(x) = f(x,u), \ \ 0 \le x \le 1, \\ u(0) = 0,\ u'(0) = 0,\ u''(1) = 0, \end{cases} \tag{2.1}$$

where $f(x,u) = \int_1^x h(s,u)ds + g(u(1))$.

Table 2 Maximum absolute errors in Example 4.2

n	E^N (method in [4]; N=11)	E^N (method in [4]; N=21)	E^N (PM; N=11)	E^N (PM; N=21)
1	1.6881×10^{-2}	4.9903×10^{-3}	2.0228×10^{-2}	1.5271×10^{-2}
2	1.2110×10^{-2}	2.8544×10^{-3}	7.2134×10^{-3}	2.0815×10^{-3}
3	1.3148×10^{-2}	3.3199×10^{-3}	6.8622×10^{-3}	1.7228×10^{-3}
4	1.2922×10^{-2}	3.2181×10^{-3}	6.8527×10^{-3}	1.7130×10^{-3}
5	1.2971×10^{-2}	3.2404×10^{-3}	6.8524×10^{-3}	1.7128×10^{-3}

PM, proposed method.

Obviously, Equations 1.1 and 2.1 are equivalent. Therefore, it suffices for us to solve Equation 2.1.

IRKM for Equation 2.1

Equation 2.1 can be solved using IRKM presented by Geng [5]. In order to apply IRKM, first, we construct a reproducing kernel space $W_2^4[0,1]$ in which every function satisfies the boundary conditions of Equation 2.1.

Reproducing kernel Hilbert space $W_2^4[0,1]$ is defined as $W_2^4[0,1]=\{u(x)\mid u(x),u'(x),u''(x),u'''(x)$ are absolutely continuous real value functions, $u''''(x)\in L^2[0,1]$, $u(0)=0,u'(0)=0,u''(1)=0\}$. The inner product and norm in $W_2^4[0,1]$ are given, respectively, by

$$(u(y),v(y))_{W_2^4}=u(0)v(0)+u(1)v(1)+u'(0)v'(0)+u'(1)v'(1)+\int_0^1 u''''v''''dy$$

and

$$\| u \|_{W^4}=\sqrt{(u,u)_{W^4}},\ u,v\in W^4[0,1].$$

According to [5-7], it is easy to obtain its reproducing kernel

$$k(x,y)=\begin{cases} k_1(x,y),\ y\le x,\\ k_1(y,x),\ y>x, \end{cases} \tag{3.1}$$

where $k_1(x,y)=\frac{1}{330281280}y^2(-3(3x^7-7x^6+3381x^3-10101x^2+21844)y^5+7x(3x^6-7x^5+25225x^2-75633x+65532)y^4-7x^2(1449x^5-25225x^4+327660x^2-4308545x+11307621)y+21x^2(1443x^5-25211x^4+65532x^3-3769207x+11000163))$.

In Equation 2.1, put $Lu(x)=u'''(x)$, it is clear that $L: W_2^4[0,1]\to W_2^1[0,1]$ is a bounded linear operator. Put $\varphi_i(x)=\overline{k}(x_i,x)$ and $\psi_i(x)=L^*\varphi_i(x)$, where $\overline{k}(x_i,x)$ is the RK of $W_2^1[0,1]$ and L^* is the adjoint operator of L. The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^4[0,1]$ can be derived from the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x)=\sum_{k=1}^{i}\beta_{ik}\psi_k(x),(\beta_{ii}>0,i=1,2,\ldots). \tag{3.2}$$

Through the RKM presented in [5-7], we have the following theorems:

Theorem 3.1. *For Equation 2.1, if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^4[0,1]$ and $\psi_i(x)=L_yk_\alpha(x,y)|_{y=x_i}$.*

Theorem 3.2. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0,1]$ and the solution of Equation 2.1 is unique, then the solution of Equation 2.1 satisfies the form*

$$u(x)=L^{-1}f(x,u(x))=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}f(x_k,u(x_k))\overline{\psi}_i(x). \tag{3.3}$$

Remark:
Case (1): Equation 2.1 is linear, that is, $f(x,u(x))=f(x)$. Then, the analytical solution to Equation 2.1 can be obtained directly from Equation 3.3.
Case (2): Equation 2.1 is nonlinear. In this case, the approximate solution to Equation 2.1 can be obtained using the following method.

According to Equation 3.3, we construct the following iteration formula:

$$\begin{cases} u_0(x)=0,\\ u_{n+1}(x)=L^{-1}f(x,u_n(x))\\ \qquad =\sum\limits_{i=1}^{\infty}\sum\limits_{k=1}^{i}\beta_{ik}f(x_k,u_n(x_k))\overline{\psi}_i(x),\\ \qquad n=0,1,\cdots. \end{cases} \tag{3.4}$$

For the proof of convergence of the iterative formula (Equation 3.4), see [5].

Remark: In the iteration process of Equation 3.4, we can guarantee that the approximation $u_n(t)$ always satisfies the boundary conditions of Equation 2.1.

Now, the approximate solution $u_n^N(x)$ can be obtained by finitely taking many terms in the series representation of $u_n(x)$ and

$$u_n^N(x)=\sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik}f(x_k,u_{n-1}(x_k),u'_{n-1}(x_k))\overline{\psi}_i(x). \tag{3.5}$$

Competing interests
The author declare that they have no competing interests.

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