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Invariant naturally reductive Randers metrics on homogeneous spaces

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Abstract

Purpose: The purpose of this paper is to study the geometric properties of naturally reductive homogeneous Randers spaces.

Methods: We use Lie theory methods in the study of Finsler geometry.

Results: We first prove that if a Randers metric is naturally reductive, then the underlying Riemannian metric is naturally reductive. Then, we show that, for Berwald type Randers metric, if the underlying Riemannian metric is naturally reductive, then the Randers metric is naturally reductive. Finally, we give a geometric criterion of homogeneous naturally reductive Randers spaces.

Conclusions: This paper provides a convenient method to construct naturally reductive Randers metrics on homogeneous Riemannian manifolds.

Keywords: Invariant randers metric, Naturally reductive metric, Homogeneous geodesic, Geodesic vector

MSC: 53C60; 53C30

Introduction

Nowadays, the concept of *homogeneity* is one of the fundamental notions in geometry although its means must be specified for the concrete situations. In this paper, we consider the homogeneity of Randers manifolds. This kind of homogeneity means that, for every smooth Randers manifold (M, F), its group of isometries $I(M, F)$ is acting transitively on M.

The notion of naturally reductive Riemannian metrics was first introduced by Kobayashi and Nomizu [1]. Among the Riemannian homogeneous metrics, the naturally reductive ones are the simplest kind. They have nice and simple geometric properties but still form a large enough class to be of interest. The naturally reductive spaces have been investigated by a number of authors as a natural generalization of Riemannian symmetric spaces. A general theory with many examples was well-developed by D'Atri and Ziller [2]. Among non-symmetric examples, there are all isotropy irreducible homogeneous spaces studied by Wolf [3] and all three-symmetric spaces from the classification list by Gray [4]. D'Atri and Ziller have classified naturally reductive compact Lie groups with left-invariant metrics in [2]. For a treatment of the non-compact semisimple case, see Gordon [5].

The purpose of this paper is to study the naturally reductive homogeneous Randers spaces. The definition of naturally reductive homogeneous Finsler spaces is a natural generalization of the definition of naturally reductive Riemannian homogeneous spaces. This definition was given by the first author in [6]. We study the geometric properties of naturally reductive homogeneous Randers spaces.

Methods

In this section, we shall recall some well-known facts about Finsler geometry (see [7] for more details). Let M be an n-dimensional smooth manifold without boundary and TM denote its tangent bundle. A Finsler structure on M is a map $F : TM \longrightarrow [0, \infty)$ which has the following properties:

1. F is smooth on $\widetilde{TM} := TM \backslash \{0\}$.
2. $F(x, \lambda y) = \lambda F(x, y)$, for any $x \in M, y \in T_xM$ and $\lambda > 0$.

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3. F^2 is strongly convex, i.e.,

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x,y)$$

is positive definite for all $(x,y) \in \widetilde{TM}$.

For any Finsler metric F, we define

$$C_{ijk} = \frac{1}{4}[F^2]_{y^i y^j y^k}.$$

Then, for any $y \neq 0$, we can define two tensor, namely,

$$g_y(u,v) = g_{ij}u^i v^j,$$

$$C_y(u,v,w) = C_{ijk}(y)u^i v^j w^k.$$

They are called the fundamental form and the Cartan torsion, respectively.

An important family of Finsler metrics introduced by Randers [8] in 1941 is Randers metrics with the following form:

$$F(x,y) = \sqrt{a_{ij}(x)y^i y^j} + b_i(x)y^i,$$

where $a = a_{ij}dx^i \otimes dx^j$ is a Riemannian metric, and $b = b_i dx^i$ is a 1-form on M. It has been shown that F is a Finsler metric if and only if $\| b \| = b_i(x)b^i(x) < 1$ [7]. The Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ induces the musical bijections between 1-form and vector fields on M. In this way, the 1-form b corresponds to a vector field $b^\sharp$ on M. Thus, a Randers metric F with Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and 1-form b can be showed by the following:

$$F(x,y) = \sqrt{a_x(y,y)} + a_x(b^\sharp, y) \qquad x \in M, y \in T_xM,$$

where $a_x(b^\sharp, b^\sharp) < 1 \qquad \forall x \in M.$

Let G be a Lie group and H be a closed subgroup of G. The coset space G/H has a unique smooth structure such that G is a Lie transformation group of G/H. It is called reductive if there exists a subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of G such that

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m},$$

where $\mathfrak{h}$ is the Lie algebra of H and $Ad(h)\mathfrak{m} \subset \mathfrak{m}, \forall h \in H$.

The study of invariant structures on homogeneous manifolds is an important problem of differential geometry. Deng and Hou studied invariant Finsler metrics on homogeneous spaces and gave some descriptions of these metrics [9]. Also, in [6,10-13], we have studied the homogeneous Finsler spaces and the homogeneous geodesics in homogeneous Finsler spaces. In [14], we study the geometry of Lie groups with bi-invariant Randers metrics.

Let G/H be a reductive homogeneous manifold with the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then, there is a one-to-one correspondence between the G-invariant Finsler metric on G/H and the Minkowski norm on $\mathfrak{m}$ satisfying $F(Ad(h)x) = F(x), \forall h \in H, x \in \mathfrak{m}$ [9].

Let G/H be a reductive homogeneous manifold. Let $a = a_{ij}dx^i \otimes dx^j$ be an invariant Riemannian metric on G/H. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to the inner product induced on $\mathfrak{g}$ by $a = a_{ij}dx^i \otimes dx^j$. Then, there exists a bijection between the set of all invariant Randers metrics on G/H with the underlying Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and the set [9]

$$V = \{X \in \mathfrak{m} | Ad(h)X = X, \langle X, X \rangle < 1, \forall h \in H\},$$

where $\langle , \rangle$ is the inner product induced by $a = a_{ij}dx^i \otimes dx^j$. Then, for any $X \in V$, there exists an invariant Randers metric on G/H by the following formula:

$$F(xH, y) = \sqrt{a_{xH}(y,y)} + a_{xH}(\tilde{X}, y), \qquad y \in T_{xH}(G/H),$$

where $\tilde{X}$ is the corresponding invariant vector field on G/H to X.

Naturally reductive spaces

A Riemannian homogeneous space $(G/H, F)$ with its origin $p = \{H\}$ is always a reductive homogeneous space in the following sense [1,15]: we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebra of G and H, respectively, and consider the adjoint representation $Ad : H \times \mathfrak{g} \longrightarrow \mathfrak{g}$ of H on $\mathfrak{g}$. There is a direct sum decomposition (reductive decomposition) of the form $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ where $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $Ad(H)(\mathfrak{m}) \subset \mathfrak{m}$. For a fixed reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, there is a natural identification of $\mathfrak{m} \subset \mathfrak{g} = T_eG$ with the tangent space T_pM via the projection $\pi : G \longrightarrow G/H = M$. Using this natural identification and the scalar product g_p on T_pM, we obtain a scalar product $\langle , \rangle$ on $\mathfrak{m}$ which is obviously $Ad(H)$-invariant.

Furthermore, note that if $(M = G/H, g)$ is a homogeneous space, for each $Z \in \mathfrak{g}$, the mapping $(t,q) \in R \times M \longrightarrow \tau(\exp(tZ))(q)$ is a one-parameter group of isometries. Consequently, it induces a Killing vector field Z^*, called the fundamental vector field on M, given by

$$Z^*_q = \frac{d}{dt}|_0(\tau(\exp(tZ))(q)), \quad q \in M.$$

The following definition is well known from [1].

Definition 3.1. A Riemannian homogeneous space $(G/H, g)$ is said to be naturally reductive if there exists a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ of $\mathfrak{g}$ satisfying the condition

$$\langle [X,Z]_\mathfrak{m}, Y \rangle + \langle X, [Z,Y]_\mathfrak{m} \rangle = 0 \ \textit{for all} \ X, Y, Z \in \mathfrak{m}. \tag{1}$$

Here, the subscript $\mathfrak{m}$ indicates the projection of an element of $\mathfrak{g}$ into $\mathfrak{m}$.

Remark 3.1. It is also well known that the condition (1) is equivalent to the following more geometrical property:

For any vector $X \in \mathfrak{m}\backslash\{0\}$, the curve $\gamma(t) = \tau(\exp tX)(p)$ is a geodesic with respect to the Riemannian connection.

Here, *exp* and $\tau(h)$ denote the Lie exponential map of G and the left transformation of G/H induced by $h \in G$, respectively. Thus, for a naturally reductive homogeneous space, every geodesic on $(G/H, g)$ is an orbit of a one-parameter subgroup of the group of isometries.

Let $(G/H, g)$ be a homogeneous Riemannian manifold with a fixed origin p, and

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$$

a reductive decomposition. A homogeneous geodesic through the origin $p \in G/H$ is a geodesic $\gamma(t)$ which is an orbit of a one-parameter subgroup of G, that is

$$\gamma(t) = \exp(tZ)(p), \quad t \in R,$$

where Z is a nonzero vector of $\mathfrak{g}$.

A non-zero vector $Z \in \mathfrak{g}$ is called a geodesic vector if the curve $\exp(tZ)(p)$ is a geodesic. Kowalski and Vanhecke [16,17] proved the following characterization of geodesic vectors.

Lemma 3.1. *A non-zero vector $Z \in \mathfrak{g}$ is a geodesic if and only if*

$$\langle [Z, Y]_{\mathfrak{m}}, Z_{\mathfrak{m}} \rangle = 0$$

for all $Y \in \mathfrak{m}$.

Results and discussion

The scheme is based on treating the geometry of coset manifolds G/H as a generalization of the geometry of Lie group G (since G/H reduces to G when $H = \{0\}$). From this viewpoint, the isomorphism $\mathfrak{m} \simeq T_o(G/H)$ generalizes the canonical isomorphism $\mathfrak{g} \simeq T_eG$, and a G-invariant Riemannian metric on G/H generalizes a left-invariant metric on G. The notion of a bi-invariant Riemannian metric on G generalizes as the notion of a naturally reductive homogeneous Riemannian space. In fact, when $H = \{e\}$, and hence $\mathfrak{m} = \mathfrak{g}$, the condition (1) is just the condition

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0, \tag{2}$$

for a bi-invariant Riemannian metric on G.

Let G be a connected Lie group. Then, there exists a bi-invariant Finsler metric on G if and only if there exists a Minkowski norm F on $\mathfrak{g}$ such that

$$g_y([x, u], v) + g_y(u, [x, v]) + 2C_y([x, y], u, v) = 0, \tag{3}$$

where $y \in \mathfrak{g} - \{0\}, x, u, v \in \mathfrak{g}$.

It is easy to see that the condition (3) is the natural generalization of (2). The following definition of naturally reductive homogeneous Finsler space was introduced by the first author in [6].

Definition 4.1. ([6]). A homogeneous manifold G/H with an invariant Finsler metric F is called naturally reductive if there exists an $Ad(H)$ - invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that

$$g_y([x, u]_{\mathfrak{m}}, v) + g_y(u, [x, v]_{\mathfrak{m}}) + 2C_y([x, y]_{\mathfrak{m}}, u, v) = 0, \tag{4}$$

where $y \neq 0, x, u, v \in \mathfrak{m}$.

Evidently, this definition is the natural generalization of (1). On the other hand, when $H = \{e\}$, hence $\mathfrak{m} = \mathfrak{g}$, this formula is just (3).

Theorem 4.1. *Let (M, F) be a homogeneous Randers space with F defined by the Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and the vector field X. If (M, F) is naturally reductive, then the underlying Riemannian metric (M, a) is naturally reductive.*

Proof. Let (M, F) be naturally reductive, i.e., for all $y \neq 0, u, v, z \in \mathfrak{m}$,

$$g_y([z, u]_{\mathfrak{m}}, v) + g_y(u, [z, v]_{\mathfrak{m}}) + 2C_y([z, y]_{\mathfrak{m}}, u, v) = 0.$$

Thus,

$$g_y([y, u]_{\mathfrak{m}}, v) + g_y(u, [y, v]_{\mathfrak{m}}) = 0.$$

(M, F) and (M, a) have the same geodesics and for all $0 \neq y \in \mathfrak{m}$, y is a geodesic vector, so for all $y \in \mathfrak{m}$, we have

$$a(X, [y, \mathfrak{g}]_{\mathfrak{m}}) = 0.$$

□

By definition,

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial r \partial s} F^2(y + ru + sv)|_{r=s=0}.$$

Thus, by a direct computation, we get

$$\begin{aligned} g_y(u, v) = {} & a(u, v) + a(X, u)a(X, v) + \frac{a(u, v)a(X, y)}{\sqrt{a(y, y)}} \\ & - \frac{a(v, y)a(u, y)a(X, y)}{a(y, y)\sqrt{(}a(y, y))} \\ & + \frac{a(X, v)a(u, y)}{\sqrt{a(y, y)}} + \frac{a(X, u)a(v, y)}{\sqrt{a(y, y)}}. \end{aligned} \tag{5}$$

Therefore,

$$g_y([y, u]_{\mathfrak{m}}, v) = a([y, u]_{\mathfrak{m}}, v)\left(1 + \frac{a(X, y)}{\sqrt{a(y, y)}}\right),$$

$$g_y(u,[y,v]_{\mathfrak{m}}) = a(u,[y,v]_{\mathfrak{m}})\left(1+\frac{a(X,y)}{\sqrt{a(y,y)}}\right).$$

Thus, we have

$$\begin{aligned} 0 &= g_y([y,u]_{\mathfrak{m}},v) + g_y(u,[y,v]_{\mathfrak{m}}) \\ &= (a([y,u]_{\mathfrak{m}},v) + a(u,[y,v]_{\mathfrak{m}}))\left(1+\frac{a(X,y)}{\sqrt{a(y,y)}}\right). \end{aligned}$$

We easily see that $\left(1+\frac{a(X,y)}{\sqrt{a(y,y)}}\right) \neq 0$. Thus,

$$a([y,u]_{\mathfrak{m}},v) + a(u,[y,v]_{\mathfrak{m}}) = 0.\square$$

Theorem 4.2. *Let $(G/H,F)$ be a homogeneous Randers space with F defined by the Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and the vector field X which is of Berwald type. If $(G/H,a)$ is naturally reductive, then $(G/H,F)$ is naturally reductive.*

Proof. Let $(G/H,F)$ be naturally reductive. We show that for all $y \neq 0, z, u, v \in \mathfrak{m}$

$$g_y([z,u]_{\mathfrak{m}},v) + g_y(u,[z,v]_{\mathfrak{m}} + 2C_y([z,y]_{\mathfrak{m}}),u,v) = 0.$$

□

Since F is of Berwald type, $(G/H,F)$ and $(G/H,a)$ have the same connection. Thus, by a direct computation from (5), we get

$$\begin{aligned} g_y([z,u]_{\mathfrak{m}},v) &= a([z,u]_{\mathfrak{m}},v)\left(1+\frac{a(X,y)}{\sqrt{a(y,y)}}\right) \\ &\quad + a([z,u]_{\mathfrak{m}},y)\left(\frac{a(X,v)}{\sqrt{a(y,y)}} - \frac{a(v,y)a(X,y)}{a(y,y)^{3/2}}\right), \end{aligned} \tag{6}$$

$$\begin{aligned} g_y(u,[z,v]_{\mathfrak{m}}) &= a(u,[z,v]_{\mathfrak{m}})\left(1+\frac{a(X,y)}{\sqrt{a(y,y)}}\right) \\ &\quad + a([z,v]_{\mathfrak{m}},y)\left(\frac{a(X,u)}{\sqrt{a(y,y)}} - \frac{a(u,y)a(X,y)}{a(y,y)^{3/2}}\right), \end{aligned} \tag{7}$$

By definition,

$$C_y(z,u,v) = \frac{1}{2}\frac{d}{dt}[g_{y+tv}(z,u)]\,|_{t=0}.$$

Thus, we have

$$\begin{aligned} C_y([z,y]_{\mathfrak{m}},u,v) &= a([z,y]_{\mathfrak{m}},u)\left(\frac{a(X,v)}{\sqrt{a(y,y)}} - \frac{a(y,v)a(X,y)}{a(y,y)^{3/2}}\right) \\ &\quad + a([z,y]_{\mathfrak{m}},v)\left(\frac{a(X,u)}{\sqrt{a(y,y)}} - \frac{a(X,y)a(u,y)}{a(y,y)^{3/2}}\right). \end{aligned} \tag{8}$$

Since $(G/H,a)$ is naturally reductive, substituting (6), (7), and (8) in

$$g_y([z,u]_{\mathfrak{m}},v) + g_y(u,[z,v]_{\mathfrak{m}}) + 2C_y([z,y]_{\mathfrak{m}},u,v)$$

completes the proof. □

For the result on homogeneous geodesics in homogeneous Finsler manifold, we refer to [6], [11], [12]. The basic formula characterizing geodesic vector (Geodesic Lemma) in the Finslerian case was given in [6]. Let $(G/H,g)$ be a homogeneous Finslerian manifold with a fixed origin p, and

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$$

a reductive decomposition.

Lemma 4.1. *([6]). A vector $Z \in \mathfrak{g}$ is a geodesic vector if and only if*

$$g_{Z_{\mathfrak{m}}}([Z,Y]_{\mathfrak{m}},Z_{\mathfrak{m}}) = 0$$

for all $Y \in \mathfrak{m}$.

We now present a geometric criterion of homogeneous naturally reductive Randers spaces as Remark 3.1.

Theorem 4.3. *Let $(M = G/H,F)$ be a homogeneous Randers space with F defined by the Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ and the vector field X. Then, the homogeneous Randers space $(M = G/H,F)$ with the origin $p = \{H\}$ and with an $Ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{m}+\mathfrak{h}$ is naturally reductive with respect to this decomposition if and only if for any vector $X \in \mathfrak{m} \setminus \{0\}$, the curve $\gamma(t) = exptX(p)$ is a geodesic of (G/H,F).*

Proof. First, we suppose that (M,F) is naturally reductive. Then, the underlying Riemannian metric $a = a_{ij}dx^i \otimes dx^j$ is naturally reductive, and the connection of F and $a = a_{ij}dx^i \otimes dx^j$ coincides. This means that F is a Berwald metric. Let $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ be the naturally reductive decomposition such that

$$g_y([x,u],v) + g_y(u,[x,v]) + 2C_y([x,y],u,v) = 0,$$

where $y \neq 0, x, u, v \in \mathfrak{m}$.

Then, for $Z \in \mathfrak{m}$,

$$g_Z(Z, [Z, Y]_{\mathfrak{m}}) = a(Z, [Z, Y]_{\mathfrak{m}}) = 0,$$

for all $Y \in \mathfrak{m}$. □

Then, according to [6], each geodesic of $(G/H, F)$ emanating from the origin $p = \{H\}$ is just $\exp(tZ).p$, $Z \in \mathfrak{m}$. On the other hand, suppose that for any vector $X \in \mathfrak{m}\backslash\{0\}$, the curve $\gamma(t) = exptX(p)$ is a geodesic of (G/H,F). We first show that F must be a Berwald metric. In fact, the tangent space T_xM can be identified with $\mathfrak{m}$ by the correspondence

$$X \in \mathfrak{m} \longrightarrow \frac{d}{dt}\exp(tX).x|_{t=0}.$$

Then, by the condition on the geodesics of (M, F), we see that the exponential mapping $Exp|_x$ of (M, F) at x is just

$$Exp(X) = \pi(\exp(X)), \quad X \in \mathfrak{m},$$

where π is the natural projection of G onto G/H. Since the mapping exp and π are smooth, $Exp|_x$ is a smooth mapping. Since (M, F) is homogeneous, the exponential mapping is smooth everywhere. Thus, F must be a Berwald metric [7].

Since F is of Berwald type, (M, F) and (M, a) have the same connection. Then, they have the same geodesics. Hence, for any $Z \in \mathfrak{m}$, the curve $\exp(tX).p$ is a homogeneous geodesic of (M, a). By Lemma 3.1, the vector $Z \in \mathfrak{m}$ is a geodesic vector, so

$$\langle [Z, Y]_{\mathfrak{m}}, Z\rangle = 0, \quad \forall Y \in \mathfrak{m}.$$

Now, for any $Y, Z, W \in \mathfrak{m}$, set $W' = W + Y$, $Y' = Y + Z$. Then, we have

$$0 = \langle [W', Y']_{\mathfrak{m}}, W'\rangle = \langle [W, Z]_{\mathfrak{m}}, Y\rangle + \langle [Y, Z]_{\mathfrak{m}}, W\rangle.$$

This implies that $(G/H, a)$ is naturally reductive. Now, using the Theorem 4.2, we have that $(G/H, F)$ is naturally reductive.

Conclusions

In this paper, we study the naturally reductive Randers metrics on homogeneous manifolds. We first prove that if a Randers metric is naturally reductive, then the underlying Riemannian metric is naturally reductive. Then, we show that for Berwald type Randers metric if the underlying Riemannian metric is naturally reductive, then the Randers metric is naturally reductive. Finally, we give a geometric criterion of homogeneous naturally reductive Randers spaces. The concept is new and may be useful in the Lie group theory and Finsler geometry.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
DL and MT carried out the proof. Both authors read and approved the final manuscript.

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