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# The edge-labeling and vertex-colors of $K_n$

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**Abstract**

A labeling of the edges of a graph is called vertex-coloring if the labeled degrees of the vertices yield a proper coloring of the graph. In this paper, we show that such a labeling is possible from the label set 1,2,3 for the complete graph $K_n$, $n \geq 3$.

**Keywords:** Edge-labeling, Vertex-coloring, Complete graph

**Subject classification:** 05C15, 05C78



## Background

All graphs in this note are simple and finite. For notation is not defined here, we refer the reader to [1].

For some $k \in \mathbb{N}$, let $f : E(G) \longrightarrow \{1, 2, \ldots, k\}$ be an integer labeling of the edges of a graph $G = (V(G), E(G))$ where $V(G) = \{v_1, \ldots, v_n, \}$. This labeling is called vertex-coloring if the labeled degrees $S_i := \sum_{v_j \in N(v_i)} f(v_i v_j)$ for all integer $1 \leq i \leq n$ of the vertices yield a proper vertex-coloring of the graph; ( i.e. the color of vertex $v_i$ is $S_i$). It is easy to see that for every graph which does not have a component isomorphic to $K_2$, there exists such a labeling for some positive integer $k$.

In 2002, Karonski et. al. [2] conjectured that such a labeling with. $k = 3$ is possible for all such graphs ($k = 2$ is not sufficient as seen for instance in complete graphs and cycles of length not divisible by 4). At first constant bound of $k = 30$ was proved by Addario-Berry et. al. [3], which was later improved to $k = 16$ in 2008 by Addario-Berry et. al. [4], to $k = 13$ by Wang and Yu [5], and to $k$= 6 by Kalkowski, et. al. [6], Also, Kalkowski et. al. introduced the best bound for $k$= 5 in [7].

In this paper, we will show an algorithm for the complete graph $K_n$ labeling that improves the bound to $k = 3$ that it is final value for $k$ in the all graphs which are simple and finite.

The following theorem is the main result of this paper.

**Theorem 1.** *Let n be a positive integer. Then for the complete graph $K_n$, there is a labeling $f : E(G) \longrightarrow \{1, 2, 3\}$, such that the induced vertex weights $S_i := \sum_{v_j \in N(v_i)} f(v_i v_j)$ properly color $V(G)$, where $1 \leq i \leq n$.*

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## The algorithm

It is trivial that the run of the algorithms in a regular graph specially in the complete graphs is harder than any other graphs, because these graphs have one case. In the following lemma, we will give an algorithm for the complete graph $K_n$.

**Lemma 1.** *There exists an edge-labeling with numbers 1, 2 and 3 for the complete graph $K_n$ such that the sum of the labels in all vertices is different.*

**Proof.** These following algorithms calculates the minimum sum that there exist for all vertices of the complete graph $K_n$. Indeed, this lower bound is for the regular graphs, which is the harder case and here we have only one case. We will prove that the lower bound of $S_n$ in the complete graph $K_n$ is $S_n \geq n - 1$. □

**Algorithm for** $K_n$ **:** For $n = 1, 2$, there is no such edge-labeling. For $K_3$, we label the edge $v_1v_2$, $v_2v_3$ and $v_1v_3$ with 3,2 and 1, respectively, see Figure 1. So, Suppose that $n \geq 4$. Let $V(K_n) = \{1, \ldots, n\}$. Note that every vertex of $K_n$ is adjacent with other $n - 1$ vertices. Now label every edge with 1. Then, for $1 \leq i \leq n$, $S_i = n - 1$.

Now for each $v_i$, $2 \leq i \leq n - 1$, set $k = i - 1$. We add $k$ times 1unit to some edges as following. Start with $v_iv_n$. If the label of edge $v_iv_n < 3$ then add one unit, otherwise, add one unit to next desired edge, i.e. that if the label of edge $v_iv_{n-1} < 3$ then add one unit, otherwise, add one unit to next edge. After $k$ stages, if the label of edge

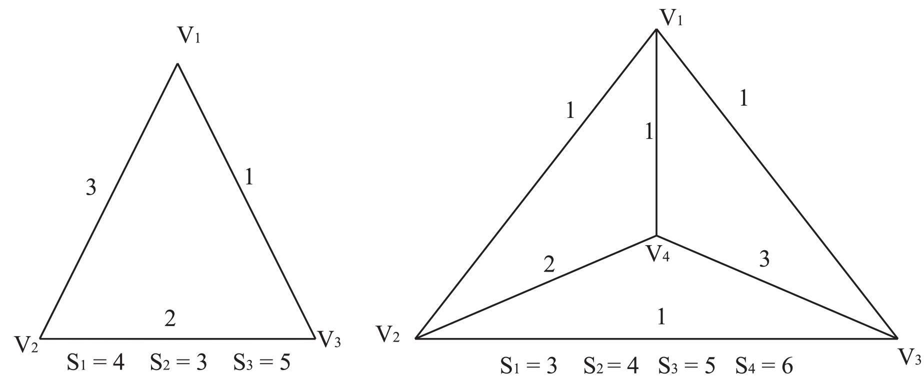


**Figure 1 1-2-3-edge labeling for graph $K_3$ and $K_4$ by the algorithm on $K_n$.**

$v_iv_{i+1} < 3$, add one unit, otherwise, if $S_i \neq S_{i-1}$ then there is no problem in the process and continue. But if $S_i = S_{i-1}$,set the label of edge $v_iv_{i-2}$ with 3 and continue.

With this algorithm for computing of $S_2$, we have $k = 1$. The process start from $v_2v_n$. One can see that $v_2v_n = 2$ and $v_2v_1 = v_2v_3 = \cdots = v_2v_{n-1} = 1$. Then, $S_2 = n - 2 + 2 = n$.

With this algorithm for computing of $S_3$, we have $k = 2$. The process start from $v_3v_n$ and before process the label $v_3v_n = 1$ less than 3 and can add one unit. Now $v_3v_n = 2$. By doing again this process $v_3v_n = 3$. Finally $v_3v_n = 3$ and $v_3v_1 = v_3v_2 = \cdots = v_3v_{n-1} = 1$, then $S_3 = n - 2 + 3 = n + 1$.

With this algorithm for computing of $S_4$, we have $k = 3$. The process start from $v_4v_n$ and before process the label of $v_4v_n = 1$, with 2 time process $v_4v_n = 3$. Therefor, need to add one unit to it's edges to finish process. We can continue adding with next edge that is $v_4v_{n-1}$. By doing this work $v_4v_n = 3$, $v_4v_{n-1} = 2$ and $v_4v_1 = v_4v_2 = \cdots = v_4v_{n-2} = 1$, then $S_4 = n - 3 + 3 + 2 = n + 2$.

This algorithm must be do until, $i = n - 1$ and the algorithm will be finished, and do not need to process for $i = n$, because this unit set in last units, with $v_1v_n = 1$, $v_2v_n = 2$ and $v_3v_n = v_4v_n = \cdots = v_{n-1}v_n = 3$ and $S_n = 3(n-3) + 1 + 2 = 3n - 6$.

Finally, we obtain $S_1, S_2, \ldots, S_n$. Therefore, all edges of the complete graph $K_n$ labeled with 1,2 and 3 and having the desired property. In fact for the graph $K_n$ we find $n$ series with finite sequences of numbers 1,2 and 3. In other words, for every $i$,$1 \leq i \leq n$ set $S_i := \sum_{j=1}^{n} f(v_iv_j)$ such that $f(v_iv_i) = 0$ and $f(v_iv_j) \in \{1, 2, 3\}$ for $i \neq j$.

Then $S_1 = n - 1$ and $S_n = 3n - 6$ are the minimum and maximum labeled degrees of $K_n$, respectively. Now the proof of Theorem 1 is complete.

Now one can ask that:

**Question 1.** Is there any such algorithm for arbitrary graph $G$?

In the rest of this paper we put the code of the algorithm of $K_n$, such that with run of its we get the result of each complete graph $K_n$.

The code is :

```
set 1 all labels of edge
calcudeAllSum\\computing all S_i
for(i = 1; i < Vertex.Size() − 1; i + +)
{
··· Vertex vi = Vertex.elementAt(i);
··· Vector < Edge > e = getAllEdgesOfThisVertex(vi);
··· k = 0;
··· for(j = e.Size() − 1; j ≥ i and k < i; j − −)
··· \\startfromv_iv_n
··· {
······ for(; k < i; k + +)
······ {
········· if(! e.elementAt(j).plasPlasLabelIfSmallerThan
Three())
········· \\if added one unit retuen true else return false
········· {
············ break;
········· }
······ }
··· }
··· if(k < i)
··· {
······ calcudeAllSum
······ if(vi.sum == Vertex.elementAt(i − 1).sum)
······ {
········· Edge e = GetEdgeOFVertex(vi, Vertex.elementAt
(i − 1))
········· \\Find edge of v_i and v_{i−1}.
········· e.label = 3;
······ }
··· }
}
```

### Competing interests
The author did not provide this information.

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