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Convergence in probability and almost surely convergence in probabilistic normed spaces

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Abstract

Our purpose in this paper is researching about characteristics of convergent in probability and almost surely convergent in *Šerstnev* space. We prove that if two sequences of random variables are convergent in probability (almost surely), then, sum, product and scalar product of them are also convergent in probability (almost surely). Meanwhile, we will prove that each continuous function of every sequence convergent in probability sequence is convergent in probability too. Finally, we represent that for independent random variables, every almost surely convergent sequence is convergent in probability. In this paper, we conclude results in *Šerstnev* space are similar to probability space.

Keywords: Probabilistic normed space, Convergence in probability, Almost surely convergence, *Šerstnev* space

Background

Menger introduced probabilistic metric space in 1942 [1]. The notion of probabilistic normed space was introduced by *Šerstnev* [2]. Alsina et al. generalized the definition of probabilistic normed space [3,4]. Lafuerza-Guillén and Sempi for probabilistic norms of probabilistic normed space induced the convergence in probability and almost surely convergence [5].

The structure of paper is as follows: the 'Preliminaries' section recalls some notions and known results in probabilistic metric space, probabilistic normed space and special case of probabilistic normed space which is called Šerstnev space. In the 'Main results' section, we prove some theorems which show convergence in probability for sum, product, scaler product and division of two sequences in Šerstnev space. We prove whether every continuous function of a sequence converges if it converges in probability. Also, we prove almost surely convergence for sum, product and scaler product in Šerstnev space. At the end, the relationship between almost surely convergence and convergence in probability is proved in *Šerstnev* space.

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Preliminaries

We recall some concepts from probabilistic metric space [6,7] and convergence concept. For more details, we refer the reader to Chung's study [8]. A distance distribution function is a mapping $F : [0,\infty] \to [0,1]$ which is nondecreasing, left continuous on $(0,\infty)$ and $F(0) = 0$. The class of all distribution functions is denoted by Δ_+. D_+ is the subset of Δ_+ containing all functions F which satisfy the condition $\lim_{t\to\infty} F(t) = 1$. If S is a nonempty set, a mapping F from $S \times S$ to Δ_+ is called a probabilistic distance on S, and $F(x, y)$ is denoted by F_{xy}. The function ϵ_0 is defined 0 if $t = 0$ and 1 if $t > 0$.

A triangular norm (shorter t-norm) is a binary operation on the unit interval $[0, 1]$ which the following conditions are satisfied [9]:

1) $T(a,1) = a$ for every $a \in [0,1]$,
2) $T(a,b) = T(b,a)$ for every $a,b \in [0,1]$,
3) $a \geq b,\ c \geq d \Rightarrow T(a,c) \geq T(b,d)\ \ a,b,c,d \in [0,1]$,
4) $T(T(a,b),c) = T(a,T(b,c))$.

Basic examples are t-norms T_L (Lukasiewicz t-norm), T_P and T_M, defined by $T_L(a,b) = \max\{a+b-1, 0\}$, $T_P(a,b) = ab$ and $T_M(a,b) = \min\{a,b\}$.

Definition 2.1. *A generalized Menger space is a triple (S, F, T) where T is a t-norm, satisfying the following conditions:*

1) $F_{xy} = \epsilon_0 \Leftrightarrow x = y$,

2) $\forall\, x, y \in S,\ F_{xy} = F_{yx}$,
3) $\forall\, x, y, z \in S,\ \forall\, s, t \geq 0,\ F_{xz}(s+t) \geq T(F_{xy}(s), F_{yz}(t))$.

Definition 2.2. *A triangle function is a binary operation on Δ_+ that is commutative, associative, nondecreasing in place and has ϵ_0 as identity.*

For more details about this subject, we refer the reader to Hadzic and Pap's study [10].

Definition 2.3. *A probabilistic normed space is a quadruple $(V, \Theta, \tau, \tau^*)$, where V is a real vector space, τ and τ^* are continuous triangle functions, and Θ is a mapping from V into Δ_+ such that for every p, $q \in V$ the following conditions hold:*

1) $\Theta_p = \epsilon_0$ iff $p = 0$ (p is the null set vector in V),
2) $\Theta_{-p} = \Theta_p$,
3) $\Theta_{p+q} \geq \tau(\Theta_p, \Theta_q)$,
4) $\Theta_p \leq \tau^*(\Theta_{\alpha p}, \Theta_{(1-\alpha)p})$ for each $\alpha \in [0, 1]$.

If $\tau^* = \tau_M$ and equality holds in (4), then $(V, \Theta, \tau, \tau_M)$ is a Šerstnev probabilistic normed space [2]. In fact, in Šerstnev probabilistic normed space, $\Theta_p = \tau_M(\Theta_{\alpha p}, \Theta_{(1-\alpha)p})$ for each $\alpha \in [0, 1]$, and instead of second condition, we have:

$$\Theta_{\lambda p}(x) = \Theta_p\left(\frac{x}{|\lambda|}\right) \text{ for all } \lambda \in \mathbb{R} - \{0\} \text{ and } x \in \mathbb{R}.$$

Main results and discussion

Now, we recall some concepts and theorems about random variables and convergence from Lafuerza-Guillén and Sempi's work [5]. Let $(\Omega, \mathcal{A}, P)$ be a probability space. $S = L^0(\mathcal{A})$ is called the linear space of equivalence classes of random variables. Suppose that $\nu : S \to \Delta_+$ be defined for all $X \in L^0(\mathcal{A})$ and for each $x \in \overline{\mathbb{R}}_+$ by

$$\nu_X(x) = P\{\omega \in \Omega : |X(\omega)| < x, \},$$

the pair $(L^0(\mathcal{A}), \nu)$ is called an equivalence normed space. In Lafuerza-Guillén et al.'s study [11], an equivalence relationship in the equivalence normed space $(L^0(\mathcal{A}), \nu)$ was defined by $X \sim Y$ if and only if $\nu_X = \nu_Y$.

Convergence in probability

We start this paragraph with a theorem from Lafuerza-Guillén and Sempi's study [5].

Theorem 3.1. *For a sequence of (equiv alence classes of) E-valued random variables $\{X_n\}$, the following statements are equivalent:*

1) *$\{X_n\}$ converges in probability to θ_S, $X_n \to^P \theta_S$ (θ_S is the null element of S), when $n \to \infty$*

2) *the corresponding sequence $\{\nu_{X_n}\}$ of probability norms converges weakly to ϵ_0 it means $d_S(\nu_{X_n}, \epsilon_0) \to 0$ when $n \to \infty$*

3) *$\{X_n\}$ converges to θ_S in the strong topology of Šerstnev space (L^0, ν, τ_L)*

We will study the relation of two sequences of E-valued random variables in the probabilistic normed space, specially about their convergence in probability and almost surely. Note that, in probability space, we know that if two sequences of random variables $\{X_n\}$, $\{Y_n\}$ are convergent in probability then the sequences $\{X_n + Y_n\}$, $\{X_n Y_n\}$, $\{\alpha X_n\}$ ($\alpha \in \mathbb{R}$) also converge in probability. The same results hold for almost sure convergence. An interesting consequence in probability space is convergence in probability of all continuous functions on every convergent in probability sequence. Also, convergence almost surely implies convergence in probability.

Now, we show in the same way the consequence in the space which Lafuerza-Guillén and Sempi introduced means $S = L^0(\mathcal{A})$.

Real and complex valued random variables are examples of E-valued random variables.

Example 3.2. *Let $\Omega = [0, 1]$, and P is Lebesgue measure on $[0, 1]$. Set $X_n(\omega)$ is one if $\omega \in [0, \frac{1}{n}]$ and is zero, otherwise. Therefore, $X_n = I(A_n)$, where $A_n = \{\omega \mid \omega \in [0, 1]\}$. Since $P(A_n) = \frac{1}{n}$, we have $P(A_n) \to 0$ where $n \to \infty$ and*

$$\nu_{X_n}(\epsilon) = 1 - P\{|\ X_n\ | > \epsilon\} = 1 - P(A_n) = 1 - \frac{1}{n} \to 1.$$

It means that X_n is convergent in probability to zero. Then, $\lim X_n(\omega)$ is equal to one when $\omega = 0$ and is equal to zero when $\omega \in (0, 1)$. So, $\lim X_n = I(0) = X$ and $\{|\ X_n - X\ | \geq \epsilon\} = (0, \frac{1}{n})$. The probability of the above event tends to zero when $n \to \infty$. It means that $\nu_{X_n - X}(\epsilon) \to 1$.

Theorem 3.3. *For two sequences of (equivalence class of) E-valued random variables $\{X_n\}$ and $\{Y_n\}$, and $\alpha \in \mathbb{R}$, if $\{X_n\}$ and $\{Y_n\}$ converge in probability to θ_s, then we have the following statements:*

(a) *$\{X_n + Y_n\}$ converges in probability to θ_s*
(b) *$\{\alpha X_n\}$ converges in probability to θ_s*
(c) *$\{X_n Y_n\}$ converges in probability to θ_s.*

Proof. By theorem 0.4, the sequences $\{X_n\}$ and $\{Y_n\}$ converge to θ_s in probability if and only if for every $x > 0$, $y > 0$, $\lim_{n\to\infty} \nu_{X_n}(x) = 1$ and $\lim_{n\to\infty} \nu_{Y_n}(y) = 1$. On the other hand, for each $x > 0$ and $t \in [0, x]$,

$$\begin{aligned} 1 - \nu_{X_n+Y_n}(x) &= P(|X_n + Y_n| \geq x) \\ &\leq P(|X_n| + |Y_n| \geq x) \\ &\leq P(\{|X_n| \geq t\} \cup \{|Y_n| \geq x - t\}) \end{aligned}$$

$$\le P(\{|X_n| \ge t\} + \{|Y_n| \ge x - t\})$$
$$= [1 - \nu_{X_n}(t)] + [1 - \nu_{Y_n}(x - t)]$$

If $n \to \infty$, then the right-hand side of the above inequality tends to zero. Since $\nu_X(x) \le 1$ for all $X \in L^0(\mathcal{A})$ and $x \in \mathbb{R}^+$, the proof of (a) is complete.

For proof of (b), we know in Šerstnev space;

$$\forall\, \alpha \in \mathbb{R} - \{0\} \quad \forall x > 0 \;\; \nu_{\alpha X_n} = \nu_{X_n}\left(\frac{x}{|\alpha|}\right)$$

proof of (c) is analogue to that of (a). In fact for every $x > 0,\ t > 0$

$$1 - \nu_{X_n Y_n}(x) = P(|X_n Y_n| \ge x)$$
$$= P(|X_n||Y_n| \ge x)$$
$$\le P\left(\{|X_n| \ge t\} \cup \left\{|Y_n| \ge \frac{x}{t}\right\}\right)$$
$$\le P\left(\{|X_n| \ge t\} + \left\{|Y_n| \ge \frac{x}{t}\right\}\right)$$
$$= [1 - \nu_{X_n}(t)] + \left[1 - \nu_{Y_n}\left(\frac{x}{t}\right)\right]$$
$$1 - \nu_{X_n Y_n}(x) \le [1 - \nu_{X_n}(t)] + \left[1 - \nu_{Y_n}\left(\frac{x}{t}\right)\right].$$

Theorem 3.4. *Let $\{X_n\}$ be a sequence of E-valued random variables and $\{X_n\}$ converges in probability to X ($X \ne \theta_s$ and X_n, X are bounded) then the sequence $\{\frac{1}{X_n}\}$ converges in probability to $\frac{1}{X}$.*

Proof.

$$1 - \nu_{\left(\frac{1}{X_n} - \frac{1}{X}\right)}(x) = P\left(\left|\frac{1}{X_n} - \frac{1}{X}\right| \ge x\right)$$
$$= P(|X_n - X| \ge |X_n||X|x)$$
$$= 1 - \nu_{(X_n - X)}(x|X_n||X|)$$

Since $\{X_n\}$ and $\{X\}$ are bounded, then the right-hand side tends to zero when $n \to \infty$, and the proof is complete.

After showing convergence in probability for the sum of two convergent sequences, scalar product and product of two sequence, we will show that each continuous function of convergent in probability sequence is convergent in probability.

Theorem 3.5. *Let $\{X_n\}$ be a sequence of E-valued random variable and φ is a continuous function from $\mathcal{R}$ to $\mathcal{R}$. If $\{X_n\}$ converges in probability to X, then the sequence $\{\varphi(X_n)\}$ converges in probability to $\varphi(X)$.*

Proof. Since φ is continuous, the following relation is derived:

$$\forall \epsilon > 0,\ \exists \delta > 0, \quad |X_n(\omega) - X(\omega)| < \delta \Rightarrow |\varphi(X_n(\omega)) - \varphi(X(\omega))| < \epsilon,$$

therefore,

$$P(|X_n - X| < \delta) \le P(|\varphi(X_n) - \varphi(X)| < \epsilon)$$

or

$$\nu_{(X_n - X)}(\delta) \le \nu_{(\varphi(X_n) - \varphi(X))}(\epsilon),$$

but $\lim_{n\to\infty} \nu_{X_n - X}(\delta) = 1$, thus $\lim_{n\to\infty} \nu_{\varphi(X_n) - \varphi(X)}(\epsilon) = 1 \;\; \forall \epsilon > 0$.

Almost surely convergence

Let the family $V = \{L^0(\mathcal{A})\}^N$ of all the sequences of (equivalence classes of) E-valued random variables. The set V is a real vector space with respect to the componentwise operations; specifically, if $s = \{X_n\}$ and $s' = \{Y_n\}$ are two sequences in V and if α is a real number, then $s \otimes s'$, $s \oplus s'$ of s and s' and the scalar product $\alpha \odot s$ of α and s are defined via

$$s \otimes s' = \{X_n\} \otimes \{Y_n\} := \{X_n Y_n\},$$
$$s \oplus s' = \{X_n\} \oplus \{Y_n\} := \{X_n + Y_n\},\ \alpha \odot s = \alpha \odot \{X_n\} := \{\alpha X_n\}$$

A mapping $\varphi : V \to \Delta_+$ will be defined on V via

$$\varphi_s(x) := P(\sup_{n\in N} |X_n| < x) = P(\bigcap_{n\in N} \{|X_n| < x\}),$$

where $x > 0$ and $s = \{X_n\}$. Then, Lafuerza-Guillén and Sempi [5] proved the next theorem.

Theorem 3.6. *The triple (V, φ, τ_L) is a Šerstnev space.*

If s is an element of V, which defined by $s = \{X_k : k \in N\}$ of E-valued random variables such that $X_k \in L^0(\mathcal{A})$ for all $n \in Z_+$, we can consider the relation of the n-shift s_n of s, $s_n = \{X_{k+n} :\ k \in N\}$ which is an element of V. They proved the next theorem, too.

Theorem 3.7. *A sequence $s = \{X_k : k \in N\}$ of E-valued random variable converges almost surely to θ_S, the null vector of S, if and only if, the sequence $\{\phi_{s_n} :\ n \in Z_+\}$ of the probabilistic norms of the n-shifts of s converges weakly to ϵ_0 or equivalently, if and only if the sequence $\{s_n\}$ of the n-shifts of s converges to $O := \{\theta_S, \theta_S, \ldots\}$ in the strong topology of (V, φ, τ_L).*

Like the theorem we showed for convergence in probability, we will prove for almost surely convergence.

Theorem 3.8. *Let two sequences $s = \{X_k;\ k \in \mathcal{N}\}$ and $s' = \{Y_k;\ k \in \mathcal{N}\}$ of E-valued random variable converge* *a.s. to θ_S the null vector of S and $\alpha \in \mathcal{R}$, then the following statements are satisfied:*

(a) The sequence $s \otimes s'$ is converges a.s. to θ_S,
(b) the sequence $s \oplus s'$ is converges a.s. to θ_S,
(c) the sequence $\alpha \odot s$ is converges a.s. to θ_S.

Proof. Let $x > 0$ and $t > 0$ then we have:

$$P(\bigcup_{k\in\mathcal{N}}\{|X_{k+n}Y_{k+n}| \geq x\}) \leq P(\bigcup_{k\in\mathcal{N}}\{|X_{k+n}||Y_{k+n}| \geq x\})$$

$$\leq P[\,(\bigcup_{k\in\mathcal{N}}\{|X_{k+n}| \geq t\} \cup \{|Y_{k+n}| \geq \frac{x}{t}\}]$$

$$= P(\bigcup_{k\in\mathcal{N}}\{|X_{k+n}| \geq t\}) + P(\bigcup_{k\in\mathcal{N}}\{|Y_{k+n}| \geq \frac{x}{t}\})$$

$$-P(\bigcup_{k\in\mathcal{N}}\{|X_{k+n}| \geq t\}) \cap P(\bigcup_{k\in\mathcal{N}}\{|Y_{k+n}| \geq \frac{x}{t}\})$$

so that

$$P(\bigcap_{k\in\mathcal{N}}\{|X_{k+n}Y_{k+n}| < x\}) = P(\bigcap_{k\in\mathcal{N}}\{|X_{k+n}||Y_{k+n}| < x\})$$

$$= 1 - P(\bigcup_{k\in\mathcal{N}}\{|X_{k+n}||Y_{k+n}| \geq x\})$$

$$\geq 1 - P(\bigcup_{k\in\mathcal{N}}\{|X_{k+n}| \geq t\} - P(\bigcup_{k\in\mathcal{N}}\{|Y_{k+n}| \geq \frac{x}{t}\})$$

$$+P[\,(\bigcup_{k\in\mathcal{N}}\{|X_{k+n}| \geq t\}) \cap (\bigcup_{k\in\mathcal{N}}\{|Y_{k+n}| \geq \frac{x}{t}\})]$$

$$= P(\bigcap_{k\in\mathcal{N}}\{|X_{k+n}| < t\} + P(\bigcap_{k\in\mathcal{N}}\{|Y_{k+n}| < \frac{x}{t}\})$$

$$-1 + P[\,(\bigcup_{k\in\mathcal{N}}\{|X_{k+n}| \geq t\}) \cap (\bigcup_{k\in\mathcal{N}}\{|Y_{k+n}| \geq \frac{x}{t}\})]$$

$$\geq P(\bigcap_{k\in\mathcal{N}}\{|X_{k+n}| < t\} + P(\bigcap_{k\in\mathcal{N}}\{|Y_{k+n}| < \frac{x}{t}\}) - 1.$$

Thus, the following relation for every $x > 0$ and $t > 0$ is reached;

$$\varphi_{s_n \otimes s'_n}(x) \geq \varphi_{s_n}(t) + \varphi_{s'_n}\left(\frac{x}{t}\right) - 1.$$

If $n \to \infty$, then $\varphi_{s_n \otimes s'_n}(x) = 1$ which gives (a).
As in the above proof for $x > 0$ and $t \in (0, x)$,

$$\varphi_{s_n \oplus s'_n}(x) \geq \varphi_{s_n}(t) + \varphi_{s'_n}(x - t) - 1$$

According to assumptions $\lim_{n\to\infty} \varphi_{s_n}(t) = 1, \lim_{n\to\infty} \varphi_{s'_n}(x - t) = 1$ we have $\lim_{n\to\infty} \varphi_{s_n \oplus s'_n}(x) = 1$ and the proof of (b) is complete.

Since $\varphi_{\alpha \odot s_n}(x) = \varphi_{s_n}(\frac{x}{|\alpha|})$, the proof of (c) is immediately derived.

Relation between almost surely convergence and convergence in probability

Now, let us turn to the relation between almost surely convergence and convergence in probability in this space.

Theorem 3.9. *Suppose that $s = \{X_k;\ k \in \mathcal{N}\}$ is a sequence of E-valued independent random variable which converges almost surely to θ_S, then $\{X_k\}$ is convergent in probability to θ_S, too.*

Proof. Suppose that $\{X_k\}$ converges almost surely to θ_S, so for every $x > 0$,

$$1 = \lim_{n\to\infty} \varphi_{s_n}(x) = \lim_{n\to\infty} P(\bigcap_{k\in\mathcal{N}} |X_{k+n}| < x)$$

$$= \lim_{n\to\infty} P(\bigcap_{k\geq n} |X_k| < x)$$

Since $\{X_n\}$ are independent, then:

$$= \lim_{n\to\infty} \prod_{k=n}^{\infty} P(|X_k| < x)$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \prod_{k=n}^{m} [\,1 - P(|X_k| \geq x)]$$

We know that if $0 \leq a \leq 1$, then $(1 - a) \leq e^{-a}$. This implies that:

$$1 \leq \lim_{n\to\infty} \lim_{m\to\infty} e^{-\Sigma_{k=n}^{m} P(|X_k| \geq x)}$$

$$= e^{-\lim_{n\to\infty} \Sigma_{k=n}^{\infty} P(|X_k| \geq x)}$$

Thus, $\lim_{n\to\infty} \Sigma_{k=n}^{\infty} P(|X_k| \geq x) = 0$, namely the series $\Sigma_{k=0}^{\infty} P(|X_k| \geq x) = \Sigma_{k=0}^{\infty}[\,1 - \nu_{X_k}(x)]$ converges, and we obtain that $\lim_{n\to\infty}[\,1 - \nu_{X_n}(x)] = 0$. It means that $\lim_{n\to\infty} \nu_{X_n}(x) = 1$. This proves the result.

Conclusion

We proved convergent in probability and almost surely convergent under algebraic operations on *Šerstnev* space are closed. In addition, every continuous function of each sequence convergent in probability sequence is convergent in probability. Also, if a sequence of independent random variables is almost surely convergent, then it is convergent in probability. There are some interesting problems that we have not solved in *Šerstnev* space, for example, proof of above theorems about another type of convergence like L^p.

Competing interests
The authors declare that they have no competing interests.

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Author's contributions

PA defined the definitions and wrote the introduction, preliminaries and abstract. AB proved the theorems. AB has approved the final manuscript. Also PA has verified the final manuscript. Both authors read and approved the final manuscript.

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