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Singular solitons and numerical analysis of Φ–four equation

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Abstract

Purpose: This paper studies the Φ-four equation that appears in relativistic quantum mechanics. The primary purpose of this paper is to address the numerical simulations of this model.

Methods: The singular soliton solution is also obtained by the ansatz method.

Results: The constraint conditions are indicated for the existence of the soliton. Numerical solutions are obtained by using the spectral method where rational Chebyshev functions are used as basis functions.

Conclusions: Results are validated by finding error estimates.

Keywords: Solitons, Spectral methods, Rational Chebyshev function

AMS: 35Q51; 35Q53; 37K10; 78A60.

PAC Codes: 02.30.Ik; 02.30.Jr; 42.81.Dp; 52.35.Sb.

Introduction

The Φ-four (P4) equation is a nonlinear evolution equation that is studied in the context of relativistic quantum mechanics. This equation was studied by several scientists in the past. The soliton perturbation theory as well as other integrability aspects and bifurcation analysis are all addressed before. The primary focus of this paper is to develop numerical algorithms in order to 'visualize' the soliton solution that supports the P4 model. There is yet a lesser known soliton solution that is occasionally studied. This is the singular soliton. The ansatz method will be first implemented in order to extract the singular soliton solution of this equation. The constraint conditions are also going to be identified for its existence.

The P4 equation is a special case of Klein-Gordon equation (KGE) that is studied with several forms of nonlinearity that includes quadratic nonlinearity, power law nonlinearity, as well as log law nonlinearity. It is primarily the perturbation theory, numerical simulation, and integrability issues that have been addressed thus far in such models [1-14].

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Governing equation

The P4 equation that is going to be studied in this paper is given by

$$q_{tt} - k^2 q_{xx} = aq + bq^3. \tag{1}$$

In Equation 1, the dependent variable $q(x,t)$ is the wave profile, while the spatial and temporal independent variables, respectively, are x and t. The coefficients a, b, and k^2 are real-valued parameters. As mentioned before, this equation was studied earlier on several ocassions. The topological 1-soliton solution was lately obtained by the ansatz method, and the corresponding bifurcation analysis was also studied. The soliton perturbation theory was studied for this equation where the adiabatic dynamics of the soliton velocity was obtained [3]. Subsequently, this equation was extended to (1+2)-dimensions where soliton solutions were also obtained. The numerical simulation was touched upon by the aid of variational iteration method although that was for quadratic nonlinear KGE [1].

Singular soliton solution

The starting hypothesis for the singular soliton solution is given by

$$q(x,t) = A\,\mathrm{csch}^p \tau, \tag{2}$$

where

$$\tau = B(x - vt).$$

In Equation 2, the constant parameters are A and B, while v is the velocity of the soliton. Now, substituting Equation 2 into Equation 1 gives

$$\left(v^2 - k^2\right) p^2 AB^2 \operatorname{csch}^p \tau + \left(v^2 - k^2\right) p(p+1) AB^2 \operatorname{csch}^{p+2} \tau = aA \operatorname{csch}^p \tau + bA^2 \operatorname{csch}^{3p} \tau. \quad (3)$$

It is well known that the solitons evolve due to a delicate balance between dispersion and nonlinearity. Therefore, by the aid of this balancing principle, equating the exponents np and $p+2$ implies

$$3p = p + 2$$

that gives

$$p = 1.$$

Now, the linearly independent functions in Equation 3 are $\operatorname{csch}^{p+j} \tau$ for $j = 0, 2$. Therefore, setting their respective coefficients to zero implies

$$A = \sqrt{\frac{3a}{2b}} \quad (4)$$

and

$$B = \sqrt{\frac{a}{v^2 - k^2}}. \quad (5)$$

Now Equation 4 poses the restriction

$$ab > 0, \quad (6)$$

while Equation 5 introduces the constraint

$$a\left(v^2 - k^2\right) > 0. \quad (7)$$

Therefore, the singular 1-soliton solution to Equation 1 is given by

$$q(x,t) = A \operatorname{csch}[\, B(x - vt)] \,,$$

where the parameters A and B are given by Equations 4 and 5, respectively. The domain restriction that is given by restriction (6) and constraint (7) must also hold in order for the singular soliton to exist.

Methods

Numerical analysis

The following equation is known as the cubic Klein-Gordon equation

$$u_{tt} = u_{xx} + 2u - u^3, \quad (8)$$

and the initial conditions are given as follows:

$$u(x,0) = \frac{1}{2}\left(\tanh \frac{x - x_1}{\sqrt{1 - c_1^2}} + \tanh \frac{x - x_2}{\sqrt{1 - c_2^2}}\right)$$

and

$$u_t(x,0) = -\frac{1}{2}\frac{c_1}{\sqrt{1 - c_1^2}}\left(\operatorname{sech} \frac{x - x_1}{\sqrt{1 - c_1^2}}\right)^2 - \frac{1}{2}\frac{c_2}{\sqrt{1 - c_2^2}}\left(\operatorname{sech} \frac{x - x_2}{\sqrt{1 - c_2^2}}\right)^2.$$

In the next sections, this partial differential equation will be solved by transforming it into a system of ordinary differential equations (ODEs) by a judiciously chosen spectral method whose basis functions are rational Chebyshev functions.

Spectral method

The rational Chebyshev functions (TB functions), introduced by Boyd (for complete reference, see [15]), are well known as radiation basis functions as nonlocal solitions as well as radiation function basis for quantum scattering (continuous spectrum). These functions are defined as

$$\mathrm{TB}_n\,(x) = \cos(n\theta), \quad \text{where} \quad \cot\theta = x.$$

The rational Chebyshev functions are orthogonal on $(-\infty, \infty)$ with the weight function chosen as $w(x) = 1/(1+x^2)$. The set of the first 11 basis functions, i.e., from TB_0 to TB_{10} can be found in [15]. In Appendix 1, the necessary formulas of the first 21 basis functions are cited to make the paper self-content.

By choosing TB functions as basis functions, one can express $u(x,t)$ as

$$u(x,t) = \sum_{l=0}^{N} a_l(t)\, \mathrm{TB}_l\,(x).$$

Now, by differentiating u twice with respect to t, we get,

$$u_{tt}(x,t) = \sum_{l=0}^{N} \ddot{a}_l(t)\, \mathrm{TB}_l\,(x),$$

where $\ddot{a}_l(t) = \frac{d^2 a}{d\, t^2}$.

Again, by differentiating u twice with respect to x, we get,

$$u_{xx}(x,t) = \sum a_l(t)\, \mathrm{TB}_l''\,(x).$$

One can express $\mathrm{TB}_l''(x)$ as $\mathrm{TB}_l''(x) = \sum \Phi_l(x)$, where $\Phi_l(x)$ will render a nine-diagonal matrix when we will take the inner product with respect to another TB function multiplied by suitable weight function (please refer to Appendix 2 for the required formula of $\mathrm{TB}_l''(x)$ in terms of other TB functions).

So, the second derivative of $u(x,t)$ with respect to x can be expressed as

$$u_{xx}(x,t) = \sum_{m}^{N} a_m(t)\,\Phi_{ml}(x).$$

Also, $u^3(x,t)$ can be written as

$$u^3(x,t) = \sum_{m_1=1}^{N}\sum_{m_2=1}^{N}\sum_{m_3=1}^{N} a_{m_1}(t)\,\mathrm{TB}_{m_1}(x)\,a_{m_2}(t) \times \mathrm{TB}_{m_2}(x)\,a_{m_3}(t)\,\mathrm{TB}_{m_3}(x).$$

Since TB functions are actually cosine functions, we can use the additive and multiplicative properties of cosine functions for TB functions in a similar fashion.

$$\cos(m_1x)\cos(m_2x) = \frac{1}{2}\left[\cos(m_1+m_2)x + \cos(m_1-m_2)x\right].$$

This yields

$$\mathrm{TB}_{m_1}(x)\mathrm{TB}_{m_2}(x) = \frac{1}{2}\mathrm{TB}_{m_1+m_2}(x) + \frac{1}{2}\mathrm{TB}_{m_1-m_2}(x).$$

Similarly, for the product of three TB functions,

$$\begin{aligned}\mathrm{TB}_{m_1}(x)\mathrm{TB}_{m_2}(x)\mathrm{TB}_{m_3}(x) =& \frac{1}{4}\mathrm{TB}_{m_1} + m_2 + m_3(x)\\ &+ \frac{1}{4}\mathrm{TB}_{m_1+m_2-m_3}(x)\\ &+ \frac{1}{4}\mathrm{TB}_{|}m_2 + m_3 - m_1|(x)\\ &+ \frac{1}{4}\mathrm{TB}_{|}m_1 - |m_2 - m_3||(x).\end{aligned}$$

So, the product of three TB functions can be written as

$$\mathrm{TB}_{m_1}(x)\mathrm{TB}_{m_2}(x)\mathrm{TB}_{m_3}(x) = \sum_{l} \Psi_{m_1\,m_2\,m_3\,l}\mathrm{TB}_l(x).$$

Equation 8 can be written as

$$\begin{aligned}\sum_{l=1}^{N} \ddot{a}_l(t)\,\mathrm{TB}_l(x) - \sum_{l=1}^{N}\sum_{m}^{N} a_m(t)\,\Phi_{ml}(x)\\ - 2\sum_{l=1}^{N} a_l(t)\,\mathrm{TB}_l(x)\\ + \sum_{l} \Psi_{m_1\,m_2\,m_3\,l}\mathrm{TB}_l(x) = 0,\end{aligned} \tag{9}$$

where

$$\begin{aligned}\ddot{a}_l(t) =& \sum_{m=1}^{N} a_m(t) * M_{ml} + 2a_l(t)\\ &- \sum_{m_1=1}^{N}\sum_{m_2=1}^{N}\sum_{m_3=1}^{N} a_{m_1}(t)\,a_{m_2}(t)\,a_{m_3}(t)\Psi_{m_1\,m_2\,m_3\,l}.\end{aligned} \tag{10}$$

Now, by using the orthogonality property of TB functions, we have the inner product:

$$(\mathrm{TB}_l, \mathrm{TB}_j) = \int_{-\infty}^{\infty} W(x)\,\mathrm{TB}_l(x)\,\mathrm{TB}_j(x)\,dx = \begin{cases}\pi/2 & \text{if} \quad l = j > 0,\\ 0 & \text{if} \quad l \neq j,\\ \pi & \text{if} \quad l = j = 0.\end{cases}$$

We also observe that the fourth term in Equation 10 can be simplified in the following way:

$$\Psi_{m_1\,m_2\,m_3\,l} = \begin{cases}1/4 & \text{for} \quad l = m_1 + m_2 + m_3,\\ 1/4 & \text{for} \quad l = m_1 - m_2 - m_3,\\ 1/4 & \text{for} \quad l = m_1 + m_2 - m_3,\\ 1/4 & \text{for} \quad l = m_1 - |m_2 - m_3|.\end{cases}$$

It was noted by Christov [16] that the derivative of a rational orthogonal function is the sum of some basis functions, at most three basis functions for the first derivative. It is argued by Boyd [17] that while using a Galerkin's method to transform a partial differential equation into a matrix problem, one eventually encounters the matrix with only a few nonzero elements in each row. The inner product $(\Phi_{ml}(x), \mathrm{TB}_j(x))$ yields the aforementioned nine-diagonal matrix M_{ml}.

Equation 10 is a system of ordinary differential equations of variable t. So, if N is chosen to be 20, Equation 9 turns out to be a system of 20 ordinary differential equations of order two. For solving this system of equations, we need 40 initial conditions, i.e., for each l, two initial conditions are required since this is ODE of order two. These initial conditions for each l are constructed in the following way:

$$a_l(0) = \int_{-\infty}^{\infty} W(x)\,\mathrm{TB}_l(x)u(x,0)dx,$$

and

$$a_l'(0) = \int_{-\infty}^{\infty} W(x)\,\mathrm{TB}_l(x)u_t(x,0)dx.$$

$u(x,0)$ and $u_t(x,0)$ are the initial conditions for Equation 8.

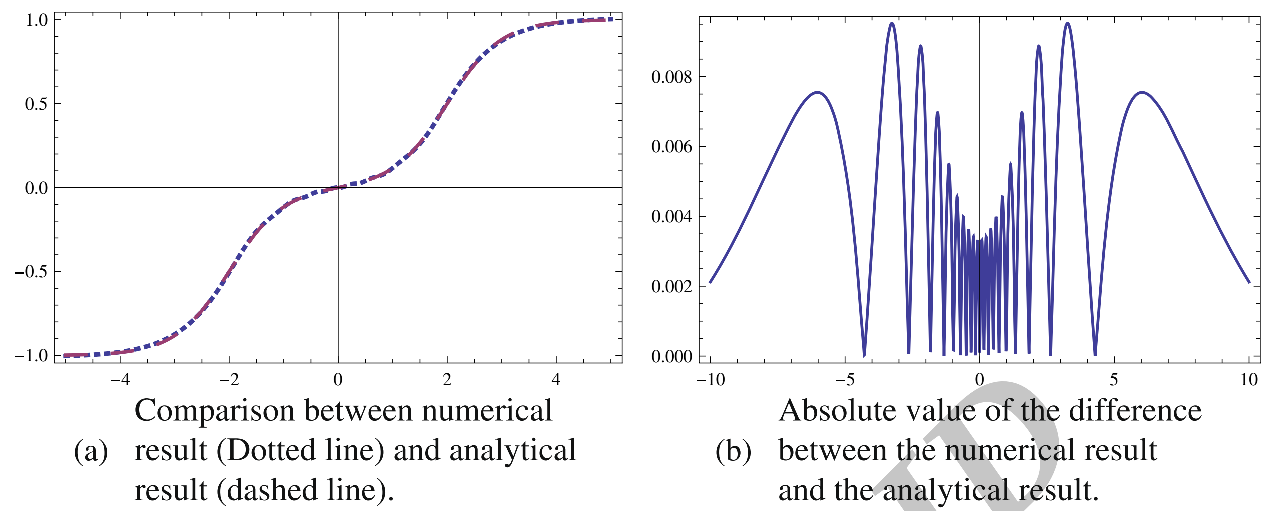


Figure 1 Comparison between the analytical solution and numerically obtained solution at $t = 0$ for $x_1 = -2, x_2 = 2, c_1 = 0.1$, and $c_2 = -0.1$. (**a**) Comparison between the numerical result (dotted line) and analytical result (dashed line). (**b**) Absolute value of the difference between the numerical result and the analytical result.

Results and discussion

Numerical simulation and results

Here, we can use any standard numerical ODE integration algorithm, e.g., MATHEMATICA's NIntegrate, to calculate all the integrations for finding initial values as well as for determining the numerical value of all the integrations which are defined to find the inner product. Then, we have used NDSolve for solving the initial value problem in Equation 10 and subsequently Equation 9.

Numerical results are provided by taking two sets of values for x_1, x_2, c_1, and c_2. In Figures 1 and 2, the comparison between the chosen boundary condition $u(x,0) = u_{\text{anal}}(x,0)$ and numerically obtained $u(x,0) = u_{\text{num}}(x,0)$ is shown. In the left hand panels of Figures 1 and 2, $u_{\text{anal}}(x,0)$ and $u_{\text{num}}(x,0)$ are drawn together in the same window. The right hand panels of these figures are used to show the absolute value of their difference, i.e., the error estimates. Calculations for Figures 1 and 2 are performed by assuming $x_1 = -2, x_2 = 2, c_1 = 0.1, c_2 = -0.1$ and $x_1 = -1, x_2 = 1, c_1 = 0.01, c_2 = -0.005$, respectively.

In Figures 3 and 4, the comparison between the chosen boundary condition $u_t(x,0) = \frac{d}{dt}u_{\text{anal}}(x,0)$ and

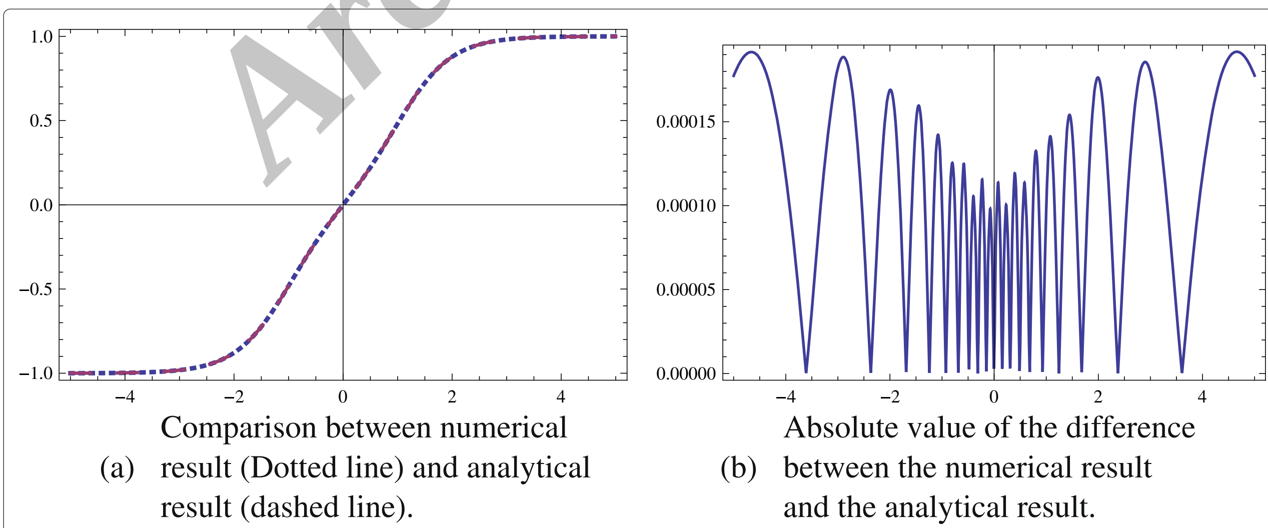
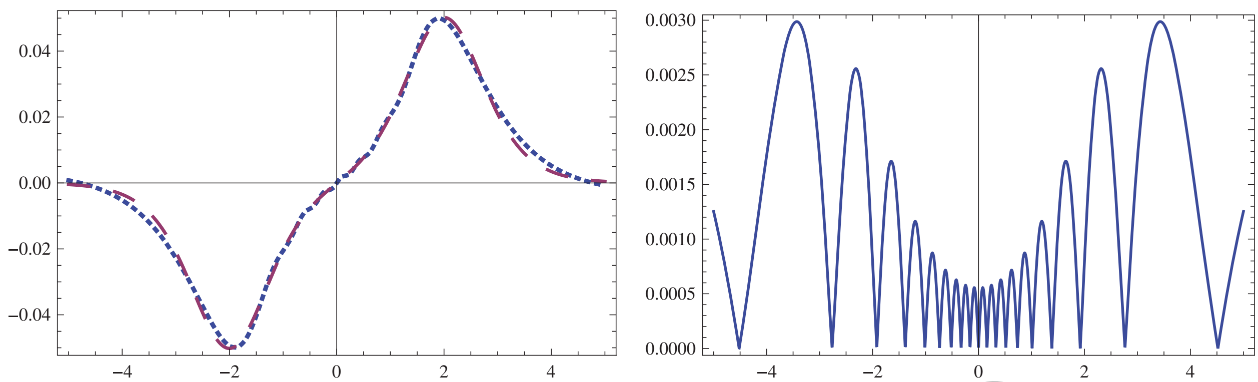


Figure 2 Comparison between the analytical solution and numerically obtained solution at $t = 0$ for $x_1 = -1, x_2 = 1, c_1 = 0.01$, and $c_2 = -0.005$. (**a**) Comparison the between numerical result (dotted line) and analytical result (dashed line). (**b**) Absolute value of the difference between the numerical result and the analytical result.



(a) Comparison between numerical result (Dotted line) and analytical result (dashed line).

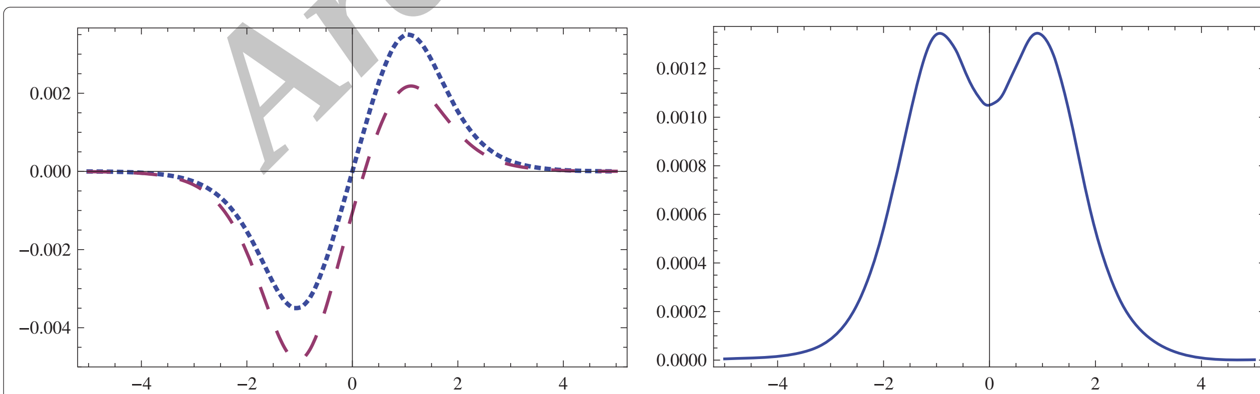
(b) Absolute value of the difference between the numerical result and the analytical result.

Figure 3 Comparison between the profile of analytical $u_t(x,t)$ and numerically obtained profile of $u_t(x,t)$ at $t = 0$ for $x_1 = -2, x_2 = 2,$ $c_1 = 0.1$, and $c_2 = -0.1$. (**a**) Comparison between numerical result (dotted line) and analytical result (dashed line). (**b**) Absolute value of the difference between the numerical result and the analytical result.

numerically obtained $u_t(x,0) = \frac{d}{dt}u_{\text{num}}(x,0)$ is shown. In the left hand panels of Figures 3 and 4, both $\frac{d}{dt}u_{\text{anal}}(x,0)$ and $\dfrac{d}{dt}u_{\text{num}}(x,0)$ $u(x,0)$ are shown together in the same window. Similarly, in the right hand panels of these figures, the absolute value of their difference is shown. Calculations for Figures 3 and 4 are carried out by assuming $x_1 = -2, x_2 = 2, c_1 = 0.1, c_2 = -0.1$ and $x_1 = -1,$ $x_2 = 1, c_1 = 0.01, c_2 = -0.005$, respectively.

One can observe from Figures 1, 2, 3, and 4 that both sets of numerically obtained initial solution and the chosen initial conditions are matching with each other almost perfectly for both given sets of x_1, x_2, c_1, and c_2. As far as the error is concerned, the 'difference' graphs in the right hand panels of these figures show that the deviation is in the range of 10^{-3} to 10^{-4} which is quite reasonable considering the fact that the number of terms chosen is $N = 20$ only.

In Figure 5, numerically obtained solutions for different values of t are shown. It can be noticed that the graph of $u(x,t)$ remains bounded between $[-1,1]$ for small $t \leq 1$. The solution remains bounded even for higher t as $t =$ 40, though not necessarily between $[-1,1]$. For higher values of t, the profile of $u(x,t)$ tends to go outside the



(a) Comparison between numerical result (Dotted line) and analytical result (dashed line).

(b) Absolute value of the difference between the numerical result and the analytical result.

Figure 4 Comparison between the profile of analytical $u_t(x,t)$ and numerically obtained profile of $u_t(x,t)$ at $t = 0$ for $x_1 = -1, x_2 = 1,$ $c_1 = 0.01$, and $c_2 = -0.005$. (**a**) Comparison between numerical result (dotted line) and analytical result (dashed line). (**b**) Absolute value of the difference between the numerical result and the analytical result.

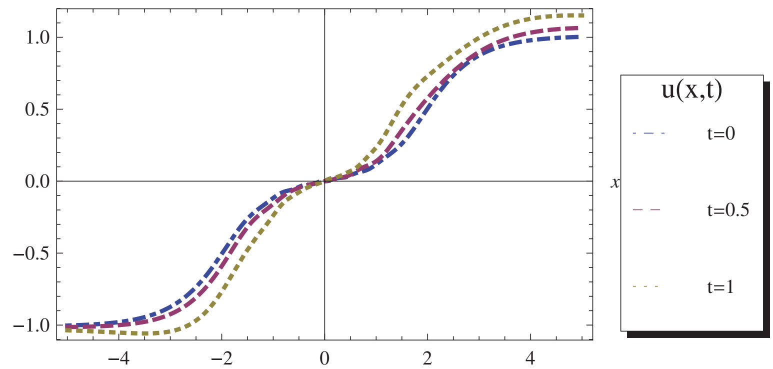


Figure 5 Solution profile $u(x,t)$ at different t.

region of $[-1,1]$ because the amount of error accumulated becomes considerably higher. It might be taken care of by choosing more TB functions. The possible significance of selecting higher numbers of spectral basis functions on reducing the error estimates will be investigated in future.

Now, it will be interesting to check our solution from a different point of view. To further validate the robustness of the spectral method used, we would like to investigate the 'TIME' part of the numerically obtained solution $u(x,t)$. In order to do that, we have considered four different points of time, say $t = 0$, $t = 0.1$, $t = 0.5$, and

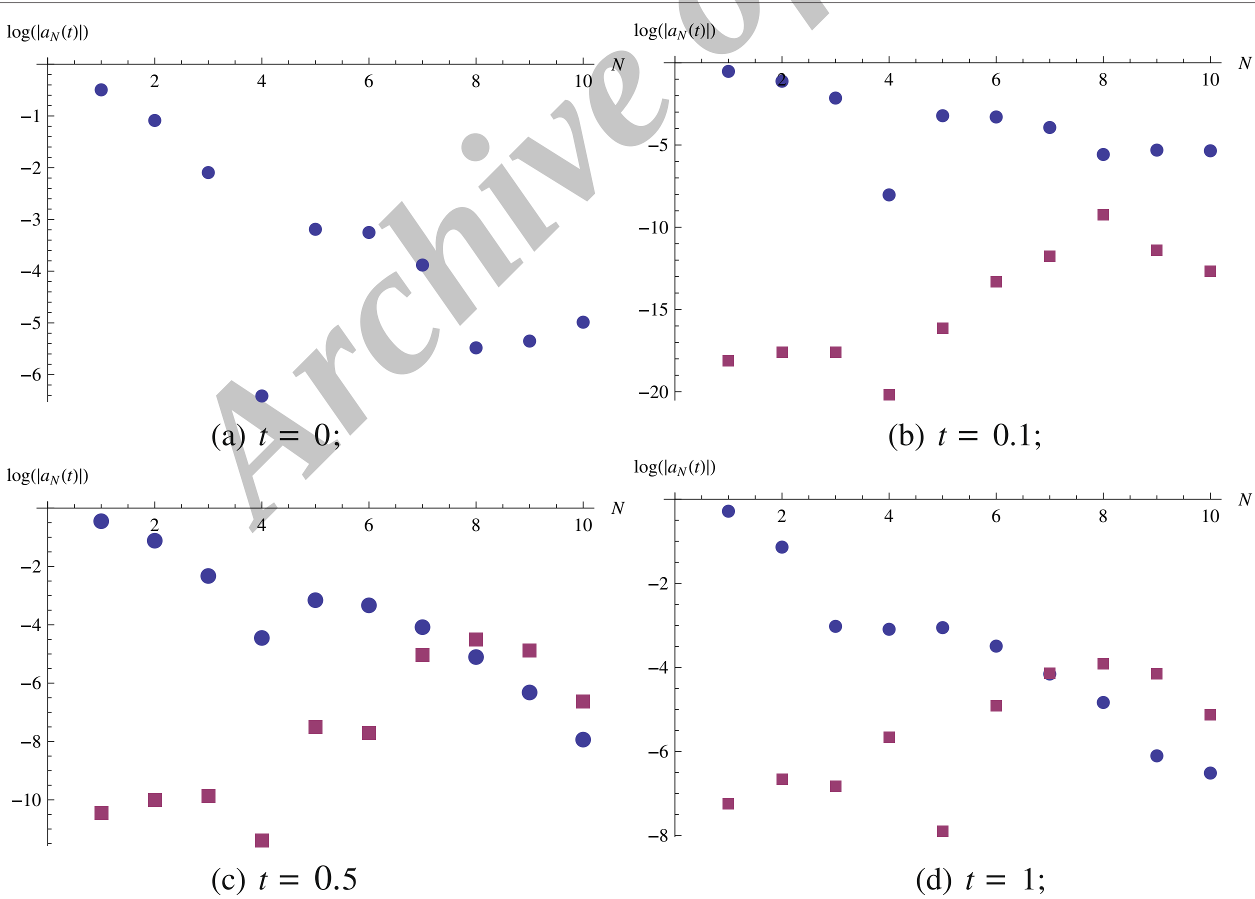


Figure 6 Scatterplot for odd and even time solutions. The odd time solutions are shown by the 'circles,' and even time solutions are shown by the 'rectangles.' (**a**) $t = 0$, (**b**) $t = 0.1$, (**c**) $t = 0.5$, and (**d**) $t = 1$.

$t = 1$ for the following set: $x_1 = -2$, $x_2 = 2$, $c_1 = 0.1$, and $c_2 = -0.1$. In the numerical method used, we have chosen $N = 20$, which means that out of these 20 terms, 10 of them are associated with the TB functions with odd degrees, which we will call *odd time solutions*. The other 10 are associated with the TB functions with even degrees, which can be called *even time solutions*. Upon obtaining and separating these odd and even time solutions, they are plotted in the same graph for different t, as shown in the Figure 6.

To ensure the boundedness of the solutions, odd time solutions should be decreasing, and the even time solutions should be increasing. For $t = 0$, only the odd time solutions are shown because at $t = 0$, the even time solutions are found to be very trivial; that is why they are considered to be equal to zero. From these scatter-plot graphs, one can observe that odd time solutions are showing a downward (decreasing) tendency, whereas even time solutions are found to be demonstrating an upward (increasing) tendency; in this process, the solutions are minimizing each others' effect to achieve boundedness. At some point though, some solutions show waywardness. This discrepancy in the time solution renders the error in the original solution $u(x, t)$.

Conclusions

This paper studied the analytical and numerical simulations for the P4 equation that is studied in relativistic quantum mechanics. The singular 1-soliton solution of the equation is obtained by the ansatz method. This leads to a couple of constraint conditions that must hold in order for the soliton solution to exist. Subsequently, the spectral method is employed to simulate the kink or topological solitons, which are also known as shock waves, by the aid of TB functions. The agreement between the exact solutions and the numerical simulations is excellent. In addition to visualizing this, the error estimate plots also additionally indicate that. Additionally, the log plots indicate an almost perfect agreement too. In the future, these studies will be extended to KGE with power law nonlinearity and also with perturbation terms. Such results will be reported in the future.

Appendices

Appendix 1

Rational Chebyshev functions (TB$_n(x)$) for the infinite interval

$$\mathrm{TB}_0(x) = 1;\;\; \mathrm{TB}_1(x) = x/\left(x^2+1\right)^{\frac{1}{2}};$$

$$\mathrm{TB}_2(x) = (x^2-1)/(x^2+1);\;\; \mathrm{TB}_3(x) = x\left(x^2-3\right)/\left(x^2+1\right)^{\frac{3}{2}};$$

$$\mathrm{TB}_4(x) = \left(x^4 - 6x^2 + 1\right)/\left(x^2+1\right)^2;$$

$$\mathrm{TB}_5(x) = x\left(x^4 - 10x^2 + 5\right)/\left(x^2+1\right)^{\frac{5}{2}};$$

$$\mathrm{TB}_6(x) = \left(x^6 - 15x^4 + 15x^2 - 1\right)/\left(x^2+1\right)^3;$$

$$\mathrm{TB}_7(x) = x\left(x^6 - 21x^4 + 35x^2 - 7\right)/\left(x^2+1\right)^{\frac{7}{2}};$$

$$\mathrm{TB}_8(x) = \left(x^8 - 28x^6 + 70x^4 - 28x^2 + 1\right)/\left(x^2+1\right)^4;$$

$$\mathrm{TB}_9(x) = x\left(x^8 - 36x^6 + 126x^4 - 84x^2 + 9\right)/\left(x^2+1\right)^{\frac{9}{2}};$$

$$\mathrm{TB}_{10}(x) = \left(x^{10} - 45x^8 + 210x^6 - 210x^4 + 45x^2 - 1\right)/\left(x^2+1\right)^5;$$

$$\mathrm{TB}_{11}(x) = x\left(x^{10} - 55x^8 + 330x^6 - 462x^4 + 165x^2 - 11\right)/\left(x^2+1\right)^{\frac{11}{2}};$$

$$\mathrm{TB}_{12}(x) = \left(x^{12} - 66x^{10} + 495x^8 - 924x^6 + 495x^4 - 66x^2 + 1\right)/\left(x^2+1\right)^6;$$

$$\mathrm{TB}_{13}(x) = x\left(x^{12} - 78x^{10} + 715x^8 - 1,716x^6 + 1,287x^4 - 286x^2 + 13\right)/\left(x^2+1\right)^{\frac{13}{2}};$$

$$\mathrm{TB}_{14}(x) = \left(x^{14} - 91x^{12} + 1,001x^{10} - 3,003x^8 + 3,003x^6 - 1,001x^4 + 91x^2 - 1\right)/\left(x^2+1\right)^7;$$

$$\mathrm{TB}_{15}(x) = x\left(x^{14} - 105x^{12} + 1,365x^{10} - 5,005x^8 + 6,435x^6 - 3,003x^4 + 455x^2 - 15\right)/\left(x^2+1\right)^{\frac{15}{2}};$$

$$\mathrm{TB}_{16}(x) = \left(x^{16} - 120x^{14} + 1,820x^{12} - 8,008x^{10} + 12,870x^8 - 8,008x^6 + 1,820x^4 - 120x^2 + 1\right)/\left(1+x^2\right)^8;$$

$$\mathrm{TB}_{17}(x) = x\left(x^{16} - 136x^{14} + 2,380x^{12} - 12,376x^{10} + 24,310x^8 - 19,448x^6 + 6,188x^4 - 680x^2 + 17\right)/\left(x^2+1\right)^{\frac{17}{2}};$$

$$\mathrm{TB}_{18}(x) = \left(x^{18} - 153x^{16} + 3,060x^{14} - 18,564x^{12} + 43,758x^{10} - 43,758x^8 + 18,564x^6 - 3,060x^4 + 153x^2 - 1\right)/\left(x^2+1\right)^9;$$

$$\mathrm{TB}_{19}(x) = x\left(x^{18} - 171x^{16} + 3,876x^{14} - 27,132x^{12} + 75,582x^{10} - 92,378x^8 + 50,388x^6 - 11,628x^4 + 969x^2 - 19\right)/\left(x^2+1\right)^{\frac{19}{2}};$$

$$\mathrm{TB}_{20}(x) = \left(x^{20} - 190x^{18} + 4,845x^{16} - 38,760x^{14} + 125,970x^{12} - 184,756x^{10} + 125,970x^8 - 38,760x^6 + 4,845x^4 - 190x^2 + 1\right)/\left(x^2+1\right)^{10}.$$

Appendix 2

Expressing the second derivative of TB functions as a linear combination of other TB functions

It can be shown that the double derivative of any TB function is indeed a linear combination of other TB functions.

We know that $\mathrm{TB}_n(x) = \cos n\theta$, where $\cot\theta = x$ and $\frac{d\theta}{dx} = -\sin^2\theta$.

The following expression for $\frac{d^2}{dx^2}\mathrm{TB}_n(x)$ involves a combination of some trigonometric functions involving various degrees of $\sin\theta$ and $\cos\theta$.

$$\mathrm{TB}_n''(x) = -\left(n^2 \cos\, n\theta\, \sin^2\theta + 2n \sin\, n\theta\, \cos\,\theta\, \sin\,\theta\right) \sin^2\theta. \quad (11)$$

It is quite obvious that to have a linear combination of TB functions of various orders, one has to express the right hand side of Equation 11 only as combination of cosine functions of the first degree, though of different angles. In order to achieve that, one has to use the following identities:

$$\sin^4\theta = \left(\frac{1}{8}\cos 4\theta - \frac{1}{2}\cos\, 2\theta + \frac{3}{8}\right);$$
$$\sin^2\theta = \left(\frac{1}{2} - \frac{1}{2}\cos\, 2\theta\right);$$
$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta);$$
$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta);$$

Basic use of these identities allows us to achieve the desired form of $\frac{d^2}{dx^2}\mathrm{TB}_n(x)$ in the following way:

$$\begin{aligned}
\mathrm{TB}_n''(x) &= -n^2\cos n\theta\sin^4\theta - 2n\sin n\theta\,\cos\theta\sin^3\theta \\
&= -n^2\cos n\theta\left(\frac{1}{8}\cos 4\theta - \frac{1}{2}\cos 2\theta + \frac{3}{8}\right) \\
&\quad - n\sin n\theta\left(\frac{1}{2}\sin 2\theta - \frac{1}{4}\sin 4\theta\right) \\
&= -\frac{n^2}{16}\cos(n+4)\theta - \frac{n^2}{16}\cos(n-4)\theta \\
&\quad + \frac{n^2}{4}\cos(n+2)\theta + \frac{n^2}{4}\cos(n-2)\theta \\
&\quad - \frac{3n^2}{8}\cos n\theta + \frac{n}{4}\cos(n+2)\theta - \frac{n}{4}\cos(n-2)\theta \\
&\quad + \frac{n}{4}\cos(n-4)\theta - \frac{n}{8}\cos(n+4)\theta \\
&= -\left(\frac{n^2}{16} + \frac{n}{8}\right)\cos(n+4)\theta \\
&\quad + \left(\frac{n^2}{4} + \frac{n}{4}\right)\cos(n+2)\theta - \frac{3n^2}{8}\cos n\theta \\
&\quad + \left(\frac{n^2}{4} - \frac{n}{4}\right)\cos(n-2)\theta \\
&\quad - \left(\frac{n^2}{16} - \frac{n}{8}\right)\cos(n-4)\theta \\
&= -\left(\frac{n^2+2n}{16}\right)TB_{n+4}(x) \\
&\quad + \left(\frac{n^2+n}{4}\right)TB_{n+2}(x) - \frac{3n^2}{8}TB_n(x) \\
&\quad + \left(\frac{n^2-n}{4}\right)TB_{n-2}(x) - \left(\frac{n^2-2n}{16}\right)TB_{n-4}(x).
\end{aligned}$$

Competing interests
Both authors declare that they have no competing interests.

Authors' contributions
AC carried out the numerical simulation after constructing the algorithm based on the suitable spectral method. AC also validated the results using elaborate graphical representation and drafted the manuscript. AB established the motivation of this particular work by providing the singular soliton solution obtained by the ansatz method. Both authors read and approved the final manuscript.

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References

1. Basak, KC, Ray, PC, Bera, RK: Solution of non-linear Klein-Gordon equation with a quadratic non-linear term by Adomian decomposition method. Commun. Nonlinear Sci. Numer. Simul. **14**(3), 718–723 (2009)
2. Biswas, A, Zony, C, Zerrad, E: Soliton perturbation theory for the quadratic nonlinear Klein-Gordon equation. Appl. Mathematics Comput. **203**(1), 153–156 (2008)
3. Biswas, A, Yildirim, A, Hayat, T, Aldossary, OM, Sassaman, R: Soliton perturbation theory for the generalized Klein-Gordon equation with full nonlinearity. Proc. Romanian Acad. Ser. A. **13**(1), 32–41 (2012)
4. Sassaman, R, Biswas, A: Topological and non-topological solitons of the generalized Klein-Gordon equation. Mathematics Comput. App. **215**(1), 212–220 (2009)
5. Sassaman, R, Heidari, A, Majid, F, Biswas, A: Topological and non-topological solitons of the generalized Klein-Gordon equation in (1+2) dimensions. Dyn. Continuous, Discrete Impulsive Sys.: Ser. A. **17**(2a), 275–286 (2010)
6. Sassaman, R, Heidari, A, Biswas, A: Topological and non-topological solitons of the nonlinear Klein-Gordon equations by He's semi-inverse variational principle. J. Franklin Inst. **347**(7), 1148–1157 (2010)
7. Sassaman, R, Biswas, A: Topological and non-topological solitons of the Klein-Gordon equations in (1+2) dimensions. Nonlinear Dyn. **61**(1-2), 23–28 (2010)
8. Sassaman, R, Biswas, A: Soliton solutions of the generalized Klein-Gordon equation by semi-inverse variational principle. Mathematics Eng. Sci. Aerospace. **2**(1), 99–104 (2011)
9. Sassaman, R, Biswas, A: 1-soliton solution of the perturbed Klein-Gordon equation. Phys. Express. **1**(1), 9–14 (2011)
10. Shakeri, F, Dehghan, M: Numerical solution of the Klein-Gordon equation via He's variational iteration method. Nonlinear Dyn. **51**(1-2), 89–97 (2007)
11. Sirendaoreji: Exact travelling wave solutions for four forms of nonlinear Klein-Gordon equations. Phys. Lett. A. **363**, 440–447 (2007)
12. Wazwaz, AM: Solutions of compact and noncompact structures for nonlinear Klein-Gordon type equation. Appl. Mathematics Comput. **134**(2-3), 487–500 (2003)
13. Wazwaz, AM: The tanh and sine-cosine methods for compact and noncompact solutions of the nonlinear Klein-Gordon equation. Appl. Mathematics Comput. **167**(2), 1179–1195 (2005)
14. Wazwaz, AM: Generalized forms of the phi-four equation with compactons, solitons and periodic solutions. Mathematics Comput. Simul. **69**(5-6), 580–588 (2005)
15. Boyd, JP: Chebyshev and Fourier Spectral Methods. Springer, New York (1989)
16. Christov, CI: A complete orthonormal system of functions in $L^2(-\infty,\infty)$ space. SIAM J. Appl. Mathematics. **42**, 1337–1334 (1982)
17. Boyd, JP: The orthogonal rational functions of Higgins and Christov and algebraically Mapped Chebyshev Polynomials. J. Approximation Theory. **61**, 98–105 (1990)

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