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Lie symmetry analysis of the two-dimensional generalized Kuramoto-Sivashinsky equation

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Abstract

Purpose: In this paper, a detailed analysis of an important nonlinear model system, the two dimensional generalized Kuramoto-Sivashinsky (2D gKS) equation, is presented by group analysis.

Methods: The basic Lie symmetry method is applied in order to determine the general symmetry group of our analyzed nonlinear model.

Results: The symmetry group of the equation and some results related to the algebraic structure of the Lie algebra of symmetries are obtained. Also, a complete classification of the subalgebras of the symmetry algebra is resulted.

Conclusions: It is proved that the Lie algebra of symmetries admits no three dimensional subalgebra. Mainly, all the group invariant solutions and the similarity reduced equations associated to the infinitesimal symmetries are obtained.

Keywords: Two dimensional generalized Kuramoto-Sivanshsky (2D gKS) equation, Lie symmetry method, Invariant solutions, Optimal system, Similarity reduced equations

Background

The idea of studying the differential equations by applying the transformation groups implied a new theory: the symmetry group theory, which is due to Sophus Lie [1]. This method, the so called classical Lie method of infinitesimal transformations, has been applied last year to important partial differential equations (PDEs) which arise from mathematics and physics. Indeed, the symmetry group of a PDEs system can be regarded as the largest (connected) local Lie group of transformations acting on the space of the independent and dependent variables of the system, with an important property of conserving the set of solutions. This group in the Lie's theory is consists of geometric transformations which act on the set of solutions by transforming their graphs. A lot of properties both of the system and their solutions can be implied from the knowledge of the symmetry group. Determining the group invariant solutions, construction of new solutions for the system from the known ones, classification of the group invariant solutions, reduction of the order of ordinary differential equations, detection of linearizing transformations, and mapping solutions to other solutions are the other important applications of Lie groups in the theory of differential equations. For many other applications of Lie symmetries refer to [2-5].

In this paper, we analyze the problem of symmetries of the two-dimensional generalized Kuramoto-Sivashinsky (2D gKS) equation:

$$\frac{\partial H}{\partial t}-c\frac{\partial H}{\partial x}+4H\frac{\partial H}{\partial x}+\frac{\partial^2 H}{\partial x^2}+\delta\frac{\partial}{\partial x}\nabla^2 H+\nabla^4 H=0. \quad (1)$$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}$ is the Laplacian in two dimensions and already transformed with $x \longrightarrow x-ct$ to a coordinate system with constant speed c, x, and z are streamwise and spanwise surface coordinates, respectively, and $H = H(x,z,t)$ denotes the local film thickness [6].

In the work of Toh et al. [7], the original equation was presented as a model system to study the pattern formation in nonlinear systems with dispersion and dissipation. More generally, in the strongly dispersive limit, Equation (1) describes a variety of physical phenomena that involve localized structures in two dimensions including Rossby waves, solitary vortices in plasma, magmons in magma segregation in earth's mantle, and localized rolls in nematic crystals (see [6] and the references therein). In the context of thin liquid films, the 2D gKS

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equation has been derived by Frenkel and Indireshkumar [8] for a film falling down a vertical plane assuming strong face tension and near critical conditions. However, for the same problem, different limits of Equation (1) were obtained in [9-11].

The structure of the present paper is as follows: In section 2, using the basic Lie symmetry method, the most general Lie point symmetry group of the 2D gKS equation is determined. In section 3, some results obtained from the algebraic structure of the Lie algebra of symmetries are given. Section 4 is devoted to obtaining the one-parameter subgroups and the most general group-invariant solutions of 2D gKS equation. In section 5, the classification of the subalgebras of the 2D gKS equation symmetry Lie algebra is presented. In section 6, the Lie invariants and the similarity solutions of the analyzed model are computed, and its reduced form corresponding to the infinitesimal symmetries are determined. Some concluding remarks are mentioned at the end of the paper.

Lie symmetries of the 2D gKS equation

In this section, we will perform the Lie group method for Equation (1). Firstly, let us consider a one-parameter Lie group of infinitesimal transformation:

$$\begin{aligned}
\bar{x} &= x + \varepsilon\xi^1(x,z,t,H) + O(\varepsilon^2),\\
\bar{z} &= t + \varepsilon\xi^2(x,z,t,H) + O(\varepsilon^2),\\
\bar{t} &= t + \varepsilon\xi^3(x,z,t,H) + O(\varepsilon^2),\\
\bar{H} &= H + \varepsilon\eta(x,z,t,H) + O(\varepsilon^2),
\end{aligned}$$

with a small parameter $\varepsilon << 1$. The symmetry generator associated with the above group of transformations can be written as:

$$\begin{aligned}
\mathbf{V} = \xi^1(x,z,t,H)\frac{\partial}{\partial x} + \xi^2(x,z,t,H)\frac{\partial}{\partial z}\\
+ \xi^3(x,z,t,H)\frac{\partial}{\partial t} + \eta(x,z,t,H)\frac{\partial}{\partial H}.
\end{aligned} \tag{2}$$

The fourth prolongation of V is the vector field

$$\begin{aligned}
\mathbf{V}^{(4)} = \mathbf{V} &+ \eta^x\frac{\partial}{\partial H_x} + \eta^z\frac{\partial}{\partial H_z} + \eta^t\frac{\partial}{\partial H_t} + \eta^{2x}\frac{\partial}{\partial H_{2x}}\\
&+ \eta^{xz}\frac{\partial}{\partial H_{xz}} + \eta^{xt}\frac{\partial}{\partial H_{xt}} + \eta^{2z}\frac{\partial}{\partial H_{2z}}\\
&+ \eta^{zt}\frac{\partial}{\partial H_{zt}} + \eta^{2t}\frac{\partial}{\partial H_{2t}} + \eta^{3x}\frac{\partial}{\partial H_{3x}} + \ldots\\
&+ \eta^{4x}\frac{\partial}{\partial H_{4x}} + \eta^{4z}\frac{\partial}{\partial H_{4z}} + \ldots
\end{aligned} \tag{3}$$

with coefficients

$$\eta^J = D_J(\eta - \sum_{i=1}^{3}\xi^i H_i^\alpha) + \sum_{i=1}^{3}\xi^i H_{J,i}\,, \tag{4}$$

where $J = (i_1, \ldots, i_k)$, $1 \le i_k \le 3$, $1 \le k \le 4$, and the sum is over all $J's$ of order $0 < \#J \le 4$.

By theorem (6.5) in the work of Olver [12], the invariance condition for the 2D gKS equation is given by the relation:

$$\begin{aligned}
\mathbf{V}^{(4)}[H_t - cH_x + 4HH_x + H_{xx} + \delta(H_{3x} + H_{(2x)z})\\
+ H_{4x} + 2H_{(2x)(2z)} + H_{4z}] = 0
\end{aligned} \tag{5}$$

The invariance condition in Equation (5) is equivalent with the following equation:

$$\begin{aligned}
&\eta^t - c\eta^x + 4\eta\eta^x + \eta^{xx} + \delta(\eta^{3x} + \eta^{(2x)z})\\
&+ \eta^{4x} + 2\eta^{(2x)(2z)} + \eta^{4z} = 0
\end{aligned} \tag{6}$$

In substituting Equation (4) into invariance condition in Equation (6), we are left with a polynomial equation involving the various derivatives of $H(x, z, and\ t)$ whose coefficients are certain derivatives of ξ^1, ξ^2, ξ^3, and η. Since ξ^1, ξ^2, ξ^3, and η depend only on x, z, t, and H, we can equate the individual coefficients to zero, leading to the complete set of determining equations:

$$\begin{aligned}
&\xi^1_H = 0, \quad \xi^1_x = 0, \quad \xi^1_z = 0, \quad \xi^1_{2t} = 0\\
&\xi^2_t = 0, \quad \xi^2_H = 0, \quad \xi^2_x = 0, \quad \xi^2_y = 0\\
&\xi^3_t = 0, \quad \xi^3_H = 0, \quad \xi^3_x = 0, \quad \xi^3_y = 0, \quad \eta - \frac{1}{4}\xi^1_t = 0.
\end{aligned}$$

By solving this system of PDEs, we find that:

Theorem 2.1. *The Lie group of point symmetries of the 2D gKS equation has a Lie algebra generated by the vector fields* $\mathbf{V} = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial z} + \xi^3\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial H}$, *where*

$$\begin{aligned}
\xi^1(x,z,t,H) &= c_1 t + c_2, & \xi^2(x,z,t,H) &= c_4,\\
\xi^3(x,z,t,H) &= c_3, & \eta(x,z,t,H) &= \frac{1}{4}c_1.
\end{aligned}$$

and $c_i,\ i = 1, \ldots, 4$ *are arbitrary constants.*

Corollary 2.2. *The infinitesimal generators of every one-parameter Lie group of point symmetries of the 2D gKS equation are:*

$$\mathbf{V}_1 = \frac{\partial}{\partial x}, \quad \mathbf{V}_2 = \frac{\partial}{\partial z}, \quad \mathbf{V}_3 = \frac{\partial}{\partial t}, \quad \mathbf{V}_4 = t\frac{\partial}{\partial x} + \frac{1}{4}\frac{\partial}{\partial H}.$$

The commutator table of symmetry generators of the 2D gKS equation is given in Table 1, where the entry in the i^{th} *row and* j^{th} *column is defined as* $[V_i, V_j] = V_iV_j - V_jV_i$, $i, j = 1, \ldots, 4$.

The structure of the Lie algebra of symmetries

In this part, we determine the structure of symmetry Lie algebra of the 2D gKS equation. The g has no non-trivial *Levi decomposition* in the form $g = \mathrm{r} \times \mathrm{g}_1$, because g has no any non-trivial radical, i.e., if r be the radical of g, then $\mathrm{g} = \mathrm{r}$. The Lie algebra g is solvable and non-semisimple.

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Table 1 Commutation relations satisfied by infinitesimal generators

[,]	V_1	V_2	V_3	V_4
V_1	0	0	0	0
V_2	0	0	0	0
V_3	0	0	0	V_1
V_4	0	0	$-V_1$	0

It is solvable, because if $g^{(1)} =< \mathrm{V_i}, [\mathrm{V_i}, \mathrm{V_j}] >=[g,g]$, we have:

$$g^{(1)} =[g,g] =< \mathrm{V}_1 >,$$

and

$$g^{(2)} =[g^{(1)}, g^{(1)}] = 0.$$

Thus, we have the following chain of ideals $g^{(1)} \supset g^{(2)} = \{0\}$. Also, g is not semisimple because its killing form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is degenerated. Taking into account the table of commutators, g has two abelians, and four and two dimensional subalgebras which are spanned by $< V_1, V_2, V_3 >$ and $< V_4 >$, respectively, such that the first one is an ideal in g.

Symmetry transformations and group invariant solutions of 2D gKS equation

The Equation (1) can be regarded as a submanifold of the jet space $J^4(R^3, R)$. Thus, in order to obtain the group transformation which is generated by the infinitesimal generators $\mathbf{X_i} = \xi_i^1 \partial_x + \xi_i^2 \partial_z + \xi_i^3 \partial_t + \eta_i \partial_H$ for $i = 1, ..., 4$, it is necessary to solve the following four systems of differential equations:

$$\frac{d\bar{x}(\varepsilon)}{d\varepsilon} = \xi_i^1(\bar{x}(\varepsilon), \bar{z}(\varepsilon), \bar{t}(\varepsilon), \bar{H}(\varepsilon)), \quad \bar{x}(0) = x,$$

$$\frac{d\bar{z}(\varepsilon)}{d\varepsilon} = \xi_i^2(\bar{x}(\varepsilon), \bar{z}(\varepsilon), \bar{t}(\varepsilon), \bar{H}(\varepsilon)), \quad \bar{z}(0) = z, \tag{7}$$

$$\frac{d\bar{t}(\varepsilon)}{d\varepsilon} = \xi_i^3(\bar{x}(\varepsilon), \bar{z}(\varepsilon), \bar{t}(\varepsilon), \bar{H}(\varepsilon)), \quad \bar{t}(0) = t, \tag{8}$$

$$\frac{d\bar{H}(\varepsilon)}{d\varepsilon} = \eta_i(\bar{x}(\varepsilon), \bar{z}(\varepsilon), \bar{t}(\varepsilon), \bar{H}(\varepsilon)), \quad \bar{H}(0) = H, \ i = 1, ..., 7.$$

Exponentiating the infinitesimal symmetries of Equation (1), the one-parameter groups $G_k(\varepsilon)$ generated by V_k for $k = 1, ..., 4$, are determined as follows.

Theorem 4.1. *The one-parameter groups $G_i(\varepsilon)$ generated by the V_i, $i = 1, ..., 4$, are given in the following:*

$$\begin{aligned} G_1(\varepsilon) &: (x,z,t,H) \longmapsto (x+\varepsilon, z, t, H), \\ G_2(\varepsilon) &: (x,z,t,H) \longmapsto (x, z+\varepsilon, t, H), \\ G_3(\varepsilon) &: (x,z,t,H) \longmapsto (x, z, t+\varepsilon, H), \\ G_4(\varepsilon) &: (x,z,t,H) \longmapsto (x+\varepsilon t, z, t, H + \frac{1}{4}\varepsilon). \end{aligned}$$

where entries give the transformed point $\exp(\varepsilon V_i)(x, z, t, H) = (\bar{x}, \bar{z}, \bar{t}, \bar{H})$.

Recall that generally, to each one-parameter subgroups of the full symmetry group of a system, there will correspond a family of solutions called invariant solutions. Consequently, we can state the following theorem:

Theorem 4.2. *If $H = f(x,z,t)$ is a solution of Equation (1), so are the functions*

$$H^1 = G_1(\varepsilon).f(x,z,t) = f(x+\varepsilon, z, t), \tag{9}$$

$$\begin{aligned} H^2 &= G_2(\varepsilon).f(x,z,t) = f(x, z+\varepsilon, t), \\ H^3 &= G_3(\varepsilon).f(x,z,t) = f(x, z, t+\varepsilon), \\ H^4 &= G_4(\varepsilon).f(x,z,t) = f(x+\varepsilon t, z, t) - \frac{1}{4}\varepsilon. \end{aligned}$$

Now, the general group of symmetries can be obtained by considering a general linear combination $\kappa_1 V_1 + \cdots + \kappa_7 V_4$ of the given vector fields. Particularly, if G is the action of the symmetry group near identity, it can be represented in the form $G = \exp(\varepsilon_4 V_4) o \cdots o \exp(\varepsilon_1 V_1)$. Consequently, from the above theorem it is deduced that:

Corollary 4.3. *For the arbitrary combination of infinitesimal symmetry generators of the form $V = \sum_{i=1}^4 V_i \in$ g, the 2D gKS equation has the following solution*

$$H = f\,(x + \varepsilon_4 t + \varepsilon_1, z + \varepsilon_2, t + \varepsilon_3) - \frac{1}{4}\varepsilon_4.$$

where ε_i, $i = 1 \cdots 4$ are arbitrary real numbers.

Classification of subalgebras for the 2D gKS equation

The main motivation for computing the symmetries of a differential equation is the search for the so called invariant solutions. It is well known that the problem of classifying invariant solutions is equivalent to the problem of classifying the subgroups of the full symmetry group under conjugation. Let H and $\tilde{H}$ be connected, $s-$dimensional Lie subgroups of the Lie group G with corresponding Lie subalgebras h and $\tilde{h}$ of the Lie algebra g of G. Then $\tilde{H} = gHg^{-1}$ are conjugate subgroups if and only $\tilde{h} = \mathrm{Ad}(g) \cdot \mathrm{h}$ are conjugate subalgebras. Thus, the problem

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of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras, and thus, we concentrate on it [5,13]. The latter problem tends to determine a list of conjugacy inequivalent subalgebras with the property that any other subalgebra is equivalent to a unique member of the list under some element of the adjoint representation for some element of a considered Lie group.

Optimal system of one-dimensional subalgebras of the 2D gKS equation

In fact, for one-dimensional subalgebras, the classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. If we take only one representative from each family of equivalent subalgebras, an optimal set of subalgebras is created. The corresponding set of invariant solutions is then the minimal list from which we can get all other invariant solutions of one-dimensional subalgebras simply via transformations.

Each V_i, $i = 1, ..., 4$, of the basis symmetries generates an adjoint representation (or interior automorphism) $\mathrm{Ad}(\exp(\varepsilon V_i))$ defined by the Lie series

$$\mathrm{Ad}(\exp(\varepsilon.V_i).V_j) = V_j - \varepsilon.[\,V_i, V_j] + \frac{\varepsilon^2}{2}.[\,V_i, [\,V_i, V_j]\,] - \cdots \tag{10}$$

where $[\,V_i, V_j]$ is the commutator for the Lie algebra, ε is a parameter, and $i, j = 1, \cdots, 4$ ([5]). In Table 2, all the adjoints are representations of the 2D gKS Lie group, with the (i, j) entry indicating $\mathrm{Ad}(\exp(\varepsilon V_i))V_j$. Therefore, we can state the following theorem:

Theorem 5.1. *An optimal system of one-dimensional subalgebras of the 2D gKS equation Lie algebra g is given by*

$$(1): \; V^1 := V_1 + aV_2 = \frac{\partial}{\partial x} + a\frac{\partial}{\partial z},$$

$$(2): \; V^2 := aV_2 + bV_3 + cV_4 = ct\frac{\partial}{\partial x} + a\frac{\partial}{\partial z} + b\frac{\partial}{\partial t} + c\frac{1}{4}\frac{\partial}{\partial H}.$$

where $a, b, c \in \mathrm{R}$.

Table 2 Adjoint representation generated by the basis symmetries of the 2D gKS Lie algebra

Ad	V_1	V_2	V_3	V_4
V_1	V_1	V_2	V_3	V_4
V_2	V_1	V_2	V_3	V_4
V_3	V_1	V_2	V_3	$V_4 - \varepsilon V_1$
V_4	V_1	V_2	$V_3 + \varepsilon V_1$	V_4

Proof. $F_i^s : \mathrm{g} \to \mathrm{g}$ defined by $V \mapsto \mathrm{Ad}(\exp(s_i V_i).V)$ is a linear map, for $i = 1, \cdots, 4$. The matrix M_i^s of F_i^s with respect to basis $\{V_1, \cdots, V_7\}$ is,

$$M_1^s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M_2^s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$M_3^s = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s & 0 & 0 & 1 \end{pmatrix} M_4^s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ s & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

respectively. Let $V = \sum_{i=1}^{4} a_i V_i$, then

$$F_4^{s_4} \circ F_3^{s_3} \circ \cdots \circ F_1^{s_1} \; : \; V \; \mapsto a_1V_1 + a_2V_2 + (a_3 + s_4a_1)V_3 + (a_4 - s_3a_1)V_4. \tag{12}$$

Now, we can simplify V as follows:

Case (1): If $a_1 \neq 0$, then we act on V by $\mathrm{Ad}(\exp(-\frac{a_3}{a_1})V_4))$, and hence, we can make the coefficient of V_3 vanish. Then, we tend to the new form

$$V' = a_1V_1 + a_2V_2 + a_4V_4.$$

By acting $\mathrm{Ad}(\exp(\frac{a_4}{a_1})V_3))$ on V', we can make the coefficient of V_4 vanish, so we obtain:

$$V'' = a_1V_1 + a_2V_2.$$

At this stage, by acting adjoint representations $M_i^s(a_i)$ on V'', we find that no more simplification of V'' is possible. Thus, each of a_1 and a_2 are arbitrary. By scaling if necessary, we can assume that $a_1 = 1$. This assumption suggests Case (1).

Case (2): If $a_1 = 0$, no further simplification is possible, and then V is reduced to Case (2).

There are no more possible cases for investigation, and the proof is complete. ⋄

Two-dimensional optimal system

The next step is constructing the two-dimensional optimal system, i.e., the classification of two-dimensional subalgebras of g. This process is performed by selecting one of the vector fields as stated in theorem (5.1). Let us

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consider V^1 (or V^2). Corresponding to it, a vector field $V = a_1V_1 + \cdots + a_4V_4$, where a_i's are smooth functions of $(x, z, t, and\ H)$ is chosen, so we must have:

$$[V^1, V] = \lambda \mathbf{V}^1 + \mu V. \tag{13}$$

Equation (13) leads us to the system

$$C^i_{jk}\alpha_j a_k = \lambda a_i + \mu\alpha_i \qquad (i = 1, \cdots, 7). \tag{14}$$

The solutions of Equation (14) give one of the two-dimensional generator, and the second generator is V^1 ors V^2 if selected. After the construction of all two-dimensional subalgebras, for every vector fields of theorem 6, they need to be simplified by the action of adjoint matrices (5.11) in the manner analogous to the way of one-dimensional optimal system. Hence, we can state the following theorem:

Theorem 5.2. *An optimal system of two-dimensional Lie algebra of 2D gKS equation is provided by those generated by*

$$\begin{aligned} &(\mathbf{1}) : < \beta_1V_1 + \beta_2V_2, \beta_3V_3 + \beta_4V_4 > \\ &(\mathbf{2}) : < V_1, \beta_2V_2 + \beta_3V_3 > . \end{aligned} \tag{15}$$

where β_i , $i = 1, ..., 4$, are arbitrary real numbers, and all of these subalgebras are abelian.

Three-dimensional optimal system

This system can be developed by the method of the expansion of the two-dimensional optimal system. For this, we take any of the two-dimensional subalgebras in Equation (15); let us consider the first two vector fields of Equation (15) and call them Y_1 and Y_2; thus, we have a subalgebra with basis $\{Y_1, Y_2\}$; and we should find a vector field $Y = a_1\mathbf{V}_1 + \cdots + a_4\mathbf{V}_4$, where a_i's are smooth functions of $(x, z, t, and\ H)$, such that the triple $\{Y_1, Y_2, and\ Y\}$ generates a basis of a three-dimensional algebra. For this purpose, it is necessary and sufficient that the vector field Y satisfies the following equations:

$$\begin{aligned} &[Y_1, Y] = \lambda_1Y + \mu_1Y_1 + \nu_1Y_2, \\ &[Y_2, Y] = \lambda_2Y + \mu_2Y_1 + \nu_2Y_2, \end{aligned} \tag{16}$$

and following from Equation (16), we obtain the system

$$\begin{aligned} &C^i_{jk}\beta^j_r a_k = \lambda_1 a_i + \mu_1\beta^i_r + \nu_1\beta^i_s, \\ &C^i_{jk}\beta^j_s a_k = \lambda_2 a_i + \mu_2\beta^i_r + \nu_2\beta^i_s. \end{aligned} \tag{17}$$

The solutions of Equation (17) is linearly independent of $\{Y_1, Y_2\}$ and give a three-dimensional subalgebra. This process is used for another two couples of vector fields in Equation (15).

Assume that $\tilde{g} = \mathrm{Span}_{\mathbb{R}}\{Y_1, Y_2, Y\}$ be a three-dimensional Lie subalgebra of g, by performing the above procedure for all two couples of vector fields in Equation (15); hence, it is concluded that $Y = \beta_1Y_1 + \beta_2Y_2$. By a suitable change of the bases for $\tilde{g}$, we can assume that $Y = 0$ so that $\tilde{g}$ is not a three-dimensional subalgebra. Thus, we infer that:

Corollary 5.3. *The 2D gKS equation Lie algebra* g *admits no three-dimensional Lie subalgebra.*

Similarity reduction of (2D)(KS) equation

The 2D gKS Equation (1) is expressed in the coordinates $(x, z, t, and\ H)$, so we should search for the form of this equation in specific coordinates in order to reduce it. The coordinates will be constructed by looking for independent invariants $(p, q, and\ r)$ corresponding to the infinitesimal symmetry generators. Hence, by applying the chain rule, the expression of the equation in the new coordinate leads to the reduced equation. We can now compute the invariants associated with the symmetry operators. They can be obtained by integrating the characteristic equations. For example for the operator, $U_9 := V_2 + V_3 + V_4 = t\partial_x + \partial_z + \partial_t + \frac{1}{4}H$,
this means:

$$\frac{dx}{t} = \frac{dz}{1} = \frac{dt}{1} = \frac{4\,dH}{1} \tag{18}$$

The corresponding invariants are as follows:

$$p = t^2 - 2x, \qquad q = z - t, \qquad r = H - \frac{1}{4t}. \tag{19}$$

Taking into account the last invariant, we assume a similarity solution of the form:

$$H = f(p, q) + \frac{1}{4t}. \tag{20}$$

and we substitute it into (1) to determine the form of the function $f(p, q)$: We obtain that $f(p, q)$ has to be a solution of the following differential equation:

$$\begin{aligned} -4f_q + 8cf_p - 32ff_p + 16f_{pp} - 32\delta f_{ppp} - 8\delta f_{pqq} \\ + 64f_{pppp} + 32f_{ppqq} + 4f_{qqqq} + 1 = 0 \end{aligned} \tag{21}$$

Having determined the infinitesimals, the Lie invariants and similarity solutions p_j, q_j, r_j, and H_j are listed in

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Table 3 Lie invariants and similarity solutions

J	$\mathbf{U_j}$	$\mathbf{p_j}$	$\mathbf{q_j}$	$\mathbf{r_j}$	$\mathbf{H_j}$
1	$\mathbf{V}_1$	z	t	H	$f(p,q)$
2	$\mathbf{V}_2$	x	t	H	$f(p,q)$
3	$\mathbf{V}_3$	x	z	H	$f(p,q)$
4	$\mathbf{V}_4$	z	t	$H-\frac{x}{4t}$	$f(p,q)+\frac{x}{4t}$
5	$\mathbf{V}_1+\mathbf{V}_2$	$z-x$	t	H	$f(p,q)$
6	$\mathbf{V}_2+\mathbf{V}_3$	x	$t-z$	H	$f(p,q)$
7	$\mathbf{V}_2+\mathbf{V}_4$	t	$z-\frac{x}{t}$	$H-\frac{x}{4t}$	$f(p,q)+\frac{x}{4t}$
8	$\mathbf{V}_3+\mathbf{V}_4$	z	t^2-2x	$H-\frac{1}{4t}$	$f(p,q)+\frac{1}{4t}$
9	$\mathbf{V}_2+\mathbf{V}_3+\mathbf{V}_4$	t^2-2x	$z-t$	$H-\frac{1}{4t}$	$f(p,q)+\frac{1}{4t}$

Table 4 Reduced equations corresponding to infinitesimal symmetries

J	Similarity Reduced Equations
1	$f_q+f_{pppp}=0$
2	$f_q-cf_p+4ff_p+f_{pp}+\delta f_{ppp}+f_{pppp}=0$
3	$f_{pppp}+f_{qqqq}+2f_{ppqq}+\delta f_{ppp}+\delta f_{pqq}$ $+f_{pp}+4ff_p-cf_p=0$
4	$4qf_{pppp}+4qf_q+4f-c=0$
5	$4f_{pppp}-2\delta f_{ppp}+f_{pp}-4ff_p+cf_p+f_q=0$
6	$f_{pppp}+2f_{ppqq}+f_{qqqq}+\delta f_{ppp}+\delta f_{pqq}$ $+f_{pp}+4ff_p-cf-p+f_q=0$
7	$4p^4f_p+4cp^3f_q-cp^3-16p^3ff_q+4fp^3+4p^2f_{qq}$ $-4\delta pf_{qqq}-4\delta p^3f_{qqq}+4f_{qqqq}+8p^2f_{qqqq}+4p^4f_{qqqq}=0$
8	$8cf_q-32ff_q+16f_{qq}-32\delta f_{qqq}-2\delta f_{ppq}$ $+16f_{qqqq}+8f_{ppqq}+4f_{pppp}+1=0$
9	$-4f_q+8cf_p-32ff_p+16f_{pp}-32\delta f_{ppp}-8\delta f_{pqq}$ $+64f_{pppp}+32f_{ppqq}+4f_{qqqq}+1=0$

Table 3. The similarity reduced forms of the 2D gKS equation associated to symmetry generators are listed in Table 4.

Results and Discussion

In this paper, following the classical Lie method, the preliminary group classification and the algebraic structure of the symmetry group for the 2D gKS equation are obtained. The classification is deduced by constructing an optimal system with the aid of Theorems 5.1 and 5.2. The result of the work is summarized in Table 3. The corresponding reduced equations are presented in Table 4.

Conclusions

In this paper, the criterion of invariance of the equation under the infinitesimal prolonged infinitesimal generators is applied in order to determine the most general Lie point symmetry group of a well-known nonlinear dynamical system: 2D gKS equation. The algebraic structure of g, the Lie algbera of symmetries of the analyzed model is discussed and it is proved that g is a solvable, non-semisimple algebra. The one-parameter groups and the symmetry transformations associated to symmetry generators are obtained. Also, a complete classification of the subalgebras of g is presented, and it is shown that g has no three-dimensional subalgebra. Mainly, the Lie invariants and similarity reduced equations of 2D gKS equation corresponding to infinitesimal symmetries are obtained.

Methods

In this paper, the basic Lie symmetry method is performed for the comprehensive analysis of the 2D gKS equation.

Competing interests
The authors declare that they have no competing interests.

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Authors' contributions
MN conceived of the study, and participated in its design and coordination. FA carried out the related physical studies and numerical computations, participated in the design and coordination of the study and drafted the manuscript. All authors read and approved the final manuscript.

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