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Stability and common stability for the systems of linear equations and its applications

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Abstract

In this paper some results about the Hyers-Ulam-Rassias stability for the linear functional equations in general form and its Pexiderized can be proved for given functions on general domain to a complex Banach spaces under some suitable conditions. In connection with the problem of G. L. Forti in the 13st ICFEI we consider the common stability for the systems of functional equations and our aim is to establish some common Hyers-Ulam-Rassias stability for systems of homogeneous linear functional equations. The results is applied to the study of some superstability results for the exponential functional equation.

Keywords: Superstability, Common stability, Linear equation, Fixed point

Introduction

The starting point of the stability theory of functional equations was the problem formulated by S. M. Ulam in 1940 (see [1]), during a conference at Wisconsin University:

Let $(G, .)$ be a group $(B, ., d)$ be a metric group. Does for every $\varepsilon > 0$, there exists a $\delta > 0$ such that if a function $f : G \to B$ satisfies the inequality

$$d(f(xy), f(x)f(y)) \leq \delta, \quad x, y \in G,$$

there exists a homomorphism $g : G \to B$ such that

$$d(f(x), g(x)) \leq \varepsilon, \quad x \in G?$$

In 1941, D.H. Hyers [2] gave an affirmative partial answer to this problem. This is the reason for which today this type of stability is called Hyers-Ulam stability of functional equation. In 1950, Aoki [3] generalized Hyers' theorem for approximately additive functions. In 1978, Th. M. Rassias [4] generalized the theorem of Hyers by considering the stability problem with unbounded Cauchy differences. Taking this fact into account, the additive functional equation $f(x + y) = f(x) + f(y)$ is said to have the Hyers-Ulam-Rassias stability on (X, Y). This terminology is also applied to the case of other functional equations. On the other hand, J. M. Rassias [5-7] considered the Cauchy difference controlled by a a product of different powers of norm. However, there was a singular case; for this singularity a counterexample was given by P. Gavruta [8]. This stability is called Ulam-Gavruta-Rassias stability. In addition, J. M. Rassias considered the mixed product-sum of powers of norms as the control function. This stability is called J.M.Rassias stability (see also [9-12]). For more detailed definitions of such terminologies one can refer to [13] and [14]. Thereafter, the stability problem of functional equations has been extended in various directions and studied by several mathematicians (see, e.g., [15-29]).

The Hyers-Ulam stability of mappings is in development and several authors have remarked interesting applications of this theory to various mathematical problems. In fact the Hyers-Ulam stability has been mainly used to study problems concerning approximate isometries or quasi-isometries, the stability of Lorentz and conformal mappings, the stability of stationary points, the stability of convex mappings, or of homogeneous mappings, etc [30-34].

Of the most importance is the linear functional equation in general form (see [35])

$$f(\rho(x)) = p(x)f(x) + q(x) \tag{1.1}$$

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where ρ, p and q are given functions on an interval I and f is unknown. When $q(x) \equiv 0$ this equation, i.e.,

$$f(\rho(x)) = p(x)f(x) \tag{1.2}$$

is called homogeneous linear equation. We refer the reader to [35,36] for numerous results and references concerning this equation and its stability in the sense of Ulam.

In 1991 Baker [37] discussed Hyers-Ulam stability for linear equations (1.1). More concretely, the Hyers-Ulam stability and the generalized Hyers-Ulam-Rassias stability for equation

$$f(x+p) = kf(x) \tag{1.3}$$

were discussed by Lee and Jun [38]. Also the gamma functional equation is a special form of homogeneous linear equation (1.2) were discussed by S. M. Jung [39-41] proved the modified Hyers-Ulam stability of the gamma functional equation. Thereafter, the stability problem of gamma functional equations has been extended and studied by several mathematicians [42-46].

Throughout this paper, assume that X is a nonempty set, $F = \mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and B is a Banach spaces over F and also $\psi : X \to \mathbb{R}^+, f, g : X \to B, p : X \to F\backslash\{0\}$ and $q : X \to B$ are functions and also $\sigma : X \to X$ is a arbitrary map.

In the first section of this paper, we present some results about Hyers-Ulam-Rassias stability via a fixed point approach for the linear functional equation in general form (1.1) and its Pexiderized

$$f(\rho(x)) = p(x)g(x) + q(x) \tag{1.4}$$

under some suitable conditions. Note that the main results of this paper can be applied to the well known stability results for the gamma, beta, Abel, Schrӧder, iterative and G-function type's equations, and also to certain other forms.

In 1979, another type of stability was observed by J. Baker, J. Lawrence and F. Zorzitto [47]. Indeed, they proved that if a function is approximately exponential, then it is either a true exponential function or bounded. This result was the first result concerning the superstability phenomenon of functional equations see also [48-51]). Later, J. Baker [52] (see also [51]) generalized this famous result as follows:

Let $(S, \cdot)$ be an arbitrary semigroup, and let f map S into the field C of all complex numbers. Assume that f is an approximately exponential function, i.e., there exists a nonnegative number ε such that

$$\|f(x \cdot y) - f(x)f(y)\| \leq \varepsilon$$

for all $x, y \in S$. Then f is either bounded or exponential.

The result of Baker, Lawrence and Zorzitto [47] was generalized by L. Székelyhidi [53] in another way and he obtained the following result.

Theorem 1.1. *[53] Let $(G, .)$ be an Abelian group with identity and let $f, m : G \to \mathbb{C}$ be functions such that there exist functions $M_1, M_2 :\to [0, \infty)$ with*

$$\|f(x.y) - f(x)m(y)\| \leq \min\{M_1(x), M_2(y)\}$$

for all $x, y \in G$. Then either f is bounded or m is an exponential and $f(x) = f(1)g(x)$ for all $x \in G$.

During the thirty-first International Symposium on Functional Equations, Th. M. Rassias [54] introduced the term *mixed stability* of the function $f : E \to \mathbb{R}$ (or $\mathbb{C}$), where E is a Banach space, with respect to two operations 'addition' and 'multiplication' among any two elements of the set $\{x, y, f(x), f(y)\}$. Especially, he raised an open problem concerning the behavior of solutions of the inequality

$$\|f(x.y) - f(x)f(y)\| \leq \theta(\|x\|^p + \|y\|^p).$$

During the 13st International Conference on Functional Equations and Inequalities 2009, G. L. Forti posed following problem.

Problem. Consider functional equations of the form

$$\sum_{i=1}^{n} a_i f(\sum_{k=1}^{n_i} b_{ik} x_k) = 0 \qquad \sum_{i=1}^{n} a_i \neq 0 \tag{1.5}$$

and

$$\sum_{i=1}^{n} \alpha_i f(\sum_{k=1}^{n_i} \beta_{ik} x_k) = 0 \qquad \sum_{i=1}^{n} \beta_i \neq 0 \tag{1.6}$$

where all parameters are real and $f : R \to R$. Assume that the two functional equations are equivalent, i.e., they have the same set of solutions. Can we say something about the common stability? More precisely, if (1.5) is stable, what can we say about the stability of (1.6). Under which additional conditions the stability of (1.5) implies that of (1.6)?

In connection the above problem we consider the term of common stability for systems of functional equations. In this paper, Usually the functional equations

$$E_1(f) = E_2(f); \tag{1.7}$$

$$D_1(f) = D_2(f) \tag{1.8}$$

is said to have common Hyers-Ulam stability if for a common approximate solution f_s such that

$$\|E_1(f_s(x)) - E_2(f_s(x))\| \leq \delta_1; \tag{1.9}$$

$$\|D_1(f_s(x)) - D_2(f_s(x))\| \leq \delta_2 \tag{1.10}$$

for some fixed constant $\delta_1, \delta_2 \geq 0$ there exists a common solution f of equations (1.7) and (1.8) such that

$$\|f(x) - f_s(x)\| \leq \varepsilon; \tag{1.11}$$

for some positive constant ε.

In the last section of this paper, In connection with the problem of G. L. Forti we consider some systems of homogeneous linear equations and our aim is to establish some common Hyers-Ulam-Rassias stability for these systems of functional equations. As a consequence of these results, we give some superstability results for the exponential functional equation. Furthermore, in connection with problem of Th. M. Rassias, we generalized the theorem of Baker, Lawrence and Zorzitto and theorem of L. Székelyhidi.

For the reader's convenience and explicit later use, we will recall two fundamental results in fixed point theory.

Definition 1.2. The pair (X, d) is called a generalized complete metric space if X is a nonempty set and $d : X^2 \to [0, \infty]$ satisfies the following conditions:

1. $d(x, y) \geq 0$ and the equality holds if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$;
4. every d-Cauchy sequence in X is d-convergent.

for all $x, y \in X$.

Note that the distance between two points in a generalized metric space is permitted to be infinity.

Definition 1.3. Let (X, d) be a metric space. A mapping $J : X \to X$ satisfies a Lipschitz condition with Lipschitz constant $L \geq 0$ if

$$d(J(x), J(y)) \leq Ld(x, y)$$

for all $x, y \in X$. If $L < 1$, then J is called a strictly contractive map.

Theorem 1.4. *(Banach's contraction principle) Let (X, d) be a complete metric space and let $J : X \to X$ be strictly contractive mapping. Then*

1. *the mapping J has a unique fixed point $x^* = J(x^*)$;*
2. *the fixed point x^* is globally attractive, i.e.,*
$$\lim_{n\to\infty} J^n(x) = x^*$$
for any starting point $x \in X$;
3. *one has the following estimation inequalities:*
$$d(J^n(x), x^*) \leq L^n d(x, x^*),$$
$$d(J^n(x), x^*) \leq \frac{1}{1-L} d(J^n(x), J^{n+1}(x)),$$
$$d(x, x^*) \leq \frac{1}{1-L} d(J(x), x)$$

for all nonnegative integers n and all $x \in X$.

Theorem 1.5. *[55] Let (X, d) be a generalized complete metric space and $J : X \to X$ be strictly contractive mapping. Then for each given element $x \in X$, either*

$$d(J^n(x), J^{n+1}(x)) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

1. *$d(J^n(x), J^{n+1}(x)) < \infty$, for all $n \geq n_0$;*
2. *the sequence $\{J^n(x)\}$ converges to a fixed point y^* of J;*
3. *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}(x), y) < \infty\}$;*
4. *$d(y, y^*) \leq \frac{1}{1-L} d(J(y), y)$.*

Stability of the linear functional equation and its Pexiderized

In this section, First we consider the Hyers-Ulam-Rassias stability via a fixed point approach for the linear functional equation (1.1) and then applying these result we will investigate Pexiderized linear functional equation (1.4).

Theorem 2.1. *Let $f : X \to B$ be a function and*

$$\|f(\rho(x)) - p(x)f(x) - q(x)\| \leq \psi(x) \tag{2.1}$$

for all $x \in X$. If there exists a real $0 < L < 1$ such that

$$\psi(\rho(x)) \leq L|p(\rho(x))|\psi(x) \tag{2.2}$$

for all $x \in X$. Then there is an unique function $T : X \to B$ such that $T(\rho(x)) = p(x)T(x) + q(x)$ and

$$\|f(x) - T(x)\| \leq \frac{\psi(x)}{(1-L)|p(x)|}$$

for all $x \in X$.

Proof. Let us consider the set $\mathcal{A} := \{h : X \to B\}$ and introduce the generalized metric on $\mathcal{A}$:

$$d(u, h) = \sup_{\{x\in X\ ;\ \psi(x)\neq 0\}} \frac{|p(x)|\|g(x) - h(x)\|}{\psi(x)}.$$

It is easy to show that $(\mathcal{A}, d)$ is generalized complete metric space. Now we define the function $J : \mathcal{A} \to \mathcal{A}$ with

$$J(h(x)) = \frac{1}{p(x)} h(\rho(x)) - \frac{q(x)}{p(x)}$$

for all $h \in \mathcal{A}$ and $x \in X$. Since $\psi(\rho(x)) \leq L|p(\rho(x))|\psi(x)$ for all $x \in X$ and ρ is a surjection map, so

$$\begin{aligned} d(J(u), J(h)) &= \sup_{\{x\in X;\psi(x)\neq 0\}} \frac{|p(x)|\|u(\rho(x)) - h(\rho(x))\|}{|p(x)|\psi(x)} \\ &\leq \sup_{\{x\in X;\psi(\rho(x))\neq 0\}} L\frac{|p(\rho(x))|\|u(\rho(x)) - h(\rho(x))\|}{\psi(\rho(x))} \\ &= Ld(u, h) \end{aligned}$$

for all $u, h \in \mathcal{A}$, that is J is a strictly contractive self-mapping of $\mathcal{A}$, with the Lipschitz constant L (note that $0 < L < 1$). From (2.1), we get

$$\left\| \frac{f(\rho(x))}{p(x)} - \frac{q(x)}{p(x)} - f(x) \right\| \leq \frac{\psi(x)}{|p(x)|}$$

for all $x \in X$, which says that $d(J(f), f) \leq 1 < \infty$. So, by Theorem (1.4), there exists a mapping $T : X \to B$ such that

1. T is a fixed point of J, i.e.,

$$T(\rho(x)) = p(x)T(x) + q(x) \tag{2.3}$$

for all $x \in S$. The mapping T is a unique fixed point of J in the set $\tilde{\mathcal{A}} = \{h \in \mathcal{A} : d(f, h) < \infty\}$. This implies that T is a unique mapping satisfying (2.3) such that there exists $C \in (0, \infty)$ satisfying

$$\|f(x) - T(x)\| \leq C \frac{\psi(x)}{|p(x)|}$$

for all $x \in X$.

2. $d(J^n(f), T) \to 0$ as $n \to \infty$. This implies that

$$T(x) = \lim_{n\to\infty} \frac{f(\rho^n(x))}{\prod_{i=0}^{n-1} p(\rho^i(x))} - \sum_{k=0}^{n-1} \frac{q(\rho^i(x))}{\prod_{i=0}^{k} p(\rho^i(x))}$$

for all $x \in X$.

3. $d(f, T) \leq \frac{1}{1-L} d(J(f), f)$, which implies,

$$d(f, T) \leq \frac{1}{1-L}$$

or

$$\|f(x) - T(x)\| \leq \frac{\psi(x)}{(1-L)|p(x)|}$$

for all $x \in X$.

Z. Gajda in his paper [56] showed that the theorem of Th. Rassias [4] is false for some special control function and give the following co-counterexample. □

Theorem 2.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and*

$$|f(x+y) - f(x) - f(y)| \leq \theta(|x| + |y|) \tag{2.4}$$

for all $x, y \in \mathbb{R}$ and some $\theta > 0$. But there is no constant $\delta \in [0, \infty)$ and no additive function $T : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$|f(x) - T(x)| \leq \delta |x| \tag{2.5}$$

for all $x \in \mathbb{R}$.

With the above Theorem, its easy to show that the following result.

Corollary 2.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and*

$$|f(2x) - 2f(x)| \leq |x| \tag{2.6}$$

for all $x \in \mathbb{R}$. But there is no constant $\delta \in [0, \infty)$ and no function $T : \mathbb{R} \to \mathbb{R}$ satisfying the conditions

$$T(2x) = 2T(x) \tag{2.7}$$

$$|f(x) - T(x)| \leq \delta |x| \tag{2.8}$$

for all $x \in \mathbb{R}$.

Its obvious that the above corollary is a counterexample for the Theorem (2.1), when $L = 1$.

With Theorem (2.1), its easy to show that the following Corollary.

Corollary 2.4. *Let $f : X \to B$ be a function and*

$$\|f(\rho(x)) - p(x)f(x) - q(x)\| \leq \delta \tag{2.9}$$

for all $x \in X$ and some $\delta > 0$. If $a \leq |p(x)|$ for all $x \in X$ and some real $a > 1$, then there is an unique function $T : X \to B$ such that $T(\rho(x)) = p(x)T(x) + q(x)$

$$\|f(x) - T(x)\| \leq \frac{\delta}{a-1}$$

for all $x \in X$.

Similarly we prove that a Hyers-Ulam-Rassias stability for the linear functional equation with another suitable conditions.

Theorem 2.5. *Let $f : X \to B$ be a function and*

$$\|f(\rho(x)) - p(x)f(x) - q(x)\| \leq \psi(x) \tag{2.10}$$

for all $x \in X$. Let there exists a positive real $L < 1$ such that

$$|p(x)|\psi(\rho^{-1}(x)) \leq L\psi(x) \tag{2.11}$$

for all $x \in X$ and also ρ be a permutation of X. Then there is an unique function $T : X \to B$ such that $T(\rho(x)) = p(x)T(x) + q(x)$

$$\|f(x) - T(x)\| \leq \frac{1}{1-L} \psi(\rho^{-1}(x))$$

for all $x \in X$.

Proof. Let us consider the set $\mathcal{A} := \{h : X \to B\}$ and introduce the generalized metric on $\mathcal{A}$:

$$d(u, h) = \sup_{\{x \in X \,;\, \psi(x) \neq 0\}} \frac{\|g(x) - h(x)\|}{\psi(\rho^{-1}(x))}.$$

It is easy to show that $(\mathcal{A}, d)$ is generalized complete metric space. Now we define the function $J : \mathcal{A} \to \mathcal{A}$ with

$$J(h(x)) = p(\rho^{-1}(x))h(\rho^{-1}(x)) + q(\rho^{-1}(x))$$

for all $h \in \mathcal{A}$ and $x \in X$. Since $|p(x)|\psi(\rho^{-1}(x)) \leq L\psi(x)$ for all $x \in X$, so

$$d(J(u),J(h)) = \sup_{\{x\in X\ ;\ \psi(x)\neq 0\}} \frac{|p(\rho^{-1}(x))|\,\|u(\rho^{-1}(x)) - h(\rho^{-1}(x))\|}{\psi(\rho^{-1}(x))}$$

$$\leq \sup_{\{x\in X\ ;\ \psi(\rho^{-1}(x))\neq 0\}} L\frac{\|u(\rho^{-1}(x)) - h(\rho^{-1}(x))\|}{\psi(\rho^{-2}(x))}$$

$$= Ld(u,h)$$

for all $u, h \in \mathcal{A}$, that is J is a strictly contractive self-mapping of $\mathcal{A}$, with the Lipschitz constant L (note that $0 < L < 1$). From (2.10), we get

$$\left\|f(x) - p(\rho^{-1}(x))f(\rho^{-1}(x))\right\| \leq \psi(\rho^{-1}(x))$$

for all $x \in X$, which says that $d(J(f),f) \leq 1 < \infty$. So, by Theorem (1.4), there exists a mapping $T : X \to B$ such that

1. T is a fixed point of J, i.e.,
$$T(\rho(x)) = p(x)T(x) + q(x) \tag{2.12}$$
for all $x \in S$. The mapping T is a unique fixed point of J in the set $\tilde{\mathcal{A}} = \{h \in \mathcal{A} : d(f,h) < \infty\}$. This implies that T is a unique mapping satisfying (2.12) such that there exists $C \in (0,\infty)$ satisfying
$$\|f(x) - T(x)\| \leq C\frac{\psi(x)}{|p(x)|}$$
for all $x \in X$.
2. $d(J^n(f), T) \to 0$ as $n \to \infty$. This implies that
$$T(x) = \lim_{n\to\infty} \prod_{i=1}^{n} p(\rho^{-i}(x))f(\rho^{-n}(x)) - \sum_{k=1}^{n} q(\rho^k(x))$$
$$\times \prod_{i=0}^{k-1} p(\rho^{-i}(x))$$
for all $x \in X$ and in the above formula, we set $p(\rho^{-i}(x)) := 1$, when $i = 0$.
3. $d(f,T) \leq \frac{1}{1-L}d(J(f),f)$, which implies,
$$d(f,T) \leq \frac{1}{1-L}.$$
$$\|f(x) - T(x)\| \leq \frac{1}{1-L}\psi(\rho^{-1}(x))$$
for all $x \in X$.

Similar to the Corollary (2.3), we get the following result, where its counterexample for the Theorem (2.5), when $L = 1$. □

Corollary 2.6. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function and*

$$\left|f(\frac{1}{2}x) - \frac{1}{2}f(x)\right| \leq |x| \tag{2.13}$$

for all $x \in \mathbb{R}$. But there is no constant $\delta \in [0,\infty)$ and no function $T : \mathbb{R} \to \mathbb{R}$ satisfying the conditions

$$T\left(\frac{1}{2}x\right) = \frac{1}{2}T(x) \tag{2.14}$$

$$|f(x) - T(x)| \leq \delta|x| \tag{2.15}$$

for all $x \in \mathbb{R}$.

Corollary 2.7. *Let $f : X \to B$ be a function and*

$$\|f(\rho(x)) - p(x)f(x) - q(x)\| \leq \delta \tag{2.16}$$

for all $x \in X$ and some $\delta > 0$. If $|p(x)| \leq L$ for all $x \in X$ and some real $0 < L < 1$, then there is an unique function $T : X \to B$ such that $T(\rho(x)) = p(x)T(x) + q(x)$

$$\|f(x) - T(x)\| \leq \frac{\delta}{1-L}$$

for all $x \in X$.

Corollary 2.8. *Let $f : X \to B$ be a function such that X be a normed linear space over F and*

$$\|f(ax) - kf(x)\| \leq \|x\|^p \tag{2.17}$$

for all $x \in X$, in which $p \in R$, $a \in F$. If $p \leq 0$, $|a| > 1$ and $|k| > 1$ or $p \leq 0$, $|a| < 1$ and $|k| < 1$ or $p \geq 0$, $|a| > 1$ and $|k| < 1$ or $p \geq 0$, $|a| < 1$ and $|k| > 1$, then there is a unique function T such that $T(ax) = aT(x)$

$$\|f(x) - T_{(\rho,k)}(x)\| \leq \frac{\|x\|^p}{||k| - 1|}$$

for all $x \in X$.

Proof. Set $\rho(x) := ax$ and $\psi(x) := \|x\|^p$ for all $x \in X$ and then apply Theorem (2.1) and Theorem (2.5). □

Now in the following we consider the Hyers-Ulam-Rassias stability of Pexiderized linear functional equation (1.4).

Theorem 2.9. *Let $f, g : X \to B$ be a function and*

$$\|f(\rho(x)) - p(x)g(x) - q(x)\| \leq \psi(x) \tag{2.18}$$

for all $x \in X$. If there exists a positive real $L < 1$ such that

$$\psi(\rho(x)) \leq L|p(\rho(x))|\psi(x); \tag{2.19}$$

$$\|f(\rho(x)) - g(\rho(x))\| \leq L\|f(x) - g(x)\| \tag{2.20}$$

for all $x \in X$. Then there is an function T such that $T(\rho(x)) = p(x)T(x) + q(x)$

$$\|f(x) - T(x)\| \le \frac{\widetilde{\psi}(x)}{(1-L)|p(x)|}$$

$$\|g(x) - T(x)\| \le \frac{L}{1-L}\left[\frac{\widetilde{\psi}(x) + \psi(x)}{|p(x)|}\right]$$

for all $x \in X$, in which $\widetilde{\psi}(x) = \psi(x) + |p(x)|\|f(x) - g(x)\|$ for all $x \in X$.

Proof. Applying (2.18), we get

$$\|f(\rho(x)) - p(x)f(x) - q(x)\| \le \psi(x) + |p(x)|\|f(x) - g(x)\| \tag{2.21}$$

$$\le \widetilde{\psi}(x) \tag{2.22}$$

for all $x \in X$. From (2.19) and (2.20), its easy to show that the following inequality

$$\widetilde{\psi}((\rho(x)) \le L|p(\rho(x))|\widetilde{\psi}(x)$$

for all $x \in X$. So, by Theorem (2.1), there is an unique function $T : X \to B$ such that $T(\rho(x)) = p(x)T(x) + q(x)$

$$\|f(x) - T(x)\| \le \frac{\widetilde{\psi}(x)}{(1-L)|p(x)|}$$

for all $x \in X$. So from the above inequality, we have

$$\|f(\rho(x)) - T(\rho(x))\| \le \frac{\widetilde{\psi}(\rho(x))}{(1-L)|p(\rho(x))|}$$

for all $x \in X$. We show that T is a linear equation, thus from the above inequality and (2.18), we get

$$\|g(x) - T(x)\| \le \frac{L}{1-L}[\frac{\widetilde{\psi}(x) + \psi(x)}{|p(x)|}]$$

for all $x \in X$. The proof is complete. □

Common stability for the systems of homogeneous linear equations

Throughout this section, assume that $\{p_i : X \to F\backslash\{0\}\}_{i\in I}$, $\{\rho_i : X \to X\}_{i\in I}$ and $\{\psi_i : X \to \mathbb{R}^+\}_{i\in I}$ be three family of functions. Here i is a variable ranging over the arbitrary index set I. Also we define the functions $P_{i,n} : X \to F\backslash\{0\}$ and $\theta_{i,n}(x) : X \to \mathbb{R}^+$ with

$$P_{i,n}(x) = \prod_{k=0}^{n-1} p_i(\rho_i^k(x))$$

and

$$\theta_{i,n}(x) = \frac{(1-L_i^n)\psi_i(x)}{(1-L_i)|p_i(x)|}$$

for a family of positive reals $\{L_i\}_{i\in I}$, all $x \in X$, any index i and positive integer n.

In this section, we consider some systems of homogeneous linear equations

$$f(\rho_i(x)) = p_i(x)f(x), \tag{3.1}$$

and our aim is to establish some common Hyers-Ulam-Rassias stability for these systems of functional equations. As a consequence of these results, we give some generalizations of well-known Baker's superstability result for exponential functional equation to the a family of functional equations. Note that the following Theorem is partial affirmative answer to problem 1, in the 13st ICFEI.

Theorem 3.1. *Let $f : X \to B$ be a function and*

$$\|f(\rho_i(x)) - p_i(x)f(x)\| \le \psi_i(x) \tag{3.2}$$

for all $x \in X$ and $i \in I$. Assume that

1. *there exists a family of positive reals $\{L_i\}_{i\in I}$ such that $L_i < 1$ and*
$$\psi_i(\rho_i(x)) \le L_i|p_i(\rho_i(x))|\psi_i(x)$$
for all $x \in X$ and $i \in I$;
2. *$\rho_i \circ \rho_j = \rho_i \circ \rho_j$ for all $i, j \in I$;*
3. *$p_i(\rho_j(x)) = p_i(x)$ for all distinct $i, j \in I$;*
4. *$\lim_{n\to\infty} \frac{\theta_{i,n}(\rho_j^n(x))}{|P_{j,n}(x)|} = 0$ for all $x \in X$ and every distinct $i, j \in I$.*

Then there is a unique function T such that

$$T(\rho_i(x)) = p_i(x)T(x)$$

for all $x \in X$ and $i \in I$ and also

$$\|f(x) - T(x)\| \le \inf_{i\in I}\left\{\frac{\psi_i(x)}{(1-L_i)|p_i(x)|}\right\}$$

for $x \in X$.

Proof. It follows from (2.1), there is an unique set of functions $T_i : X \to B$ such that $T_i(\rho_i(x)) = p_i(x)T_i(x)$

$$\|f(x) - T_i(x)\| \le \frac{\psi_i(x)}{(1-L_i)|p_i(x)|}$$

for all $x \in X$. Moreover, The function T_i is given by

$$T_i(x) = \lim_{n\to\infty} \frac{f(\rho_i^n(x))}{\prod_{k=0}^{n-1} p_i(\rho_i^k(x))} = \lim_{n\to\infty} J_i^n(f)$$

for all $x \in X$ and any fixed $i \in I$. In the proof of Theorem (2.1), we show that

$$d(J_i(f), f) \le 1.$$

By induction, its easy to show that

$$d(J_i^n(f), f) \le \frac{1-L_i^n}{1-L_i},$$

which means that

$$\left\| f(\rho_i^n(x)) - \prod_{k=0}^{n-1} p_i(\rho_i^k(x)) f(x) \right\| \leq \left(\prod_{k=0}^{n-1} p_i(\rho_i^k(x)) \right) \times \frac{(1-L_i^n)\psi_i(x)}{(1-L_i)|p_i(x)|}$$

for all $x \in X$ and $i \in I$. Now we show that $T_i = T_j$ for any $i, j \in I$. Let i and j be two arbitrary fixed indexes of I. So, from last inequality, we obtain

$$\|f(\rho_i^n(x)) - P_{i,n}(x)f(x)\| \leq |P_{i,n}(x)|\theta_{i,n}(x); \tag{3.3}$$

$$\|f(\rho_j^n(x)) - P_{j,n}(x)f(x)\| \leq |P_{j,n}(x)|\theta_{j,n}(x) \tag{3.4}$$

for all $x \in X$. On the replacing x by $\rho_j^n(x)$ in (3.3) and x by $\rho_i^n(x)$ in (3.4), we have

$$\|f(\rho_i^n(\rho_j^n(x))) - P_{i,n}(\rho_j^n(x))f(\rho_j^n(x))\| \leq |P_{i,n}(\rho_j^n(x))|\theta_{i,n} \times (\rho_j^n(x)); \tag{3.5}$$

$$\|f(\rho_j^n(\rho_i^n(x))) - P_{j,n}(\rho_i^n(x))f(\rho_i^n(x))\| \leq |P_{j,n}(\rho_i^n(x))|\theta_{j,n} \times (\rho_i^n(x)) \tag{3.6}$$

for all $x \in X$. From assumption (3.1), its obvious that $f(\rho_i^n(\rho_j^n(x))) = f(\rho_j^n(\rho_i^n(x)))$, $P_{i,n}(\rho_j^n(x)) = P_{i,n}(x)$ and $P_{j,n}(\rho_i^n(x)) = P_{j,n}(x)$ for all $x \in X$. So, Combining (3.5) and (3.6), we have

$$\|P_{i,n}(x)f(\rho_j^n(x)) - P_{j,n}(x)f(\rho_i^n(x))\| \leq |P_{i,n}(x)|\theta_{i,n}(\rho_j^n(x)) + |P_{j,n}(x)|\theta_{j,n}(\rho_i^n(x))$$

or

$$\left\| \frac{f(\rho_j^n(x))}{P_{j,n}(x)} - \frac{f(\rho_i^n(x))}{P_{i,n}(x)} \right\| \leq \frac{\theta_{i,n}(\rho_j^n(x))}{|P_{j,n}(x)|} + \frac{\theta_{j,n}(\rho_i^n(x))}{|P_{i,n}(x)|}$$

for all $x \in X$. From assumption $\lim_{n\to\infty} \frac{\theta_{i,n}(\rho_j^n(x))}{|P_{j,n}(x)|} = 0$ for all $x \in X$ and every distinct $i, j \in I$, so, its implies that $T_i = T_j$.

Now set $T = T_i$ and since $\|f(x) - T_i(x)\| \leq \frac{\psi_i(x)}{(1-L_i)|p_i(x)|}$ for all $x \in X$ and all $x \in I$, there is a unique function T such that

$$T(\rho_i(x)) = p_i(x)T(x)$$

for all $x \in X$ and $i \in I$ and also

$$\|f(x) - T(x)\| \leq \inf_{i\in I} \left\{ \frac{\psi_i(x)}{(1-L_i)|p_i(x)|} \right\}$$

for $x \in X$. □

Corollary 3.2. *Let $f : X \to B$ be a function and*

$$\|f(\rho_i(x)) - c_i f(x)\| \leq \psi_i(x) \tag{3.7}$$

for all $x \in X$ and $i \in I$, where $\{c_i\}_{i\in I}$ and $\{\delta_i\}_{i\in I}$ are two family of real numbers such that $\delta_i \geq 0$ and $|c_i| > 1$. Assume that $\rho_i \circ \rho_j = \rho_i \circ \rho_j$ for all $i, j \in I$ and also

$$\psi_i(\rho_i(x)) \leq \psi_i(x)$$

for all $x \in X$ and any $i \in I$, then there is a unique function T such that

$$T(\rho_i(x)) = c_i T(x)$$

for all $x \in X$ and $i \in I$ and also

$$\|f(x) - T(x)\| \leq \inf_{i\in I} \left\{ \frac{\psi_i(x)}{c_i - 1} \right\}$$

for $x \in X$.

Proof. Sets $p_i := c_i$ and $\psi_i := \delta_i$ and applying Theorem (3.1). □

In the following, the results are applied to the study of some superstability results for the exponential functional equation.

Theorem 3.3. *Let $(S, +)$ be an commutative semigroup and $f, g : S \to \mathbb{C}$ satisfying*

$$\|f(x+y) - g(y)f(x)\| \leq \phi(x, y) \tag{3.8}$$

for all $x, y \in S$, where $\phi : S^2 \to \mathbb{R}^+$ is function. Let g be a unbounded function and

$$\phi(x+i, y) \leq \phi(x, y)$$

for all $x, y \in S$ and $i \in I$, where $I = \{i \in S \ \|\, |g(i)| > 1\}$. Then $f(x+y) = g(y)f(x)$ for all $x, y \in S$.

Proof. Let g be a unbounded function and $I = \{i \in S \ \|\ |g(i)| > 1\}$, then sets $\rho_i(x) := x + i$, $c_i := g(i)$ and $\psi_i := \phi(x, i)$ for all $x \in S$ and any $i \in I$. Since $\rho_i\rho_j = \rho_i\rho_j$ and $\psi_i(\rho_i(x)) \leq \psi_i(x)$ for all $x \in S$ and any $i \in I$, so by Corollary (3.2), there is an unique function T such that

$$T(\rho_i(x)) = c_i T(x)$$

for all $x \in X$ and $i \in I$ and also

$$\|f(x) - T(x)\| \leq \inf_{i\in I}\{\frac{\psi_i}{c_i - 1}\}$$

for $x \in X$. Since g is a unbounded function, from last inequality $T = f$, which implies that

$$f(\rho_i(x)) = c_i f(x)$$

or

$$f(x+i) = g(i)f(x) \tag{3.9}$$

for all $x \in S$ and $i \in I$. On the replacing x by $x + ni$ in (3.8)

$$\|f((x+y)+ni) - g(y)f(x+ni)\| \leq \phi(x, y)$$

or

$$\left\| \frac{f((x+y)+ni)}{g(i)^n} - \frac{g(y)f(x+ni)}{g(i)^n} \right\| \leq \frac{\phi(x+ni,y)}{|g(i)|^n}$$

for all $x, y \in S$, any fixed $i \in I$ and positive integer n. From equation (3.9), its easy to show that $f(x+ni) = g(i)^n f(x)$ and $\phi(x+ni, y) \leq \phi(x, y)$ for all $x \in S$, any fixed $i \in I$ and positive integer n. So, we have

$$\|f(x+y) - g(y)f(x)\| \leq \frac{\phi(x,y)}{|g(i)|^n}$$

for all $x, y \in S$, any fixed $i \in I$ and positive integer n (note that $|g(i)| > 1$), which implies that $f(x+y) = g(y)f(x)$ for all $x, y \in S$. The proof is complete. $\square$

With the above Theorem, its obvious that the following corollaries.

Corollary 3.4. *Let $(S, +)$ be a commutative semigroup and $f, g : S \to C$ satisfying*

$$\|f(x+y) - g(y)f(x)\| \leq \delta \tag{3.10}$$

for all $x, y \in S$ and some $\delta > 0$. Then g is either bounded or $f(x+y) = g(y)f(x)$ for all $x, y \in S$.

Corollary 3.5. *Let $(S, +)$ be a commutative semigroup and $f : S \to C$ satisfying*

$$\|f(x+y) - f(y)f(x)\| \leq \delta \tag{3.11}$$

for all $x, y \in S$. Then f is either bounded or f is exponential.

Endnote

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Competing interests

Both authors declare that they have no competing interests.

Author's contributions

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