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A comparative study of numerical integration based on block-pulse and sinc functions and Chebyshev wavelet

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Abstract

In this paper, numerical integration rules based on block-pulse functions and Chebyshev wavelet are proposed to find approximate values of definite integrals. Errors of these numerical integrations are given. These numerical integrations are compared by sinc functions numerical integration method. Some numerical examples are provided to illustrate the accuracy of proposed rules and comparison between them. The main advantage of proposed numerical integration methods are their efficiency and simple applicability.

Keywords: Numerical integration, Block-pulse functions, Sinc functions, Chebyshev wavelet

Background

Numerical integration is the approximate computation of an integral using numerical techniques. There are several reasons for carrying out numerical integration. The integrand $f(x)$ may be known only at certain points, such as obtained by sampling. Some embedded systems and other computer applications may need numerical integration for this reason. A formula for the integrand may be known, but it may be difficult or impossible to find an antiderivative which is an elementary function. An example of such an integrand is $f(x) = exp(-x^2)$, the antiderivative of which cannot be written in elementary form. It may be possible to find an antiderivative symbolically, but it may be easier to compute a numerical approximation than to compute the antiderivative. That may be the case if the antiderivative is given as an infinite series or product, or if its evaluation requires a special function which is not available. There are a wide range of methods available for numerical integration. A good source for such techniques is in the work of Press et al. [1].

Numerical integration has many applications in science and engineering. In recent years, the wavelet approach is becoming more popular in the field of numerical approximations. Different types of wavelets and approximating functions have been used in numerical approximations. Among them Chebyshev wavelets ([2,3]) and block-pulse functions ([4-6]) have gained popularity among researchers due to their useful properties. Motivated by the excellent performance of these methods, we will apply the same techniques for numerical integration by these two functions. We also present numerical integration based on sinc functions that has been mentioned by Stenger [7] and compared block-pulse and Chebyshev wavelet results by sinc method results.

The paper is organized as follows: In section 'Numerical integration using Chebyshev wavelets,' we briefly review the concept and some properties of the block-pulse functions and proposed method of numerical integration based on block-pulse functions. In section 'Numerical integration using Chebyshev wavelets,' the concepts of the Chebyshev wavelets are presented, and then numerical integration using them is described. In the 'Numerical integration using sinc functions' section, sinc functions are used for numerical integration. In the 'Error estimates' section, the convergence analysis of the described methods are discussed. It is shown that the Sinc procedure converges to the solution at an exponential rate. Numerical experiments are given in the 'Error estimates' section

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to verify the exponential convergence rate and to demonstrate the efficiency and accuracy of the proposed numerical scheme. Finally, the 'Numerical examples' section concluded the paper.

Results and discussion

Numerical integration using block-pulse functions

Block-pulse functions

Definition 1. An *m-set* of block-pulse function (BPFs) is defined as follows:

$$\phi_i(t) = \begin{cases} 1 & for\ \frac{(i-1)T}{m} \leq t < \frac{iT}{m}, \\ 0 & otherwise. \end{cases} \quad (1)$$

with $t \in [0, T)$ and $i = 1, 2, \ldots, m$ and $h = \frac{T}{m}$.

The elementary properties of BPFs are as follows:

1) *Disjointness*: The BPFs are disjoined with each other in the interval $t \in [0, T)$:

$$\phi_i(t)\phi_j(t) = \delta_{ij}\phi_i(t), \quad (2)$$

for $i, j = 1, 2, \ldots, m$.

2) *Orthogonality*: The BPFs are orthogonal with each other in the interval $t \in [0, T)$:

$$\int_0^T \phi_i(t)\phi_j(t)dt = h\delta_{ij}, \quad (3)$$

for $i, j = 1, 2, \ldots, m$.

3) *Completeness*: The BPFs set is complete when m approaches infinity. This means that for every $f \in L^2([0, T))$, when m approaches to the infinity, Parseval's identity holds,

$$\int_0^T f^2(t)dt = \sum_{i=1}^{\infty} f_i^2 \|\phi_i(t)\|^2, \quad (4)$$

where

$$f_i = \frac{1}{h}\int_0^T f(t)\phi_i(t)dt. \quad (5)$$

The orthogonality property of BPFs is the basis of expanding functions into their block pulse series. An arbitrary real bounded function $f(t)$, which is square integrable in the interval $t \in [0, T)$, can be expanded into a block pulse series in the sense of minimizing the mean square error between $f(t)$ and its approximation:

$$f(t) \simeq \hat{f}_m(t) = \sum_{i=1}^{m} f_i\phi_i(t), \quad (6)$$

where $\hat{f}_m(t)$ is the Block pulse series of the original function $f(t)$, and f_i is the Block pulse coefficient with respect to the ith BPF $\phi_i(t)$.

Method of numerical integration based on BPFs

We consider the integral $\int_a^b f(x)dx$. By using $x = (b-a)t + a$, we have

$$\int_a^b f(x)dx = (b-a)\int_0^1 f((b-a)t + a)dt.$$

Theorem 1. *The approximate value of the integral is*

$$\int_0^1 f(t)dt \simeq \frac{1}{m}\sum_{i=1}^{m} f_i. \quad (7)$$

Proof.

$$\int_0^1 f(t)dt \simeq \sum_{i=1}^{m} f_i \int_0^1 \phi_i(t)dt = \frac{1}{m}\sum_{i=1}^{m} f_i.$$

To calculate the coefficients f_i, we consider the nodal points,

$$t_k = \frac{2k-1}{2m} \quad k = 1, 2, \ldots, m. \quad (8)$$

The discretized form of Equation 6 can be written as

$$f(t_k) = \sum_{i=1}^{m} f_i\phi_i(t_k) = f_k \quad k = 1, 2, \ldots, m. \quad (9)$$

so, the approximate value of the integral based on BPFs is

$$\int_0^1 f(t)dt \simeq \frac{1}{m}\sum_{i=1}^{m} f\left(\frac{2i-1}{2m}\right), \quad (10)$$

or,

$$\int_a^b f(x)dx \simeq \frac{b-a}{m}\sum_{i=1}^{m} f\left(a + (b-a)\frac{2i-1}{2m}\right). \quad (11)$$

□

Numerical integration using Chebyshev wavelets

Chebyshev wavelets

Definition 2. The *Chebyshev wavelets*, $\psi_{n,m}(x)$, $n = 1, 2, \ldots, 2^{k-1}$ and $m = 0, 1, \ldots, M-1$, is defined on the interval $[0, 1)$ as,

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}}\widetilde{T}_m(2^k x - 2n + 1) & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}}, \\ 0 & otherwise. \end{cases} \quad (12)$$

where

$$\widetilde{T}_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}} & m = 0, \\ \sqrt{\frac{2}{\pi}}T_m(t) & m > 0. \end{cases} \quad (13)$$

and k can assume any positive integer and m is the degree of Chebyshev polynomials of the first kind. In Equation 13, the coefficients are used for orthonormality. Here, $T_m(t)$ is the Chebyshev polynomials of the first kind of degree

m which is orthogonal with respect to the weight function $W(t) = \frac{1}{\sqrt{1-t^2}}$, on the interval $[-1, 1]$ and satisfy the following recursive formula :

$$\begin{cases} T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t) \quad m = 1, 2, 3, \ldots, \\ T_0(t) = 1, \quad T_1(t) = t. \end{cases} \tag{14}$$

Any function $f(x)$, which is square integrable in the interval $x \in [0, 1)$, can be expressed as

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{n,m} \psi_{n,m}(x) \quad x \in [0, 1), \tag{15}$$

so, we can approximate $f(x)$ as

$$f(x) \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x) \quad x \in [0, 1). \tag{16}$$

Method of numerical integration based on Chebyshev wavelets

We consider the integral $\int_a^b f(x)dx$. By using $x = (b-a)t + a$, we have,

$$\int_a^b f(x)dx = (b-a) \int_0^1 f((b-a)t + a)dt.$$

Theorem 2. *The approximate value of the integral is*

$$\int_0^1 f(x)dx \simeq \frac{2^{1-\frac{k}{2}}}{\sqrt{\pi}} \sum_{n=1}^{2^{k-1}} \left[c_{n,0} + \sum_{l=1}^{\left[\frac{M-1}{2}\right]} \frac{\sqrt{2}}{1-4l^2} c_{n,2l} \right]. \tag{17}$$

Proof.

$$\int_0^1 f(x)dx \simeq \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \int_0^1 \psi_{n,m}(x)dx,$$

$$\int_0^1 \psi_{n,m}(x)dx = 2^{\frac{k}{2}} \int_{\frac{n-1}{2^{k-1}}}^{\frac{n}{2^{k-1}}} \widetilde{T}_m(2^k x - 2n + 1)dx$$

$$= 2^{-\frac{k}{2}} \int_{-1}^1 \widetilde{T}_m(t)dt,$$

since,

$$\int_{-1}^1 T_m(t)dt = \begin{cases} 0 & m \text{ is odd}, \\ \frac{-2}{m^2-1} & m \text{ is even}, \end{cases} \tag{18}$$

we have,

$$\int_0^1 \psi_{n,m}(x)dx = \begin{cases} \frac{2^{1-\frac{k}{2}}}{\sqrt{\pi}} & m = 0, \\ 0 & m \text{ is odd}, \\ \frac{2^{1-\frac{k}{2}}}{1-m^2} \sqrt{\frac{2}{\pi}} & m \text{ is even}. \end{cases} \tag{19}$$

so,

$$\int_0^1 f(x)dx \simeq \frac{2^{1-\frac{k}{2}}}{\sqrt{\pi}} \sum_{n=1}^{2^{k-1}} \left[c_{n,0} + \sum_{l=1}^{\left[\frac{M-1}{2}\right]} \frac{\sqrt{2}}{1-4l^2} c_{n,2l} \right]. \tag{20}$$

In order to calculate the coefficients $c_{n,0}$ and $c_{n,2l}$ of Chebyshev wavelets, we consider the nodal points

$$x_p = \frac{2p-1}{2^k M} \qquad p = 1, 2, 3, \ldots, 2^{k-1}M. \tag{21}$$

□

Substituting these points in Equation 16 , we obtain

$$f(x_p) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{n,m} \psi_{n,m}(x_p) \quad p = 1, 2, 3, \ldots, 2^{k-1}M. \tag{22}$$

We can calculate the coefficients $c_{n,0}$ and $c_{n,2l}$ from the above system of equations. By using the definition Chebyshev wavelets, we have

$$\frac{n-1}{2^{k-1}} \le \frac{2p-1}{2^k M} < \frac{n}{2^{k-1}},$$

so,

$$p = (n-1)M + i \qquad , \qquad i = 1, 2, 3, \ldots, M,$$

$$f(x_p) = f\left(\frac{2(n-1)M + 2i - 1}{2^k M} \right) \quad i = 1, 2, 3, \ldots, M,$$

$$\psi_{n,0}(x_p) = \frac{2^{\frac{k}{2}}}{\sqrt{\pi}},$$

$$\psi_{n,m}(x_p) = \sqrt{\frac{2}{\pi}} 2^{\frac{k}{2}} T_m \left(2^k x_p - 2n + 1 \right)$$
$$= \sqrt{\frac{2}{\pi}} 2^{\frac{k}{2}} T_m \left(\frac{2i-1}{M} - 1 \right).$$

Therefore, the above system of equations converted to the below system of equations is given as

$$f\left(\frac{2(n-1)M+2i-1}{2^kM}\right)=\frac{2^{\frac{k}{2}}}{\sqrt{\pi}}c_{n,0}+\sum_{m=1}^{M-1}\sqrt{\frac{2}{\pi}}2^{\frac{k}{2}}T_m\times\left(\frac{2i-1}{M}-1\right)c_{n,m}$$

$$i=1,2,3,\ldots,M. \tag{23}$$

The coefficients $c_{n,0}$ and $c_{n,2l}$ can be easily calculated from the below system of equations,

$$\begin{bmatrix}\frac{\sqrt{2}}{2} & T_1(\frac{1}{M}-1) & \ldots & T_{M-1}(\frac{1}{M}-1)\\ \frac{\sqrt{2}}{2} & T_1(\frac{3}{M}-1) & \ldots & T_{M-1}(\frac{3}{M}-1)\\ \vdots & \vdots & \ldots & \vdots\\ \frac{\sqrt{2}}{2} & T_1(\frac{2M-1}{M}-1) & \ldots & T_{M-1}(\frac{2M-1}{M}-1)\end{bmatrix} \tag{24}$$

$$\begin{bmatrix}c_{n,0}\\ c_{n,1}\\ \vdots\\ c_{n,M-1}\end{bmatrix}=\sqrt{\frac{\pi}{2}}2^{-\frac{k}{2}}\begin{bmatrix}f\left(\frac{2(n-1)M+1}{2^kM}\right)\\ f\left(\frac{2(n-1)M+3}{2^kM}\right)\\ \vdots\\ f\left(\frac{2(n-1)M+2M-1}{2^kM}\right)\end{bmatrix}.$$

For $M=1$, the coefficients $c_{n,0}$ and the approximate value of the integral are given as

$$c_{n,0}=\sqrt{\pi}2^{-\frac{k}{2}}f\left(\frac{2n-1}{2^k}\right), \tag{25}$$

$$\int_0^1 f(x)dx\simeq\frac{1}{2^{k-1}}\sum_{n=1}^{2^{k-1}}f\left(\frac{2n-1}{2^k}\right). \tag{26}$$

For $M=2$,

$$\int_0^1 f(x)dx\simeq\frac{1}{2^k}\sum_{n=1}^{2^{k-1}}\left[f\left(\frac{4n-3}{2^{k+1}}\right)+f\left(\frac{4n-1}{2^{k+1}}\right)\right]. \tag{27}$$

For $M=3$,

$$\int_0^1 f(x)dx\simeq\frac{1}{2^{k+2}}\sum_{n=1}^{2^{k-1}}\left[3f\left(\frac{6n-5}{3\times 2^k}\right)+2f\left(\frac{6n-3}{3\times 2^k}\right)+3f\left(\frac{6n-1}{3\times 2^k}\right)\right]. \tag{28}$$

For $M=4$,

$$\int_0^1 f(x)dx\simeq\frac{1}{3\times 2^{k+3}}\sum_{n=1}^{2^{k-1}}\left[13f\left(\frac{8n-7}{2^{k+2}}\right)+11f\left(\frac{8n-5}{2^{k+2}}\right)+11f\left(\frac{8n-3}{2^{k+2}}\right)+13f\left(\frac{8n-1}{2^{k+2}}\right)\right]. \tag{29}$$

For $M=5$,

$$\int_0^1 f(x)dx\simeq\frac{1}{9\times 2^{k+6}}\sum_{n=1}^{2^{k-1}}\left[275f\left(\frac{10n-9}{5\times 2^k}\right)+100f\left(\frac{10n-7}{5\times 2^k}\right)+402f\left(\frac{10n-5}{5\times 2^k}\right)+100f\left(\frac{10n-3}{5\times 2^k}\right)+275f\left(\frac{10n-1}{5\times 2^k}\right)\right]. \tag{30}$$

For $M=6$,

$$\int_0^1 f(x)dx\simeq\frac{1}{5\times 2^{k+7}}\sum_{n=1}^{2^{k-1}}\left[247f\left(\frac{12n-11}{3\times 2^{k+1}}\right)+139f\left(\frac{12n-9}{3\times 2^{k+1}}\right)+254f\left(\frac{12n-7}{3\times 2^{k+1}}\right)+254f\left(\frac{12n-5}{3\times 2^{k+1}}\right)+139f\left(\frac{12n-3}{3\times 2^{k+1}}\right)+247f\left(\frac{12n-1}{3\times 2^{k+1}}\right)\right]. \tag{31}$$

Numerical integration using sinc functions

Sinc functions

Definition 3. The *sinc function* is defined on the whole real line by,

$$Sinc(x)=\begin{cases}\frac{sin(\pi x)}{\pi x} & x\neq 0,\\ 1 & x=0.\end{cases} \tag{32}$$

For any $h>0$, the translated sinc functions with evenly spaced nodes are given as follows :

$$S(j,h)(x)=Sinc\left(\frac{x-jh}{h}\right),\quad j=0,\pm1,\pm2,\ldots, \tag{33}$$

which are called the jth sinc functions.
The sinc function for the interpolating points $x_k=kh$ is given by

$$S(j,h)(kh)=\delta^{(0)}_{jk}=\begin{cases}1 & k=j,\\ 0 & k\neq j.\end{cases} \tag{34}$$

If f is defined on the real line, then for $h > 0$ the series

$$f(x) = \sum_{j=-\infty}^{\infty} f(jh)S(j,h), \tag{35}$$

is called the Whittaker cardinal expansion of f whenever this series converges.

Definition 4. Let D be a simply connected domain in the complex plane having boundary ∂D. Let a and b denote two distinct points of ∂D, and ϕ denote a conformal map of D onto $D_d = \{w \in \mathcal{C} : |Im(w)| < d\}$, such that $\phi(a) = -\infty$ and $\phi(b) = +\infty$. Let $\zeta = \phi^{-1}$ denote the inverse map, and let Γ be defined by

$$\Gamma = \{z \in \mathcal{C} : z = \zeta(u), u \in \mathcal{R}\}.$$

Given ϕ, ζ and a positive number h, let us set

$$z_k = z_k(h) = \zeta(kh), \quad k = 0, \pm 1, \pm 2, \ldots.$$

Let us also define ρ by

$$\rho(z) = e^{\phi(z)}.$$

Definition 5. Let $L_\alpha(D)$ be the set of all analytic functions f, for which there exists a constant, C, such that

$$|f(z)| \leq C \frac{|\rho(z)|^\alpha}{[1 + |\rho(z)|]^{2\alpha}}, \quad z \in D, \quad 0 < \alpha \leq 1. \tag{36}$$

Theorem 3. *Let $f \in L_\alpha(D)$, let N be a positive integer, and let h be selected by the formula*

$$h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}},$$

Then there exist positive constant C_1, independent of N, such that

$$\sup_{z \in \Gamma} \left| f(z) - \sum_{j=-N}^{N} f(z_j) S(j,h) \circ \phi(z) \right| \leq C_1 e^{-(\pi d \alpha N)^{\frac{1}{2}}}. \tag{37}$$

Method of numerical integration based on sinc functions

In this section, we consider numerical integration for single integrals using sinc functions.

Let $\Gamma = (a,b)$, where $-\infty < a < b < \infty$. In this case, we take

$$\phi(x) = ln\left(\frac{x-a}{b-x}\right),$$
$$\phi'(x) = \frac{b-a}{(x-a)(b-x)}.$$

This map carries the eye-shaped complex domain

$$D_E = \left\{ z \in \mathcal{C} : \left| arg\left(\frac{z-a}{b-z}\right) \right| < d \right\},$$

onto the infinite strip D_d. The basis function on *(a,b)* is then given by

$$S(j,h) \circ \phi(x) = Sinc\left(\frac{\phi(x) - jh}{h}\right).$$

Notice that these functions exhibit Kronecker delta behavior on the grid points $x_k \in (a,b)$ defined by

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}.$$

Theorem 4. *Let $\frac{f}{\phi'} \in L_\alpha(D)$, with $0 < \alpha \leq 1$, and $0 < d \leq \pi$, let N be a positive integer, and let $h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}$. Then there exists a positive constant, C_2, which is independent of N, such that*

$$\left| \int_a^b f(t)dt - h \sum_{j=-N}^{N} \frac{f(x_j)}{\phi'(x_j)} \right| \leq C_2 e^{-(\pi d \alpha N)^{\frac{1}{2}}}. \tag{38}$$

so, the approximate value of the integral based on Sinc functions is

$$\int_a^b f(t)dt \simeq h \sum_{j=-N}^{N} \frac{f(x_j)}{\phi'(x_j)}. \tag{39}$$

Table 1 The absolute error of the approximate value of the integral for Example 1

BPFs		Sinc		Chebyshev wavelet	
m	***E***	***N***	***E***	***k, M***	***E***
16	$1.76147E-4$	16	$1.96016E-6$	$k=7, M=2$	$2.74818E-6$
64	$1.09935E-5$	32	$1.20463E-8$	$k=9, M=2$	$1.71757E-7$
256	$6.87033E-7$	48	$2.34518E-10$	$k=5, M=3$	$2.97317E-8$
512	$1.71757E-7$	64	$8.35526E-12$	$k=7, M=3$	$1.16064E-10$
1024	$4.29393E-8$	80	$4.39814E-13$	$k=7, M=4$	$6.00400E-11$
4096	$2.68371E-9$	96	$3.04756E-14$	$k=7, M=5$	$4.44089E-16$
2^{14}	$1.67732E-10$	112	$2.38697E-15$	$k=7, M=6$	$2.22044E-16$
2^{15}	$4.19344E-11$	128	$2.77555E-16$	$k=8, M=5$	$1.11022E-16$

Table 2 The absolute error of the approximate value of the integral for Example 2

BPFs		Sinc		Chebyshev wavelet	
m	*E*	*N*	*E*	*k,M*	*E*
16	5.72841*E* − 5	16	1.90339*E* − 5	$k=7, M=2$	9.35567*E* − 7
64	3.74229*E* − 6	32	4.78903*E* − 8	$k=9, M=2$	5.84728*E* − 8
256	2.33891*E* − 7	48	6.16587*E* − 10	$k=5, M=3$	3.23897*E* − 7
512	5.84728*E* − 8	64	6.16914*E* − 11	$k=7, M=3$	3.87839*E* − 12
1024	1.46182*E* − 8	80	5.06328*E* − 12	$k=7, M=4$	1.96420*E* − 12
4096	9.13637*E* − 10	96	8.93174*E* − 14	$k=7, M=5$	1.20181*E* − 13
2^{14}	5.71022*E* − 11	112	2.89213*E* − 14	$k=7, M=6$	6.97775*E* − 14
2^{15}	1.42780*E* − 11	128	5.55111*E* − 17	$k=8, M=5$	1.11022*E* − 16

Table 3 The absolute error of the approximate value of the integral for Example 3

BPFs		Sinc		Chebyshev wavelet	
m	*E*	*N*	*E*	*k,M*	*E*
16	3.90705*E* − 4	16	4.72827*E* − 6	$k=7, M=2$	6.10445*E* − 6
64	2.44179*E* − 5	32	2.94841*E* − 8	$k=9, M=2$	3.81528*E* − 7
256	1.52611*E* − 6	48	5.74079*E* − 10	$k=5, M=3$	2.23131*E* − 9
512	3.81528*E* − 7	64	2.04526*E* − 11	$k=7, M=3$	1.06799*E* − 11
1024	9.53820*E* − 8	80	1.07636*E* − 12	$k=7, M=4$	5.52957*E* − 12
4096	5.96137*E* − 9	96	7.44959*E* − 14	$k=7, M=5$	1.22124*E* − 14
2^{14}	3.72588*E* − 10	112	6.21724*E* − 15	$k=7, M=6$	7.66053*E* − 15
2^{15}	9.31439*E* − 11	128	7.77156*E* − 16	$k=8, M=5$	1.11022*E* − 16

Error estimates

Error of numerical integration based on BPFs

Since we use block pulse approximation for numerical integration, first, we compute upper bound for error of functions approximation by using block-pulse functions.

Theorem 5. *Suppose that f(t) is an arbitrary real bounded function, which is square integrable in the interval* $t \in I = [0,1)$*, and* $e(t) = f(t) - \hat{f}_m(t)$, $t \in I = [0,1)$*, which* $\hat{f}_m(t) = \sum_{i=1}^{m} f_i \phi_i(t)$ *is the block pulse series f(t). Then,*

$$\|e(t)\| \leq \frac{h^{\frac{1}{2}}}{2\sqrt{3}} \sup_{t\in I} |f'(t)|. \tag{40}$$

Proof. Let,

$$e_i(t) = f(t) - f_i, \qquad (i-1)h \leq t < ih, \qquad h = \frac{1}{m}.$$

We have,

$$e_i(t) = f(t) - \frac{1}{h}\int_{(i-1)h}^{ih} f(s)ds = \frac{1}{h}\int_{(i-1)h}^{ih} \big(f(t) - f(s)\big)\, ds$$
$$= \frac{f'(\eta_i)}{h}\int_{(i-1)h}^{ih} (t-s)ds = f'(\eta_i)\left(t + (-i + \frac{1}{2})h\right),$$
$$(i-1)h \leq \eta_i < ih. \tag{41}$$

then,

$$\|e_i(t)\|^2 = \int_{(i-1)h}^{ih} |e_i(t)|^2 dt = (f'(\eta_i))^2 \int_{(i-1)h}^{ih} \times \left(t + (-i + \frac{1}{2})h\right)^2 dt = \frac{h^3}{12}(f'(\eta_i))^2,$$

or,

$$\|e_i(t)\| = \frac{h^{\frac{3}{2}}}{2\sqrt{3}} |f'(\eta_i)|, \qquad i = 1, 2, \ldots, m.$$

Consequently,

$$\|e(t)\| \leq \frac{h^{\frac{3}{2}}}{2\sqrt{3}} \sum_{i=1}^{m} |f'(\eta_i)| \leq \frac{h^{\frac{1}{2}}}{2\sqrt{3}} \sup_{t\in I} |f'(t)|.$$

□

Theorem 6. *Suppose that f(t) is an arbitrary real bounded function, which is square integrable in the interval* $t \in I = [0,1)$*, and* $e(t) = f(t) - \hat{f}_m(t)$, $t \in I = [0,1)$*, which* $\hat{f}_m(t) = \sum_{i=1}^{m} f_i \phi_i(t)$ *is the block pulse series f(t), and* $E_m(f)$ *is error of numerical integration based on BPFs. Then,*

$$|E_m(f)| \leq \frac{h^{\frac{1}{2}}}{2\sqrt{3}} \sup_{t\in I} |f'(t)|. \tag{42}$$

Proof. we can write $E_m(f)$ as follows :

$$E_m(f) = \int_0^1 f(t)dt - h\sum_{i=1}^{m} f_i = \int_0^1 e(t)dt, \qquad h = \frac{1}{m}.$$

Thus,

$$|E_m(f)| \leq \int_0^1 \|e(t)\|dt \leq \frac{h^{\frac{1}{2}}}{2\sqrt{3}} \sup_{t\in I} |f'(t)|.$$

□

Error of numerical integration based on sinc functions

Theorem 7. *Let* $\frac{f}{\phi'} \in L_\alpha(D)$*, with* $0 < \alpha \leq 1$*, and* $0 < d \leq \pi$*, let N be a positive integer, and let* $h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}$*. Then there exists a positive constant, K, which is independent of N, such that*

$$\left|E_N(f)\right| \leq K e^{-(\pi d \alpha N)^{\frac{1}{2}}}, \qquad (43)$$

where,

$$E_N(f) = \int_a^b f(t)dt - h\sum_{j=-N}^{N} \frac{f(x_j)}{\phi'(x_j)}.$$

Proof : see [7].

Numerical examples

The following examples are given to show the accuracy and efficiency numerical integration by using BPFs and Sinc functions and Chebyshev wavelets.

Example 1.

$$\int_0^1 sin(x^2)dx.$$

The approximate value of the integral with Matlab 7 programming is $quad(sin(x^2), 0, 1) = 0.310268301723381$. The absolute error approximate value of the integral is defined as $E = |quad(sin(x^2), 0, 1) - Intf|$, where $Intf$ is the approximate value of the integral with BPFs and Sinc functions and Chebyshev wavelets. The absolute error are shown in Table 1.

Example 2. *(Improper integral)*

$$\int_0^1 \frac{e^{-\frac{1}{x}}}{x^2}dx.$$

The approximate value of the integral with Matlab 7 programming is $quad(\frac{e^{-\frac{1}{x}}}{x^2}, 0, 1) = 0.367879441171442$. The absolute error approximate value of the integral is defined as $E = |quad(\frac{e^{-\frac{1}{x}}}{x^2}, 0, 1) - Intf|$, where $Intf$ is the approximate value of the integral with BPFs and Sinc functions and Chebyshev wavelets. The absolute error are shown in Table 2.

Example 3.

$$\int_1^e \frac{cos(lnx)}{x}dx.$$

The exact value of the integral is $sin(1) = 0.841470984807897$. The absolute error approximate value of the integral is defined as $E = |sin(1) - Intf|$, where $Intf$ is the approximate value of the integral with BPFs and Sinc functions and Chebyshev wavelets. The absolute error are shown in Table 3.

Conclusions

This paper presents two numerical integration methods based on Block-Pulse function and Chebyshev wavelet. A comparative analysis of Block-Pulse function, Chebyshev wavelet and Sinc function is performed to find numerical integration. Error analysis of these methods besides numerical examples provide a solid foundation for using these functions in the context of numerical approximation of integral equations, partial differential equations and ordinary differential equations.

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