Bhrawy and Taha *Mathematical Sciences* 2012, **6**:41
http://www.iaumath.com/content/6/1/41

Mathematical Sciences
a SpringerOpen Journal

ORIGINAL RESEARCH Open Access

An operational matrix of fractional integration of the Laguerre polynomials and its application on a semi-infinite interval

Ali H Bhrawy[1,2*] and Taha M Taha[2]

Abstract

Purpose: In this paper, we construct the operational matrix of fractional integration of arbitrary order for Laguerre polynomials.

Methods: We introduce some necessary definitions and give some relevant properties of Laguerre polynomials. The fractional integration is described in the Riemann-Liouville sense. We develop a direct solution technique for solving the integrated forms of fractional differential equations (FDEs) on the half line using the Laguerre tau method based on operational matrix of fractional integration in the Riemann-Liouville sense.

Results: In order to show the fundamental importance of the Laguerre operational matrix, we apply it together with the spectral Laguerre tau method for the numerical solution of general linear multi-term FDEs on a semi-infinite interval.

Conclusions: The results obtained by the present methods reveal that the present method is very effective and convenient for linear FDEs. Illustrative examples are included to demonstrate the validity and applicability of the new technique for linear muti-term FDEs on a semi-infinite interval.

Keywords: Operational matrix, Laguerre polynomials, Tau method, Multi-term FDEs, Riemann-Liouville derivative

Introduction

Fractional calculus has been used to model physical and engineering processes that are found to be best described by fractional differential equations (FDEs) (see [1-3] and references therein). Spectral methods have high accuracy [4-8]. The usual spectral methods are only available for bounded domains for solving FDEs (see [9-13]). However, it is also interesting to consider spectral methods for FDEs on the half line. Some authors have developed the Laguerre spectral method for the half line for ordinary, partial, and delay differential equations (see [14-19]).

The operational matrix of fractional derivatives has been determined for some types of orthogonal polynomials, such as Chebyshev polynomials [20], Legendre polynomials [10], and Jacobi polynomials [21]. Moreover, the operational matrices for integer derivatives have been used for solving differential and integral equations (see for instance [22-24]). Recently, Lakestani et al. [25] constructed the operational matrix of fractional derivatives using B-spline functions. Also, Bhrawy and Alofi [26] introduced the shifted Chebyshev operational matrix of fractional integration, in the Riemann-Liouville sense, of arbitrary order and applied together with the spectral tau method for solving linear FDEs. The fractional integration is described.

The operational matrix of integer integration has been determined for several types of orthogonal polynomials, such as the Laguerre series [27], Chebyshev polynomials [28], Legendre polynomials [29], Bessel series [30], and Laguerre and Hermite poynomials [31]. Recently, Singh et al. [32] derived the Bernstein operational matrix of integration (see also [33]). Up to now, and to the best of our knowledge, most of formulae corresponding to those mentioned previously are unknown and are traceless in the literature for fractional integration in the Riemann-

*Correspondence: alibhrawy@yahoo.co.uk
[1]Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
[2]Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt

Liouville sense. This partially motivates our interest in operational matrix of fractional integration for Laguerre polynomials.

Another motivation is concerned with the direct solution techniques for solving the integrated forms of FDEs on the half line using the Laguerre tau method based on operational matrix of fractional integration in the Riemann-Liouville sense. Finally, the accuracy of the proposed algorithm is demonstrated by test problems.

The paper is organized as follows. In the 'Preliminaries and notation' subsection of the 'Methods' section, we introduce some necessary definitions and give some relevant properties of the Laguerre polynomials. In the 'Laguerre operational matrix of fractional integration' subsection, the Laguerre operational matrix of fractional integration is introduced. In the 'Application of Laguerre operational matrix for multi-order FDEs' subsection of the 'Methods' section, we apply the Laguerre operational matrix of fractional integration for solving linear multi-order FDEs. In the 'Illustrative examples' subsection in the 'Results and discussion' section, the proposed method is applied to several examples. Finally, some concluding remarks in the 'Conclusions' section.

Methods

Preliminaries and notation

Let $\Lambda = (0, \infty)$, $w(x) = e^{-x}$, and $L_\ell(x)$ be the Laguerre polynomial of degree ℓ, defined by the following:

$$L_\ell(x) = \frac{1}{\ell!} e^x \partial_x^\ell (x^\ell \, e^{-x}), \quad \ell = 0, 1, \ldots. \tag{1}$$

They satisfy the equations

$$\partial_x (x \, e^{-x} \partial_x L_\ell(x)) + \ell e^{-x} L_\ell(x) = 0 \quad x \in \Lambda,$$

and

$$L_\ell(x) = \partial_x L_\ell(x) - \partial_x L_{\ell+1}(x), \quad \ell \geq 0.$$

The set of Laguerre polynomials is the $L_w^2(\Lambda)$-orthogonal system, namely,

$$\int_\Lambda L_j(x) L_k(x) w(x) dx = \delta_{jk}, \quad \forall i, j \geq 0, \tag{2}$$

where δ_{jk} is the Kronecher function.

The special value

$$D^q L_i(0) = (-1)^q \sum_{j=0}^{i-q} \frac{(i-j-1)!}{(q-1)! \, (i-j-q)!}, \tag{3}$$

where q, a positive integer, will be of important use later.

A function $u(x)$, square integrable in Λ, may be expressed in terms of Laguerre polynomials as follows:

$$u(x) = \sum_{j=0}^{\infty} a_j L_j(x),$$

where the coefficient a_j is given as follows:

$$a_j = \int_\Lambda u(x) L_j(x) w(x) dx, \quad j = 0, 1, \ldots. \tag{4}$$

In practice, only the first $(N+1)$-term Laguerre polynomials are considered. We then have the following:

$$u_N(x) = \sum_{j=0}^{N} a_j L_j(x) = C^T \phi(x). \tag{5}$$

where the Laguerre coefficient vector C and the Laguerre vector $\phi(x)$ are given as follows:

$$C^T = [\, c_0, c_1, \ldots, c_N \,],$$

$$\phi(x) = [\, L_0(x), L_1(x), \ldots, L_N(x)]^T. \tag{6}$$

If we define q times repeated the integration of the Laguerre vector $\phi(x)$ by $J^q \phi(x)$, then

$$J^q \phi(x) \simeq \mathbf{P}^{(q)} \phi(x), \tag{7}$$

where q is an integer value, and $\mathbf{P}^{(q)}$ is the operational matrix of integration of $\phi(x)$.

There are several definitions of a fractional integration of order $\nu > 0$, and they are not necessarily equivalent to each other (see [34]). The most used definition is due to Riemann-Liouville, which is defined as follows:

$$J^\nu f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) dt, \quad \nu > 0, \quad x > 0,$$
$$J^0 f(x) = f(x). \tag{8}$$

One of the basic properties of the operator J^ν is

$$J^\nu x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\nu)} \, x^{\beta+\nu}. \tag{9}$$

The Riemann-Liouville fractional derivative of order ν will be denoted by D^ν. The next equation defines the Riemann-Liouville fractional derivative of order:

$$D^\nu f(x) = \frac{d^m}{dx^m} (J^{m-\nu} f(x)), \tag{10}$$

where $m-1 < \nu \le m,\ m \in N$, and m is the smallest integer greater than ν.

Lemma 1. *If* $m-1 < \nu \le m,\ m \in N$, *then*

$$D^{\nu}J^{\nu}f(x)=f(x),\quad J^{\nu}D^{\nu}f(x)=f(x)-\sum_{i=0}^{m-1}f^{(i)}(0^{+})\frac{x^{i}}{i!},\ x>0. \tag{11}$$

Laguerre operational matrix of fractional integration

The main objective of this section is to find the fractional integration of the Laguerre vector in the Riemann-Liouville sense.

Theorem 1. *Let* $\phi(x)$ *be the Laguerre vector and* $\nu > 0$, *then*

$$J^{\nu}\phi(x) \simeq \boldsymbol{P}^{(\nu)}\phi(x), \tag{12}$$

where $\boldsymbol{P}^{(\nu)}$ *is the* $(N+1)\times(N+1)$ *operational matrix of fractional integration of order* ν *in the Riemann-Liouville sense and is defined as follows:*

$$\boldsymbol{P}^{(\nu)}=\begin{pmatrix} \Omega_{\nu}(0,0) & \Omega_{\nu}(0,1) & \Omega_{\nu}(0,2) & \cdots & \Omega_{\nu}(0,N)\\ \Omega_{\nu}(1,0) & \Omega_{\nu}(1,1) & \Omega_{\nu}(1,2) & \cdots & \Omega_{\nu}(1,N)\\ \Omega_{\nu}(2,0) & \Omega_{\nu}(2,1) & \Omega_{\nu}(2,2) & \cdots & \Omega_{\nu}(2,N)\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ \Omega_{\nu}(i,0) & \Omega_{\nu}(i,1) & \Omega_{\nu}(i,2) & \cdots & \Omega_{\nu}(i,N)\\ \vdots & \vdots & \vdots & \cdots & \vdots\\ \Omega_{\nu}(N,0) & \Omega_{\nu}(N,1) & \Omega_{\nu}(N,2) & \cdots & \Omega_{\nu}(N,N) \end{pmatrix} \tag{13}$$

where

$$\Omega_{\nu}(i,j)=\sum_{k=0}^{i}\sum_{r=0}^{j}\frac{(-1)^{k+r}\ i!\ r!\,\Gamma(k+\nu+r+1)}{(i-k)!\ k!\ (j-r)!\ (r!)^{2}\ \Gamma(k+\nu+1)}.$$

Proof. The analytic form of the Laguerre polynomials $L_i(x)$ of degree i is given as follows:

$$L_i(x)=\sum_{k=0}^{i}(-1)^{k}\frac{i!}{(i-k)!\ (k!)^{2}}\,x^{k}, \tag{14}$$

where $L_i(0) = 1$, applying the Riemann-Liouville fractional integration of order ν of Equation 14. Using Equations 8 and 9, and since the Riemann-Liouville's fractional integration is a linear operation, we get the following:

$$\begin{aligned} J^{\nu}L_i(x) &= \sum_{k=0}^{i}(-1)^{k}\frac{i!}{(i-k)!\ (k!)^{2}}\,J^{\nu}x^{k}\\ &= \sum_{k=0}^{i}(-1)^{k}\frac{i!}{(i-k)!\ k!\ \Gamma(k+\nu+1)}\,x^{k+\nu},\\ & i=0,1,\cdots,N. \end{aligned} \tag{15}$$

□

Now, by approximating $x^{k+\nu}$ by the $N+1$ terms of the Laguerre series, we have the following:

$$x^{k+\nu}=\sum_{j=0}^{N}b_jL_j(x), \tag{16}$$

where b_j is given from Equation 4 with $u(x) = x^{k+\nu}$, that is,

$$b_j=\sum_{r=0}^{j}\frac{(-1)^{r}\ j!\ \Gamma(k+\nu+r+1)}{(j-r)!\ (r!)^{2}},\quad j=1,2,\cdots,N. \tag{17}$$

In virtue of Equations 15 and 16, we get the following:

$$J^{\nu}L_i(x)=\sum_{j=0}^{N}\Omega_{\nu}(i,j)L_j(x),\quad i=0,1,\cdots,N, \tag{18}$$

where

$$\Omega_{\nu}(i,j)=\sum_{k=0}^{i}\sum_{r=0}^{j}\frac{(-1)^{k+r}\ i!\ r!\,\Gamma(k+\nu+r+1)}{(i-k)!\ k!\ (j-r)!\ (r!)^{2}\ \Gamma(k+\nu+1)},\quad j=1,2,\cdots N.$$

Accordingly, Equation 18 can be written in a vector form as follows:

$$J^{\nu}L_i(x)\simeq\Big[\Omega_{\nu}(i,0),\Omega_{\nu}(i,1),\Omega_{\nu}(i,2),\cdots,\Omega_{\nu}(i,N)\Big]\phi(x),\quad i=0,1,\cdots,N. \tag{19}$$

Equation 19 leads to the desired result.

Application of Laguerre operational matrix for multi-order FDEs

In this section, the Laguerre tau method based on operational matrix is proposed to numerically solve the FDEs. The basic idea of this technique is as follows: (1) The FDE is converted to a fully integrated form via fractional integration in the Riemann-Liouville sense. (2) Subsequently, the integrated form equation is approximated by representing them as linear combinations of the Laguerre polynomials. (3) Finally, the integrated form equation is converted to an algebraic equation by introducing the operational matrix of fractional integration of the Laguerre polynomials.

In order to show the fundamental importance of the Laguerre operational matrix of fractional integration, we apply it to solve the following multi-order FDE:

$$D^{\nu}u(x) = \sum_{i=1}^{k} \gamma_j D^{\beta_i}u(x) + \gamma_{k+1}u(x) + f(x), \quad \text{in } \Lambda, \tag{20}$$

with initial conditions

$$u^{(i)}(0) = d_i, \qquad i = 0, \cdots, m-1, \tag{21}$$

where γ_i $(i = 1, \cdots, k+1)$ are real constant coefficients and also $m-1 < \nu \leq m$, $0 < \beta_1 < \beta_2 < \cdots < \beta_k < \nu$. Moreover, $D^{\nu}u(x) \equiv u^{(\nu)}(x)$ denotes the Riemann-Liouville fractional derivative of order ν for $u(x)$; the values of d_i $(i = 0, \cdots, m-1)$ describe the initial state of $u(x)$, and $f(x)$ is a given source function. If we apply the Riemann-Liouville integral of order ν on Equation 20 and after making use of Equation 11, we get the integrated form of Equation 20, namely

$$\begin{aligned} u(x) - \sum_{j=0}^{m-1} u^{(j)}(0^+)\frac{x^j}{j!} &= \sum_{i=1}^{k} \gamma_i J^{\nu-\beta_i}\Big[u(x) - \sum_{j=0}^{m_i-1} u^{(j)}(0^+)\frac{x^j}{j!}\Big] \\ &\quad + \gamma_{k+1}J^{\nu}u(x) + J^{\nu}f(x), \\ u^{(i)}(0) = d_i, &\qquad i = 0, \cdots, m-1, \end{aligned} \tag{22}$$

where $m_i - 1 < \beta_i \leq m_i$, $m_i \in N$. This implies that

$$\begin{aligned} u(x) &= \sum_{i=1}^{k} \gamma_i J^{\nu-\beta_i}u(x) + \gamma_{k+1}J^{\nu}u(x) + g(x), \\ u^{(i)}(0) &= d_i, \qquad i = 0, \cdots, m-1, \end{aligned} \tag{23}$$

where

$$g(x) = J^{\nu}f(x) + \sum_{j=0}^{m-1} d_j\frac{x^j}{j!} + \sum_{i=1}^{k} \gamma_i J^{\nu-\beta_i}\left(\sum_{j=0}^{m_i-1} d_j\frac{x^j}{j!}\right).$$

In order to use the tau method with the Laguerre operational matrix for solving the fully integrated problem (Equation 23) with initial conditions (Equation 21), we approximate $u(x)$ and $g(x)$ by the Laguerre polynomials as follows:

$$u_N(x) \simeq \sum_{i=0}^{N} c_i L_i(x) = C^T\phi(x), \tag{24}$$

$$g(x) \simeq \sum_{i=0}^{N} g_i L_i(x) = G^T\phi(x), \tag{25}$$

where the vector $G = [g_0, \cdots, g_N]^T$ is given, but $C = [c_0, \cdots, c_N]^T$ is an unknown vector.

Now, the Riemann-Liouville integral of orders ν and $(\nu - \beta_j)$ of the approximate solution (Equation 24), after making use of Theorem 1 (relation (12)), can be written as follows:

$$J^{\nu}u_N(x) \simeq C^T J^{\nu}\phi(x) \simeq C^T\mathbf{P}^{(\nu)}\phi(x), \tag{26}$$

and

$$\begin{aligned} J^{\nu-\beta_j}u_N(x) &\simeq C^T J^{\nu-\beta_j}\phi(x) \\ &\simeq C^T\mathbf{P}^{(\nu-\beta_j)}\phi(x), \quad j = 1, \cdots, k, \end{aligned} \tag{27}$$

respectively, where $\mathbf{P}^{(\nu)}$ is the $(N+1)\times(N+1)$ operational matrix of fractional integration of the order ν.

Employing Equations 24 to 27, the residual $R_N(x)$ for Equation 23 can be written as follows:

$$R_N(x) = (C^T - C^T\sum_{j=1}^{k} \gamma_j\mathbf{P}^{(\nu-\beta_j)} - \gamma_{k+1}C^T\mathbf{P}^{(\nu)} - G^T)\phi(x). \tag{28}$$

As in a typical tau method (see [20,26]), we generate $N - m + 1$ linear algebraic equations by applying the following:

$$\begin{aligned} \langle R_N(x), L_j(x)\rangle &= \int_0^{\infty} R_N(x)w(x)L_j(x)dx = 0, \\ j &= 0, 1, \cdots, N-m. \end{aligned} \tag{29}$$

Also, by substituting Equations 4 and 24 in Equation 21, we get the following:

$$u^{(i)}(0) = \sum_{i=0}^{N} c_i L_i^{(i)}(0) = d_i, \qquad i = 0, 1, \cdots, m-1. \tag{30}$$

Equations 29 and 30 generate the $N - m + 1$ and m set of linear equations, respectively. These linear equations can be solved for unknown coefficients of the vector C. Consequently, $u_N(x)$ given in Equation 24 can be calculated, which gives a solution of Equation 20 with the initial conditions (Equation 21).

Results and discussion

Illustrative examples

To illustrate the effectiveness of the proposed method in the present paper, some test examples are carried out in this section. The results obtained by the present methods reveal that the present method is very effective and convenient for linear FDEs.

Example 1. *As the first example, we consider the following initial value problem*

$$D^{\frac{3}{2}}u(x) + 3u(x) = 3x^3 + \frac{8}{\Gamma(0.5)}x^{1.5},$$
$$u(0) = 0, \quad u'(0) = 0, \qquad x \in \Lambda = (0, \infty), \tag{31}$$

whose exact solution is given by $u(x) = x^3$.

If we apply the technique described in the 'Application of Laguerre operational matrix for multi- order FDEs' subsection with $N = 3$, then the approximate solution can be written as follows:

$$u_N(x) = \sum_{i=0}^{3} c_i L_i(x) = C^T \phi(x),$$

and

$$\mathbf{P}^{(\frac{3}{2})} = \begin{pmatrix} 1 & \frac{-3}{2} & \frac{3}{8} & \frac{1}{16} \\ 0 & 1 & \frac{-3}{2} & \frac{3}{8} \\ 0 & 0 & 1 & \frac{-3}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; G = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix}.$$

Using Equation 29, we obtain the following:

$$\begin{aligned} &\frac{9}{8}c_0 - \frac{9}{2}c_1 + 4c_2 - g_2 = 0, \\ &\frac{3}{16}c_0 + \frac{9}{8}c_1 - \frac{9}{2}c_2 + 4c_3 - g_3 = 0, \end{aligned} \tag{32}$$

Now, applying Equation 30, we get the following:

$$\begin{aligned} &C^T \phi(0) = c_0 + c_1 + c_2 + c_3 = 0, \\ &C^T \mathbf{D}^{(1)} \phi(0) = -c_1 - 2c_2 - 3c_3 = 0. \end{aligned} \tag{33}$$

Solving the linear system, Equations 32 to 33 yield the following:

$$c_0 = 6, \qquad c_1 = -18, \qquad c_2 = 18, \qquad c_1 = -6.$$

Thereby, we can write

$$u_N(x) = \sum_{i=0}^{3} c_i L_i(x) = x^3.$$

Numerical results will not be presented since the exact solution is obtained.

Example 2. *Consider the equation*

$$D^2 u(x) + D^{\frac{3}{4}} u(x) + u(x) = x^3 + 6x + \frac{128}{15\,\Gamma(\frac{1}{4})} x^{\frac{9}{4}},$$
$$u(0) = 0, \quad u'(0) = 0, \quad x \in \Lambda, \tag{34}$$

whose exact solution is given by $u(x) = x^3$.

If we apply the technique described in the 'Application of Laguerre operational matrix for multi- order FDEs' subsection with $N = 3$, then the approximate solution can be written as follows:

$$u_N(x) = \sum_{i=0}^{3} c_i L_i(x) = C^T \phi(x),$$

and

$$\mathbf{P}^{(\frac{5}{4})} = \begin{pmatrix} 1 & \frac{-5}{4} & \frac{5}{32} & \frac{5}{128} \\ 0 & 1 & \frac{-5}{4} & \frac{5}{32} \\ 0 & 0 & 1 & \frac{-5}{4} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; \mathbf{P}^2 = \begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$
$$G = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{pmatrix}.$$

Using Equation 29, we obtain the following:

$$\begin{aligned} &(1 + \frac{5}{32})c_0 - (2 + \frac{5}{4})c_1 + 3c_2 - g_2 = 0, \\ &\frac{5}{128}c_0 + (1 + \frac{5}{32})c_1 - (2 + \frac{5}{4})c_2 + 3c_3 - g_3 = 0, \end{aligned} \tag{35}$$

Now, applying Equation 30, we get the following:

$$\begin{aligned} &C^T \phi(0) = c_0 + c_1 + c_2 + c_3 = 0, \\ &C^T \mathbf{D}^{(1)} \phi(0) = -c_1 - 2c_2 - 3c_3 = 0. \end{aligned} \tag{36}$$

By solving the linear system (Equations 35 to 36), we have the following:

$$c_0 = 6, \qquad c_1 = -18, \qquad c_2 = 18, \qquad c_1 = -6.$$

Thereby, we can write

$$u_N(x) = \sum_{i=0}^{3} c_i L_i(x) = x^3.$$

Numerical results will not be presented since the exact solution is obtained.

Example 3. *Consider the equation*

$$D^2u(x) - 2Du(x) + D^{\frac{1}{2}}u(x) + u(x) = x^7 + \frac{2048}{429\sqrt{\pi}}x^{6.5} - 14x^6 + 42x^5 - x^2 - \frac{8}{3\sqrt{\pi}}x^{1.5} + 4x - 2,$$
$$u(0) = 0, \quad u'(0) = 0, \quad x \in \Lambda, \tag{37}$$

whose exact solution is given by $u(x) = x^7 - x^2$.

Now, if we apply the technique described in Examples 1 and 2, with $N = 7$, then we have the following:

$$c_0 = 5038, \quad c_1 = -35276, \quad c_2 = 105838,$$
$$c_3 = -176400, \quad c_4 = 176400, \quad c_5 = -105840,$$
$$c_6 = 35280, \quad c_7 = -5040.$$

Thus, we can write

$$u_N(x) = \sum_{i=0}^{7} c_i L_i(x) = x^7 - x^2,$$

which is the exact solution.

Conclusions

In this article, we have presented the operational matrix of fractional integration of the Laguerre polynomials, and as an important application, we describe how to use the operational tau technique to numerically solve the general multi-term linear fractional-order differential equations with initial conditions on a semi-infinite domain. The fractional integration is described in the Riemann-Liouville sense. The numerical results given in the previous section show that the proposed algorithm with a small number of Laguerre polynomials is giving a satisfactory result.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors, AHB and TMT, have equal contributions to each part of this article. All authors read and approved the final manuscript.

Acknowledgements
The authors would like to thank the reviewers for their constructive comments and suggestions to improve the quality of the article.

Received: 30 August 2012 Accepted: 17 September 2012
Published: 3 October 2012

References

1. Das, S: Functional Fractional Calculus for System Identification and Controls. Springer, New York (2008)
2. Kilbas, AA, Srivastava, HM, Trujillo, JJ: Theory and Applications of Fractional Differential Equations. Elsevier, San Diego (2006)
3. Podlubny, I: Fractional Differential Equations, vol. 198 of Mathematics in Science and Engineering. Academic Press Inc., San Diego (1999)
4. Canuto, C, Hussaini, MY, Quarteroni, A, Zang, TA: Spectral Methods in Fluid Dynamics. Springer, New York (1988)
5. Doha, EH, Bhrawy, AH: Efficient spectral-Galerkin algorithms for direct solution of fourth-order differential equations using Jacobi polynomials. Appl. Numer. Math. **58**, 1224–1244 (2008)
6. Doha, EH, Bhrawy, AH: An efficient direct solver for multidimensional elliptic Robin boundary value problems using a Legendre spectral-Galerkin method. Comput. Math. Appl. **64**, 558–571 (2012)
7. Bhrawy, AH, Alofi, AS, El-Soubhy, SI: An extension of the Legendre-Galerkin method for solving sixth-order differential equations with variable polynomial coefficients. Math. Problems Eng. **2012**(896575), 13 (2012). doi:10.1155/2012/896575
8. Bhrawy, AH, Alofi, AS, El-Soubhy, SI: Spectral shifted Jacobi tau and collocation methods for solving fifth-order boundary value problems. Abstract Appl. Anal. **2011**(823273), 14 (2011). doi:10.1155/2011/823273
9. Doha, EH, Bhrawy, AH, Ezz-Eldien, SS: Efficient Chebyshev spectral methods for solving multi-term fractional orders differential equations. Appl. Math. Modell. **35**, 5662–5672 (2011)
10. Saadatmandi, A, Dehghan, M: A new operational matrix for solving fractional-order differential equations. Comput. Math. Appl. **59**, 1326–1336 (2010)
11. Li, Y, Sun, N: Numerical solution of fractional differential equations using the generalized block pulse operational matrix. Comput. Math. Appl. **62**, 1046–1054 (2011)
12. Bhrawy, AH, Alofi, AS, Ezz-Eldien, SS: A quadrature tau method for fractional differential equations with variable coefficients. Appl. Math. Lett. **24**, 2146–2152 (2011)
13. Bhrawy, AH, Al-Shomrani, MM: A shifted Legendre spectral method for fractional-order multi-point boundary value problems. Adv. Difference Equations. **2012**, 8 (2012)
14. Mikhailenko, BG: Spectral Laguerre method for the approximate solution of time dependent problems. Appl. Mathematics Lett. **12**, 105–110 (1999)
15. Guo, B-Y, Wang, L-L: Modified Laguerre pseudospectral method refined by multidomain Legendre pseudospectral approximation. J. Comput. Appl. Math. **190**, 304–324 (2006)
16. Funaro, D: Estimates of Laguerre spectral projectors in Sobolev spaces. In: Brezinski, C, Gori, L, Ronveaux, A (eds.) Orthogonal Polynomials and Their Applications, pp. 263–266. Scientific Publishing Co., Singapore (1991)
17. Guo, B-Y, Shen, J: Laguerre-Galerkin method for nonlinear partial differential on a semi-infinite interval. Numer. Math. **86**, 635–654 (2000)
18. Mastroianni, G, Occorsio, D: Lagrange interpolation at Laguerre zeros in some weighted uniform spaces. Acta. Math. Hungar. **91**, 27–52 (2001)
19. Gulsu, M, Gurbuz, B, Ozturk, Y, Sezer, M: Laguerre polynomial approach for solving linear delay difference equations. Appl. Mathematics Comput. **217**, 6765–6776 (2011)
20. Doha, EH, Bhrawy, AH, Ezz-Eldien, SS: A Chebyshev spectral method based on operational matrix for initial and boundary value problems of fractional order. Comput. Math. Appl. **62**, 2364–2373 (2011)
21. Doha, EH, Bhrawy, AH, Ezz-Eldien, SS: A new Jacobi operational matrix: an application for solving fractional differential equations. Appl. Math. Modell. **36**, 4931–4943 (2012)
22. Pandey, RK, Kumar, N: Solution of Lane-Emden type equations using Bernstein operational matrix of differentiation. New Astron. **17**, 303–308 (2012)
23. Maleknejad, K, Hashemizadeh, E, Basirat, B: Computational method based on Bernstein operational matrices for nonlinear Volterra-Fredholm-Hammerstein integral equations. Commun Nonlinear Sci Numer Simulat. **17**, 52–61 (2012)
24. Maleknejad, K, Basirat, B, Hashemizadeh, E: A Bernstein operational matrix approach for solving a system of high order linear Volterra-Fredholm

integro-differential equations. Math. Comput. Modell. **55**, 1363–1372 (2012)
25. Lakestani, M, Dehghan, M, Irandoust-pakchin, S: The construction of operational matrix of fractional derivatives using B-spline functions. Commun Nonlinear Sci Numer Simulat. **17**, 1149–1162 (2012)
26. Bhrawy, AH, Alofi, AS: The operational matrix of fractional integration for shifted Chebyshev polynomials. Appl. Math. Lett. **26**, 25–31 (2013)
27. Hwang, C, Shih, YP: Parameter identification via Laguerre polynomials. Int. J. Syst. Sci. **13**, 209–217 (1982)
28. Paraskevopoulos, PN: Chebyshev series approach to system identification, analysis and control. J. Franklin Inst. **316**, 135–157 (1983)
29. Paraskevopoulos, PN: Legendre series approach to identification and analysis of linear systems. IEEE Trans. Autom. Control. **30**, 585–589 (1985)
30. Paraskevopoulos, PN, Sklavounos, P, Georgiou, GCh: The operational matrix of integration for Bessel functions. J. Franklin Inst. **327**, 329–341 (1990)
31. Doha, EH, Ahmed, HM, El-Soubhy, SI: Explicit formulae for the coefficients of integrated expansions of Laguerre and Hermite polynomials and their integrals. Integral Transforms and Special Functions. **20**, 491–503 (2009)
32. Singh, AK, Singh, VK, Singh, OP: The Bernstein operational matrix of integration. Appl. Math. Sciences. **3**(49), 2427–2436 (2009)
33. Doha, EH, Bhrawy, AH, Saker, MA: Integrals of Bernstein polynomials: an application for the solution of high even-order differential equations. Appl. Math. Lett. **24**, 559–565 (2011)
34. Miller, K, Ross, B: An Introduction to the Fractional Calaulus and Fractional Differential Equations. John Wiley & amp Sons Inc., New York (1993)

doi:10.1186/2251-7456-6-41
Cite this article as: Bhrawy and Taha: **An operational matrix of fractional integration of the Laguerre polynomials and its application on a semi-infinite interval.** *Mathematical Sciences* 2012 **6**:41.