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# On $\Delta$ -statistical convergence in random 2-normed space

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## Abstract

Recently, Mursaleen introduced the concepts of statistical convergence in random 2-normed spaces. In this paper, we define and study the notion of  $\Delta$ -statistical convergence and  $\Delta$ -statistical Cauchy sequences in random 2-normed spaces, where  $\lambda = (\lambda_m)$  be a non-decreasing sequence of positive numbers tending to infinity such that  $\lambda_{m+1} \leq \lambda_m + 1$ ,  $\lambda_1 = 1$  and prove some theorems. In last section we will give the definition of the  $\Delta$ -limit and cluster points and we will show their relation between those classes.

**Keywords:** Statistical convergence,  $\lambda$ -statistical convergence, Difference sequence,  $t$ -norm, 2-norm, Random 2-normed space

**MSC:** Primary 40A05, Secondary 46A70, 40A99, 46A99

## Introduction

The concept of statistical convergence play a vital role not only in pure mathematics but also in other branches of science involving mathematics, especially in information theory, computer science, biological science, dynamical systems, geographic information systems, population modeling, and motion planning in robotics.

The notion of statistical convergence was introduced by Fast [1] and Schoenberg [2] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, and number theory. Later on it was further investigated by Fridy [3], Šalát [4], Çakalli [5], Maio and Kocinac [6], Miller [7], Maddox [8], Leindler [9], Mursaleen and Alotaibi [10], Mursaleen and Edely [11], Mursaleen and Edely [12], and many others. In the recent years, generalization of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on Stone-Čech compactification of the natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability [13].

The notion of statistical convergence depends on the density of subsets of  $\mathbf{N}$ . A subset of  $\mathbf{N}$  is said to have density  $\delta(E)$  if

$$\delta(E) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \chi_E(k) \text{ exists.}$$

**Definition 1.1.** A sequence  $x = (x_k)$  is said to be *statistically convergent* to  $\ell$  if for every  $\varepsilon > 0$

$$\delta(\{k \in \mathbf{N} : |x_k - \ell| \geq \varepsilon\}) = 0.$$

In this case, we write  $S - \lim x = \ell$  or  $x_k \rightarrow \ell(S)$  and  $S$  denotes the set of all statistically convergent sequences.

The probabilistic metric space was introduced by Menger [14] which is an interesting and important generalization of the notion of a metric space. Karakus [15] studied the concept of statistical convergence in probabilistic normed spaces. The theory of probabilistic normed spaces was initiated and developed in [16-20] and it was further extended to random/probabilistic 2-normed spaces by Goleţ [21] using the concept of 2-norm which is defined by Gähler [22], and Gürdal and Pehlivan [23] studied statistical convergence in 2-Banach spaces.

Mursaleen [24], introduced the  $\lambda$ -statistical convergence for real sequences as follows:

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Let  $\lambda = (\lambda_m)$  be a non-decreasing sequence of positive numbers tending to  $\infty$  such that

$$\lambda_{m+1} \leq \lambda_m + 1, \lambda_1 = 1.$$

The collection of such sequence  $\lambda$  will be denoted by  $\Delta$ . Let  $K \subseteq \mathbf{N}$  be a set of positive integers. Then

$$\delta_\lambda(K) = \lim_m \frac{1}{\lambda_m} |\{m - \lambda_m + 1 \leq j \leq m : j \in K\}|$$

is said to be  $\lambda$ -density of  $K$ . In case  $\lambda_m = m$ , then  $\lambda$ -density reduces to natural density, so  $S_\lambda$  is the same as  $S$ . Also, since  $(\frac{\lambda_m}{m}) \leq 1$ ,  $\delta(K) \leq \delta_\lambda(K)$  for every  $K \subseteq \mathbf{N}$ .

**Definition 1.2.** [24] A sequence  $x = (x_k)$  is said to be  $\lambda$ -statistically convergent or  $S_\lambda$ -convergent to  $\ell$  if for every  $\varepsilon > 0$ , the set  $\{k \in I_m : |x_k - \ell| \geq \varepsilon\}$  has  $\lambda$ -density of zero. In this case we write  $S_\lambda - \lim x = \ell$  or  $x_k \rightarrow \ell(S_\lambda)$  and

$$S_\lambda = \{x = (x_k) : \exists \ell \in \mathbf{R}, S_\lambda - \lim x = \ell\}.$$

**Definition 1.3.** Let  $\mu = (\mu_k)_{k=0}^\infty$  be a strictly increasing sequence of positive real numbers tending to the infinity which is

$$0 < \mu_0 < \mu_1 < \dots \text{ and } \mu_k \rightarrow \infty \text{ as } k \rightarrow \infty.$$

Mursaleen and Noman [25] introduced the notion of  $\mu$ -convergent sequences as follows: A sequence  $x = (x_k)$  is said to be  $\mu$ -convergent to the number  $l$  if  $\Lambda x_k \rightarrow l$  as  $k \rightarrow \infty$ , where

$$\Lambda x_k = \frac{1}{\mu_i} \sum_{i=0}^k (\mu_i - \mu_{i-1}) x_i$$

A sequence  $x = (x_k)$  is said to be  $\Lambda$ -statistically convergent to  $\ell$  if for every  $\varepsilon > 0$  the set  $\{k \in \mathbf{N} : |\Lambda x_k - \ell| \geq \varepsilon\}$  has natural density zero, i.e.,

$$\lim_{m \rightarrow \infty} \frac{1}{m} |\{k \leq m : |\Lambda x_k - \ell| \geq \varepsilon\}| = 0.$$

The existing literature on statistical convergence and its generalizations appears to have been restricted to real or complex sequences, but in recent years these ideas have been also extended to the sequences in fuzzy normed [26] and intuitionistic fuzzy normed spaces [27-31]. Further details on generalization of statistical convergence can be found in [11,12,32,33].

## Methods

### Preliminaries

**Definition 2.1.** A function  $f : \mathbf{R} \rightarrow \mathbf{R}_0^+$  is called a *distribution function* if it is a non-decreasing and left continuous with  $\inf_{t \in \mathbf{R}} f(t) = 0$  and  $\sup_{t \in \mathbf{R}} f(t) = 1$ . By  $D^+$

we denote the set of all distribution functions such that  $f(0) = 0$ . If  $a \in \mathbf{R}_0^+$ , then  $H_a \in D^+$ , where

$$H_a(t) = \begin{cases} 1, & \text{if } t > a; \\ 0, & \text{if } t \leq a \end{cases}$$

It is obvious that  $H_0 \geq f$  for all  $f \in D^+$ .

A  $t$ -norm is a continuous mapping  $* : [0, 1] \times [0, 1] \rightarrow [0, 1]$  such that  $([0, 1], *)$  is abelian monoid with unit one and  $c * d \geq a * b$  if  $c \geq a$  and  $d \geq b$  for all  $a, b, c \in [0, 1]$ . A triangle function  $\tau$  is a binary operation on  $D^+$ , which is commutative, associative, and  $\tau(f, H_0) = f$  for every  $f \in D^+$ .

In [22], Gähler introduced the following concept of 2-normed space.

**Definition 2.2.** Let  $X$  be a real vector space of dimension  $d > 1$  ( $d$  may be infinite). A real-valued function  $\|\cdot, \cdot\|$  from  $X^2$  into  $\mathbf{R}$  satisfying the following conditions:

- (1)  $\|x_1, x_2\| = 0$  if and only if  $x_1, x_2$  are linearly dependent,
- (2)  $\|x_1, x_2\|$  is invariant under permutation,
- (3)  $\|\alpha x_1, x_2\| = |\alpha| \|x_1, x_2\|$ , for any  $\alpha \in \mathbf{R}$ , and
- (4)  $\|x + \bar{x}, x_2\| \leq \|x, x_2\| + \|\bar{x}, x_2\|$

is called an 2-norm on  $X$ , and the pair  $(X, \|\cdot, \cdot\|)$  is called an 2-normed space.

A trivial example of an 2-normed space is  $X = \mathbf{R}^2$ , equipped with the Euclidean 2-norm  $\|x_1, x_2\|_E =$  the volume of the parallelogram spanned by the vectors  $x_1, x_2$  which may be given explicitly by the formula

$$\|x_1, x_2\|_E = |\det(x_{ij})| = \text{abs}(\det(\langle x_i, x_j \rangle))$$

where  $x_i = (x_{i1}, x_{i2}) \in \mathbf{R}^2$  for each  $i = 1, 2$ .

Recently, Golet [21] used the idea of 2-normed space to define the random 2-normed space.

**Definition 2.3.** Let  $X$  be a linear space of dimension  $d > 1$  ( $d$  may be infinite),  $\tau$  a triangle, and  $\mathcal{F} : X \times X \rightarrow D^+$ . Then  $\mathcal{F}$  is called a *probabilistic 2-norm* and  $(X, \mathcal{F}, \tau)$  a *probabilistic 2-normed space* if the following conditions are satisfied:

1.  $(P2N_1)$   $\mathcal{F}(x, y; t) = H_0(t)$  if  $x$  and  $y$  are linearly dependent, where  $\mathcal{F}(x, y; t)$  denotes the value of  $\mathcal{F}(x, y)$  at  $t \in \mathbf{R}$ ,
2.  $(P2N_2)$   $\mathcal{F}(x, y; t) \neq H_0(t)$  if  $x$  and  $y$  are linearly independent,
3.  $(P2N_3)$   $\mathcal{F}(x, y; t) = \mathcal{F}(y, x; t)$ , for all  $x, y \in X$ ,
4.  $(P2N_4)$   $\mathcal{F}(\alpha x, y; t) = \mathcal{F}(x, y; \frac{t}{|\alpha|})$ , for every  $t > 0, \alpha \neq 0$  and  $x, y \in X$ ,
5.  $(P2N_5)$   $\mathcal{F}(x + y, z; t) \geq \tau(\mathcal{F}(x, z; t), \mathcal{F}(y, z; t))$ , whenever  $x, y, z \in X$ .

If  $(P2N_5)$  is replaced by  $(P2N_6)$   $\mathcal{F}(x + y, z; t_1 + t_2) \geq \mathcal{F}(x, z; t_1) * \mathcal{F}(y, z; t_2)$ , for all  $x, y, z \in X$  and  $t_1, t_2 \in \mathbf{R}_0^+$ ; then  $(X, \mathcal{F}, *)$  is called a *random 2-normed space* (for short, R2NS).

**Remark 2.1.** Every 2-normed space  $(X, \|\cdot, \cdot\|)$  can be made a random 2-normed space in a natural way, by setting

(i)  $\mathcal{F}(x, y; t) = H_0(t - \|x, y\|)$ , for every  $x, y \in X, t > 0$  and  $a * b = \min\{a, b\}, a, b \in [0, 1]$ ;

(ii)  $\mathcal{F}(x, y; t) = \frac{t}{t + \|x, y\|}$ , for every  $x, y \in X, t > 0$  and  $a * b = ab, a, b \in [0, 1]$ .

In [34], Grdal and Pehlivan studied statistical convergence in 2-normed spaces and in 2-Banach spaces in [23]. In fact, Mursaleen [35] studied the concept of statistical convergence of sequences in random 2-normed space. Recently in [36], Esi and zdemir introduced and studied the concept of generalized  $\Delta^m$ -statistical convergence of sequences in probabilistic normed space. In [37], Hazarika introduced the generalized statistical convergence in random 2-normed spaces.

**Definition 2.4.** [35] A sequence  $x = (x_k)$  in a random 2-normed space  $(X, \mathcal{F}, *)$  is said to be *statistical convergent* or  *$S^{R2N}$  convergent* to some  $\ell \in X$  with respect to  $\mathcal{F}$  if for each  $\varepsilon > 0, \theta \in (0, 1)$  and for non zero,  $z \in X$ , such that

$$\delta(\{k \in \mathbf{N} : \mathcal{F}(x_k - \ell, z; \varepsilon) \leq 1 - \theta\}) = 0,$$

In other words we can write the sequence  $(x_k)$  *statistical converges* to  $\ell$  in random 2-normed space  $(X, \mathcal{F}, *)$  if

$$\lim_{m \rightarrow \infty} \frac{1}{m} |\{k \leq m : \mathcal{F}(x_k - \ell, z; \varepsilon) \leq 1 - \theta\}| = 0.$$

or equivalently

$$\delta(\{k \in \mathbf{N} : \mathcal{F}(x_k - \ell, z; \varepsilon) > 1 - \theta\}) = 1,$$

i.e.,

$$S - \lim_{k \rightarrow \infty} \mathcal{F}(x_k - \ell, z; \varepsilon) = 1.$$

In this case, we write  $S^{R2N} - \lim x = \ell$  and  $\ell$  is called the  $S^{R2N} - \text{limit}$  of  $x$ . Let  $S^{R2N}(X)$  denotes the set of all statistical convergent sequences in random 2-normed space  $(X, \mathcal{F}, *)$ .

**Definition 2.5.** [37] A sequence  $x = (x_k)$  in a random 2-normed space  $(X, \mathcal{F}, *)$  is said to be  $\lambda$ -*statistical convergent* or  $S_\lambda^{R2N}$ -*convergent* to some  $\ell \in X$  with respect to  $\mathcal{F}$  if for each  $\varepsilon > 0, \theta \in (0, 1)$  and for non zero  $z \in X$  such that

$$\delta_\lambda(\{k \in \mathbf{N} : \mathcal{F}(x_k - \ell, z; \varepsilon) \leq 1 - \theta\}) = 0,$$

In other words, we can write the sequence  $(x_k)$   $\lambda$ -

*statistical converges* to  $\ell$  in random 2-normed space  $(X, \mathcal{F}, *)$  if

$$\lim_{m \rightarrow \infty} \frac{1}{\lambda_m} |\{k \in I_m : \mathcal{F}(x_k - \ell, z; \varepsilon) \leq 1 - \theta\}| = 0,$$

or equivalently,

$$\delta_\lambda(\{k \in \mathbf{N} : \mathcal{F}(x_k - \ell, z; \varepsilon) > 1 - \theta\}) = 1,$$

i.e.,

$$S_\lambda - \lim_{k \rightarrow \infty} \mathcal{F}(x_k - \ell, z; \varepsilon) = 1.$$

In this case, we write  $S_\lambda^{R2N} - \lim x = \ell$  and  $\ell$  is called the  $S_\lambda^{R2N} - \text{limit}$  of  $x$ . Let  $S_\lambda^{R2N}(X)$  denotes the set of all statistical convergent sequences in random 2-normed space  $(X, \mathcal{F}, *)$ .

In this paper, we define and study  $\Lambda$ -statistical convergence in random 2-normed space which is quite a new and interesting idea to work with. We show that some properties of  $\Lambda$ -statistical convergence of real numbers also hold for sequences in random 2-normed spaces. We find some relations of  $\Lambda$ -statistical convergent sequences in random 2-normed spaces. Also, we find out the relation between  $\Lambda$ -statistical convergent and  $\Lambda$ -statistical Cauchy sequences in this spaces.

## Results and discussion

### $\Lambda$ -statistical convergence in random 2-normed space

In this section, we define  $\Lambda$ -statistical convergent sequence in random 2-normed  $(X, \mathcal{F}, *)$ . Also, we obtained some basic properties of this notion in random 2-normed space.

**Definition 3.1.** A sequence  $x = (x_k)$  in a random 2-normed space  $(X, \mathcal{F}, *)$  is said to be  $\Lambda$ -*convergent* to  $\ell \in X$  with respect to  $\mathcal{F}$  if for each  $\varepsilon > 0, \theta \in (0, 1)$  there exists a positive integer  $n_0$  such that  $\mathcal{F}(\Lambda x_k - \ell, z; \varepsilon) > 1 - \theta$ , whenever  $k \geq n_0$  and for non zero  $z \in X$ . In this case, we write  $\mathcal{F} - \lim_k \Lambda x_k = \ell$ , and  $\ell$  is called the  $\mathcal{F}_\Lambda$ -*limit* of  $x = (x_k)$ .

**Definition 3.2.** A sequence  $x = (x_k)$  in a random 2-normed space  $(X, \mathcal{F}, *)$  is said to be  $\Lambda$ -*Cauchy* with respect to  $\mathcal{F}$  if for each  $\varepsilon > 0, \theta \in (0, 1)$  there exists a positive integer  $n_0 = n_0(\varepsilon)$  such that  $\mathcal{F}(\Lambda x_k - \Lambda x_s, z; \varepsilon) < 1 - \theta$ , whenever  $k, s \geq n_0$  and for non zero  $z \in X$ .

**Definition 3.3.** A sequence  $x = (x_k)$  in a random 2-normed space  $(X, \mathcal{F}, *)$  is said to be  $\Lambda$ -*statistically convergent* or  $S_\Lambda$ -*convergent* to  $\ell \in X$  with respect to  $\mathcal{F}$  if for every  $\varepsilon > 0, \theta \in (0, 1)$  and for non zero  $z \in X$  such that

$$\delta_\Lambda(\{k \in I_m : \mathcal{F}(\Lambda x_k - \ell, z; \varepsilon) \leq 1 - \theta\}) = 0.$$

In other ways, we can write

$$\lim_{m \rightarrow \infty} \frac{1}{\lambda_m} |\{k \in I_m : \mathcal{F}(\Lambda x_k - \ell, z; \varepsilon) \leq 1 - \theta\}| = 0,$$

or equivalently,

$$\delta_\Lambda(\{k \in I_m : \mathcal{F}(\Lambda x_k - \ell, z; \varepsilon) > 1 - \theta\}) = 1,$$

i.e.,

$$S_\Lambda - \lim_{k \rightarrow \infty} \mathcal{F}(\Lambda x_k - \ell, z; \varepsilon) = 1.$$

In this case, we write  $S_\Lambda^{R2N} - \lim x = \ell$  or  $x_k \rightarrow \ell(S_\Lambda^{R2N})$  and

$$S_\Lambda^{R2N}(X) = \{x = (x_k) : \exists \ell \in \mathbf{R}, S_\Lambda^{R2N} - \lim x = \ell\}.$$

Let  $S_\Lambda^{R2N}(X)$  denotes the set of all  $\Lambda$ -statistical convergent sequences in random 2-normed space  $(X, \mathcal{F}, *)$ .

**Definition 3.4.** A sequence  $x = (x_k)$  in a random 2-normed space  $(X, \mathcal{F}, *)$  is said to be  $\Lambda$ -statistical Cauchy with respect to  $\mathcal{F}$  if for every  $\varepsilon > 0, \theta \in (0, 1)$  and for non zero  $z \in X$ , there exists a positive integer  $n = n(\varepsilon)$  such that for all  $k, s \geq n$

$$\delta_\Lambda(\{k \in I_m : \mathcal{F}(\Lambda x_k - \Lambda x_s, z; \varepsilon) \leq 1 - \theta\}) = 0,$$

or equivalently,

$$\delta_\Lambda(\{k \in I_m : \mathcal{F}(\Lambda x_k - \Lambda x_s, z; \varepsilon) > 1 - \theta\}) = 1.$$

Definition 3.3 immediately implies the following Lemma.

**Lemma 3.1.** Let  $(X, \mathcal{F}, *)$  be a random 2-normed space. If  $x = (x_k)$  is a sequence in  $X$ , then for every  $\varepsilon > 0, \theta \in (0, 1)$  and for non zero  $z \in X$ , then the following statements are equivalent:

- (i)  $S_\Lambda - \lim_{k \rightarrow \infty} x_k = \ell$ .
- (ii)  $\delta_\Lambda(\{k \in I_m : \mathcal{F}(\Lambda x_k - \ell, z; \varepsilon) \leq 1 - \theta\}) = 0$ .
- (iii)  $\delta_\Lambda(\{k \in I_m : \mathcal{F}(\Lambda x_k - \ell, z; \varepsilon) > 1 - \theta\}) = 1$ .
- (iv)  $S_\Lambda - \lim_{k \rightarrow \infty} \mathcal{F}(\Lambda x_k - \ell, z; \varepsilon) = 1$ .

**Theorem 3.2.** Let  $(X, \mathcal{F}, *)$  be a random 2-normed space. If  $x = (x_k)$  is a sequence in  $X$  such that  $S_\Lambda^{R2N} - \lim x_k = \ell$  exists, then it is unique.

*Proof.* Suppose that there exist elements  $\ell_1, \ell_2$  ( $\ell_1 \neq \ell_2$ ) in  $X$  such that

$$S_\Lambda - \lim_{k \rightarrow \infty} x_k = \ell_1; S_\Lambda^{R2N} - \lim_{k \rightarrow \infty} x_k = \ell_2.$$

□

Let  $\varepsilon > 0$  be given. Choose  $r > 0$  such that

$$(1 - r) * (1 - r) > 1 - \varepsilon. \tag{3.1}$$

Then, for any  $t > 0$  and for non zero  $z \in X$ , we define

$$K_1(r, t) = \left\{ k \in I_m : \mathcal{F} \left( \Lambda x_k - \ell_1, z; \frac{t}{2} \right) \leq 1 - r \right\};$$

$$K_2(r, t) = \left\{ k \in I_m : \mathcal{F} \left( \Lambda x_k - \ell_2, z; \frac{t}{2} \right) \leq 1 - r \right\}.$$

Since  $S_\Lambda^{R2N} - \lim_{k \rightarrow \infty} x_k = \ell_1$  and  $S_\Lambda^{R2N} - \lim_{k \rightarrow \infty} x_k = \ell_2$ , we have

$$\delta_\Lambda(K_1(r, t)) = 0 \text{ and } \delta_\Lambda(K_2(r, t)) = 0 \text{ for all } t > 0.$$

Now let  $K(r, t) = K_1(r, t) \cup K_2(r, t)$ , then it is easy to observe that  $\delta_\Lambda(K(r, t)) = 0$ . But we have  $\delta_\Lambda(K^c(r, t)) = 1$ .

Now if  $k \in K^c(r, t)$ , then we have

$$\begin{aligned} \mathcal{F}(\ell_1 - \ell_2, z; t) &\geq \mathcal{F} \left( \Lambda x_k - \ell_1, z; \frac{t}{2} \right) * \mathcal{F} \left( \Lambda x_k - \ell_2, z; \frac{t}{2} \right) \\ &> (1 - r) * (1 - r). \end{aligned}$$

It follows by (3.1) that

$$\mathcal{F}(\ell_1 - \ell_2, z; t) > (1 - \varepsilon).$$

Since  $\varepsilon > 0$  was arbitrary, we get  $\mathcal{F}(\ell_1 - \ell_2, z; t) = 1$  for all  $t > 0$  and non zero  $z \in X$ . Hence  $\ell_1 = \ell_2$ . The next theorem gives the algebraic characterization of  $\Lambda$ -statistical convergence on random 2-normed spaces.

**Theorem 3.3.** Let  $(X, \mathcal{F}, *)$  be a random 2-normed space, and  $x = (x_k)$  and  $y = (y_k)$  be two sequences in  $X$ :

- (a) If  $S_\Lambda^{R2N} - \lim x_k = \ell$  and  $c (\neq 0) \in \mathbf{R}$ , then  $S_\Lambda^{R2N} - \lim cx_k = c\ell$ .
- (b) If  $S_\Lambda - \lim x_k = \ell_1$  and  $S_\Lambda^{R2N} - \lim y_k = \ell_2$ , then  $S_\Lambda^{R2N} - \lim (x_k + y_k) = \ell_1 + \ell_2$ .

The proof of the theorem is straightforward, thus omitted.

**Theorem 3.4.** Let  $(X, \mathcal{F}, *)$  be a random 2-normed space. If  $x = (x_k)$  be a sequence in  $X$  such that  $\mathcal{F}_\Lambda - \lim x_k = \ell$ , then  $S_\Lambda^{R2N} - \lim x_k = \ell$ .

*Proof.* Let  $\mathcal{F}_\Lambda - \lim x_k = \ell$ . Then for every  $\varepsilon > 0, t > 0$  and non zero  $z \in X$ , there is a positive integer  $n_0$  such that

$$\mathcal{F}(\Lambda x_k - \ell, z; t) > 1 - \varepsilon$$

for all  $k \geq n_0$ . Since the set

$$K(\varepsilon, t) = \{k \in I_m : \mathcal{F}(\Lambda x_k - \ell, z; t) \leq 1 - \varepsilon\}$$

has, at most, many finite terms. Also, since every finite subset of  $\mathbf{N}$  has  $\delta_\Lambda$ -density zero, consequently, we have  $\delta_\Lambda(K(\varepsilon, t)) = 0$ . This shows that  $S_\Lambda^{R2N} - \lim x_k = \ell$ . □

**Remark 3.5.** The converse of the above theorem is not true in general. It follows from the following example.

**Example 3.6.** Let  $X = \mathbb{R}^2$ , with the 2-norm  $\|x, z\| = |x_1z_2 - x_2z_1|$ ,  $x = (x_1, x_2)$ ,  $z = (z_1, z_2)$ , and  $a \times b = ab$  for all  $a, b \in [0, 1]$ . Let  $\mathcal{F}(x, y; t) = \frac{t}{t + \|x, y\|}$ , for all  $x, z \in X$ ,  $z_2 \neq 0$ , and  $t > 0$ . Now we define a sequence  $x = (x_k)$  by

$$\Delta x_k = \begin{cases} (k, 0), & \text{if } m - [\sqrt{\lambda_m}] + 1 \leq k \leq m; \\ (0, 0), & \text{otherwise.} \end{cases}$$

Nor for every  $0 < \varepsilon < 1$  and  $t > 0$ , we write

$$\begin{aligned} K(\varepsilon, t) &= \{k \in I_m : \mathcal{F}(\Delta x_k - \ell, z; t) \leq 1 - \varepsilon, \ell = (0, 0)\} \\ &= \{k \in I_m : \frac{t}{t + |\Delta x_k|} \leq 1 - \varepsilon\} \\ &= \{k \in I_m : |\Delta x_k| \geq \frac{t\varepsilon}{1 - \varepsilon} > 0\} \\ &= \{k \in I_m : m - [\sqrt{\lambda_m}] + 1 \leq k \leq m\}, \end{aligned}$$

so we get

$$\begin{aligned} \frac{1}{\lambda_m} |K(\varepsilon, t)| &\leq \frac{1}{\lambda_m} |\{k \in I_m : m - [\sqrt{\lambda_m}] + 1 \leq k \leq m\}| \\ &\leq \frac{[\sqrt{\lambda_m}]}{\lambda_m}. \end{aligned}$$

Taking the limit  $m$  which approaches to  $\infty$ , we get

$$\delta_\Delta(K(\varepsilon, t)) = \lim_{m \rightarrow \infty} \frac{1}{\lambda_m} |K(\varepsilon, t)| \leq \lim_{m \rightarrow \infty} \frac{[\sqrt{\lambda_m}]}{\lambda_m} = 0.$$

This shows that  $x_k \rightarrow 0(S_\Delta^{R2N}(X))$ .

On the other hand, the sequence is not  $\mathcal{F}_\Delta$ -convergent to zero as

$$\begin{aligned} \mathcal{F}(\Delta x_k - \ell, z; t) &= \frac{t}{t + |\Delta x_k|} \\ &= \begin{cases} \frac{t}{t + kz_2}, & \text{if } m - [\sqrt{\lambda_m}] + 1 \leq k \leq m; \\ 1, & \text{otherwise.} \end{cases} \leq 1. \end{aligned}$$

**Theorem 3.7.** Let  $(X, \mathcal{F}, *)$  be a random 2-normed space. If  $x = (x_k)$  be a sequence in  $X$ , then  $S_\Delta^{R2N} - \lim x_k = \ell$  if and only if there exists a subset  $K \subseteq \mathbb{N}$ , such that  $\delta_\Delta(K) = 1$  and  $\mathcal{F}_\Delta - \lim x_k = \ell$ .

*Proof.* Suppose first that  $S_\Delta^{R2N} - \lim x_k = \ell$ . Then for any  $t > 0$ ,  $r = 1, 2, 3, \dots$  and non zero  $z \in X$ , let

$$A(r, t) = \left\{ k \in I_m : \mathcal{F}(\Delta x_k - \ell, z; t) > 1 - \frac{1}{r} \right\}$$

and

$$K(r, t) = \left\{ k \in I_m : \mathcal{F}(\Delta x_k - \ell, z; t) \leq 1 - \frac{1}{r} \right\}.$$

□

Since  $S_\Delta^{R2N} - \lim x_k = \ell$ , it follows that

$$\delta_\Delta(K(r, t)) = 0.$$

Now for  $t > 0$  and  $r = 1, 2, 3, \dots$ , we observe that

$$A(r, t) \supset A(r + 1, t)$$

and

$$\delta_\Delta(A(r, t)) = 1. \tag{3.2}$$

Now we have to show that for  $k \in A(r, t)$ ,  $\mathcal{F}_\Delta - \lim x_k = \ell$ . Suppose that for  $k \in A(r, t)$ ,  $(x_k)$  is not convergent to  $\ell$  with respect to  $\mathcal{F}_\Delta$ . Then, there exists some  $s > 0$  such that

$$\{k \in I_m : \mathcal{F}(\Delta x_k - \ell, z; t) \leq 1 - s\}$$

for infinitely many terms  $x_k$ . Let

$$A(s, t) = \{k \in I_m : \mathcal{F}(\Delta x_k - \ell, z; t) > 1 - s\}$$

and

$$s > \frac{1}{r}, r = 1, 2, 3, \dots$$

Then we have

$$\delta_\Delta(A(s, t)) = 0.$$

Furthermore,  $A(r, t) \subset A(s, t)$  implies that  $\delta_\Delta(A(r, t)) = 0$ , which contradicts (3.2) as  $\delta_\Delta(A(r, t)) = 1$ . Hence,  $\mathcal{F}_\Delta - \lim x_k = \ell$ .

Conversely, suppose that there exists a subset  $K \subseteq \mathbb{N}$  such that  $\delta_\Delta(K) = 1$  and  $\mathcal{F}_\Delta - \lim x_k = \ell$ .

Then for every  $\varepsilon > 0$ ,  $t > 0$  and non zero  $z \in X$ , we can find out a positive integer  $n$  such that

$$\mathcal{F}(\Delta x_k - \ell, z; t) > 1 - \varepsilon$$

for all  $k \geq n$ . If we take

$$K(\varepsilon, t) = \{k \in I_m : \mathcal{F}(\Delta x_k - \ell, z; t) \leq 1 - \varepsilon\},$$

then it is easy to see that

$$K(\varepsilon, t) \subset \mathbb{N} - \{n_{k+1}, n_{k+2}, \dots\}$$

and consequently,

$$\delta_\Delta(K(\varepsilon, t)) \leq 1 - \varepsilon.$$

Hence,  $S_\Delta^{R2N} - \lim x_k = \ell$ .

Finally, we establish the Cauchy convergence criteria in random 2-normed spaces.

**Theorem 3.8.** Let  $(X, \mathcal{F}, *)$  be a random 2-normed space. Then a sequence  $(x_k)$  in  $X$  is  $\Delta$ -statistically convergent if and only if it is  $\Delta$ -statistically Cauchy.

*Proof.* Let  $(x_k)$  be a  $\Delta$ -statistically convergent sequence in  $X$ . We assume that  $S_\Delta^{R2N} - \lim x_k = \ell$ . Let  $\varepsilon > 0$  be given. Choose  $r > 0$  such that (3.1) is satisfied. For  $t > 0$  and for non zero  $z \in X$ , we define

$$A(r, t) = \left\{ k \in I_m : \mathcal{F}(\Delta x_k - \ell, z; \frac{t}{2}) \leq 1 - r \right\}$$

and

$$A^c(r, t) = \left\{ k \in I_m : \mathcal{F}(\Lambda x_k - \ell, z; \frac{t}{2}) > 1 - r \right\}.$$

□

Since  $S_{\Lambda}^{R2N} - \lim x_k = \ell$ , it follows that  $\delta_{\Lambda}(A(r, t)) = 0$  and consequently,  $\delta_{\Lambda}(A^c(r, t)) = 1$ . Let  $p \in A^c(r, t)$ . Then

$$\mathcal{F}(\Lambda x_k - \ell, z; \frac{t}{2}) \leq 1 - r. \tag{3.3}$$

If we take

$$B(\varepsilon, t) = \{k \in I_m : \mathcal{F}(\Lambda x_k - \ell, z; t) \leq 1 - \varepsilon\}$$

then to prove the result, it is sufficient to prove that  $B(\varepsilon, t) \subseteq A(r, t)$ . Let  $n \in B(\varepsilon, t)$ , then for non zero  $z \in X$

$$\mathcal{F}(\Lambda x_n - \Lambda x_p, z; t) \leq 1 - \varepsilon. \tag{3.4}$$

If  $\mathcal{F}(\Lambda x_n - \Lambda x_p, z; t) \leq 1 - \varepsilon$ , then we have  $\mathcal{F}(\Lambda x_n - \ell, z; \frac{t}{2}) \leq 1 - r$  and therefore  $n \in A(r, t)$ . As otherwise i.e., if  $\mathcal{F}(\Lambda x_n - \ell, z; \frac{t}{2}) > 1 - r$  then by (3.1), (3.3), and (3.4) we get

$$\begin{aligned} 1 - \varepsilon &\geq \mathcal{F}(\Lambda x_n - \Lambda x_p, z; t) \\ &\geq \mathcal{F}(\Lambda x_n - \ell, z; \frac{t}{2}) * \mathcal{F}(\Lambda x_p - \ell, z; \frac{t}{2}) \\ &> (1 - r) * (1 - r) > (1 - \varepsilon) \end{aligned}$$

which is not possible. Thus,  $B(\varepsilon, t) \subset A(r, t)$ . Since  $\delta_{\Lambda}(A(r, t)) = 0$ , it follows that  $\delta_{\Lambda}(B(\varepsilon, t)) = 0$ . This shows that  $(x_k)$  is  $\Lambda$ -statistically Cauchy. Conversely, suppose  $(x_k)$  is  $\Lambda$ -statistically Cauchy but not  $\Lambda$ -statistically convergent. Then there exists positive integer  $p$  and for non zero  $z \in X$  such that if we take

$$A(\varepsilon, t) = \{k \in I_m : \mathcal{F}(\Lambda x_k - \Lambda x_p, z; t) \leq 1 - \varepsilon\}$$

and

$$B(\varepsilon, t) = \{k \in I_m : \mathcal{F}(\Lambda x_k - \ell, z; \frac{t}{2}) > 1 - \varepsilon\}$$

then

$$\delta_{\Lambda}(A(\varepsilon, t)) = 0 = \delta_{\Lambda}(B(\varepsilon, t))$$

and consequently,

$$\delta_{\Lambda}(A^c(\varepsilon, t)) = 1 = \delta_{\Lambda}(B^c(\varepsilon, t)). \tag{3.5}$$

Since

$$\mathcal{F}(\Lambda x_n - \Lambda x_p, z; t) \geq 2\mathcal{F}(\Lambda x_k - \ell, z; \frac{t}{2}) > 1 - \varepsilon;$$

if  $\mathcal{F}(\Lambda x_k - \ell, z; \frac{t}{2}) > \frac{1-\varepsilon}{2}$ , then we have

$$\delta_{\Lambda}(\{k \in I_m : \mathcal{F}(\Lambda x_n - \Lambda x_p, z; t) > 1 - \varepsilon\}) = 0$$

i.e.,  $\delta_{\Lambda}(A^c(\varepsilon, t)) = 0$ , which contradicts (3.5) as  $\delta_{\Lambda}(A^c(\varepsilon, t)) = 1$ . Hence,  $(x_k)$  is  $\Lambda$ -statistically convergent.

Combining Theorem 3.7 and Theorem 3.8 we get the following corollary.

**Corollary 3.9.** *Let  $(X, \mathcal{F}, *)$  be a random 2-normed space and  $x = (x_k)$  be a sequence in  $X$ . Then the following statements are equivalent:*

- (a)  $x$  is  $\Lambda$ -statistically convergent.
- (b)  $x$  is  $\Lambda$ -statistically Cauchy.
- (c) There exists a subset  $K \subseteq \mathbb{N}$  such that  $\delta_{\Lambda}(K) = 1$  and  $\mathcal{F}_{\Lambda} - \lim x_k = \ell$ .

**$\lambda$ -statistical limit points and statistical cluster points on random-2-normed space**

In this section, we will define the  $\Lambda$ - statistical limit points and cluster point and we will give connection between this classes.

**Definition 4.1.** Let  $(X, \mathcal{F}, *)$  be a  $R2N$ -space.  $l \in X$  is called a  $\Lambda$ - limit point of the sequence  $x = (x_k)$  with respect to  $\mathcal{F}$  provided that there is a subsequence of  $x$  that  $\Lambda$ - converges to  $l$  with respect to  $\mathcal{F}$ .

We will denote by  $L_{\Lambda}^{R2N}(x)$  the set of all  $\Lambda$ - limit points of the sequence  $x = (x_k)$ .

**Definition 4.2.** Let  $(X, \mathcal{F}, *)$  be a  $R2N$ -space. Then  $\xi \in X$  is called a  $\Lambda$ - statistical limit point of the sequence  $x = (x_k)$  with respect to  $\mathcal{F}$  provided that there is a subsequence  $\{x_{k(j)}\}$  of  $x = (x_k)$  that  $\Lambda$ - converges to  $l$  with respect to  $\mathcal{F}$  and  $\delta_{\Lambda}(K) \neq 0$ , where  $K = \{k(j) \in \mathbb{N} : j \in \mathbb{N}\}$ .

We will denote by  $\Lambda_{\Lambda}^{R2N}(x)$  the set of all  $\Lambda$ - statistical limit points of the sequence  $x = (x_k)$ .

**Definition 4.3.** Let  $(X, \mathcal{F}, *)$  be a  $R2N$ -space. Then  $\vartheta \in X$  is called a  $\Lambda$ - statistical cluster point of the sequence  $x = (x_k)$  with respect to  $\mathcal{F}$  provided that for every  $\varepsilon > 0, \lambda \in (0, 1)$ , and  $z \in X \setminus \{0\}$ ,

$$\overline{\delta_{\Lambda}}(\{k \in \mathbb{N} : \mathcal{F}(x_k - \vartheta, z; \varepsilon) > 1 - \lambda\}) > 0.$$

We will denote by  $\Gamma_{\Lambda}^{R2N}(x)$  the set of all  $\Lambda$ - statistical cluster of the sequence  $x = (x_k)$ .

**Theorem 4.1.** *Let  $(X, \mathcal{F}, *)$  be a  $R2N$ -space. For any  $x \in X, \Lambda_{\Lambda}^{R2N}(x) \subset \Gamma_{\Lambda}^{R2N}(x)$ .*

**Theorem 4.2.** *Let  $(X, \mathcal{F}, *)$  be a  $R2N$ -space. For any  $x \in X, \Gamma_{\Lambda}^{R2N}(x) \subseteq L_{\Lambda}^{R2N}(x)$ .*

**Theorem 4.3.** *Let  $(X, \mathcal{F}, *)$  be a  $R2N$ -space. For a sequence  $x = (x_k)$ , if  $S_{\Lambda}^{R2N} - \lim x = x_0$ , then we get  $\Gamma_{\Lambda}^{R2N}(x) = \Lambda_{\Lambda}^{R2N}(x) = \{x_0\}$ .*

**Conclusions**

In this paper, we have define and studied the notion of  $\Lambda$ -statistical convergence and  $\Lambda$ -statistical Cauchy sequences in random 2-normed spaces , where  $\lambda = (\lambda_m)$

be a non-decreasing sequence of positive numbers tending to infinity such that  $\lambda_{m+1} \leq \lambda_m + 1$ ,  $\lambda_1 = 1$  and we have proved some theorems. In last section we have given the definition of the  $\Lambda$ -limit and cluster points and we have shown their relation between those classes.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

AE has introduced and studied the concept  $\lambda$ -statistical convergence in random 2-normed space. NB has introduced and studied the concept of  $\lambda$ -statistical limit points and statistical cluster points on random-2-normed space. Both authors read and approved the final manuscript.

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