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An analysis on classifications of hyperbolic and elliptic PDEs

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Abstract

Purpose: Our aim in this study is to generate some partial differential equations (PDEs) with variable coefficients by using the PDEs with non-constant coefficients.

Methods: Then by applying the single and double convolution products, we produce some new equations having polynomials coefficients. We then classify the new equations on using the classification method for the second order linear partial differential equations.

Results: Classification is invariant under single and double convolutions by applying some conditions, that is, we identify some conditions where a hyperbolic equation will be hyperbolic again after single and double convolutions.

Conclusions: It is shown that the classifications of the new PDEs are related to the coefficients of polynomials which are considered during the process of convolution product.

Keywords: Hyperbolic equation, Elliptic equation, Single and double convolution, Classification of PDE

MSC: 35L05; 44A35

Introduction

The topic of partial differential equations (PDEs) is a very important subject, yet there is no general method to solve all the PDEs. The behavior of the solutions very much depends essentially on the classification of PDEs; therefore, the problem of classifying partial differential equations is very natural and well known since the classification governs the sufficient number and the type of the conditions in order to determine whether the problem is well posed and has a unique solution.

It is also well known that some second-order linear partial differential equations can be classified as parabolic, hyperbolic, or elliptic; however, if a PDE has coefficients which are not constant, it is rather a mixed type. In many applications of partial differential equations, the coefficients are not constant; in fact, they are a function of two or more independent variables and possible dependent variables. Therefore, the analysis that we have for the equations having constant coefficients to describe the solution may not be held globally for equations with variable coefficients.

On the other side, there are some very useful physical problems whose type can be changed. One of the best known example is for the transonic flow, where the equation is in the form of

$$\left(1-\frac{u^2}{c^2}\right)\phi_{xx}-\frac{2uv}{c^2}\phi_{xy}+\left(1-\frac{v^2}{c^2}\right)\phi_{yy}+f(\phi)=0$$

where u and v are the velocity components, and c is a constant (see [1]).

Similarly, partial differential equations with variable coefficients are also used in finance, for example, the arbitrage-free value C of many derivatives

$$\frac{\partial C}{\partial \tau}+s^2\frac{\sigma^2(s,\tau)}{2}\frac{\partial^2 C}{\partial s^2}+b(s,\tau)\frac{\partial C}{\partial s}-r(s,\tau)C=0$$

with three variable coefficients $\sigma(s,\tau)$, $b(s,\tau)$, and $r(s,\tau)$. In fact, this partial differential equation holds whenever C is twice differentiable with respect to s and once with respect to τ; see [2]. However, in the literature, there was no systematic way to generate partial differential equations with variable coefficients by using the equations

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with constant coefficients; most of the partial differential equations with variable coefficients depend on the nature of particular problems.

Recently, Kılıçman and Eltayeb [3] studied the classifications of hyperbolic and elliptic equations with non-constant coefficients, which was extended by the same authors [4] to the finite product of convolutions and classifications of hyperbolic and elliptic PDEs where the authors consider the coefficients of polynomials with positive coefficients. In fact, in their paper [5], the same authors proposed a systematic way to generate PDEs with variable coefficients by using the convolution product. In this study, we extend the current classification to the arbitrary coefficients. During this study, we use the following convolution notations: Single convolution between two continues functions $F(x)$ and $G(x)$ given by

$$F(x) * G(x) = \int_0^x F(x-\theta,)G(\theta)d\theta,$$

and double is convolution defined by

$$F_1(x,y) ** F_2(x,y) = \int_0^y \int_0^x F_1(x-\theta_1, y-\theta_2)F_2(\theta_1,\theta_2)d\theta_1 d\theta_2$$

for further details we refer to [6,7].

Methods

The classification problem for the partial differential equations are well known, that is, the classification of second order PDEs is suggested by the classification of the quadratic equations in the analytic geometry, that is, the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \quad (1)$$

is hyperbolic, parabolic, or elliptic accordingly as

$$B^2 - 4AC.$$

Now similarly, consider the equation

$$au_{xx} + 2bu_{xy} + cu_{yy} + F(x,y,u,u_x,u_y) = 0 \quad (2)$$

where $a, b, c, d, e,$ and f are of class $C^2(\Omega)$, $\Omega \subseteq \mathbb{R}^2$ is the domain, $(a,b,c) \neq (0,0,0)$, and the expression $au_{xx} + 2bu_{xy} + cu_{yy}$ is called the principal part of Equation 2. Since the principal part mainly determines the properties of the solution, it is well known that

(1) If $b^2 - 4ac > 0$, Equation 2 is called a hyperbolic equation.
(2) If $b^2 - 4ac < 0$, Equation 2 is called a parabolic equation.
(3) If $b^2 - 4ac = 0$, Equation 2 is called an elliptic equation.

Results and discussion

Now, let us consider the general linear second order partial differential equation with non-constant coefficients in the form of

$$a(x,y)u_{xx} + b(x,y)u_{xy} + c(x,y)u_{yy} + d(x,y)u_x + e(x,y)u_y + f(x,y)u = 0, \quad (3)$$

and the almost linear equation in two variables

$$au_{xx} + bu_{xy} + cu_{yy} + F(x,y,u,u_x,u_y) = 0, \quad (4)$$

where $a, b,$ and c, are polynomials and defined by

$$a(x,y) = \sum_{\beta=1}^{n}\sum_{\alpha=1}^{m} a_{\alpha\beta}x^{\alpha}y^{\beta}, \quad b(x,y) = \sum_{\zeta=1}^{n}\sum_{\eta=1}^{m} b_{\zeta\eta}x^{\zeta}y^{\eta},$$

$$c(x,y) = \sum_{l=1}^{n}\sum_{k=1}^{m} c_{kl}x^{k}y^{l}$$

and $(a,b,c) \neq (0,0,0)$, where the expression $au_{xx} + 2bu_{xy} + cu_{yy}$ is called the principal part of Equation 4 since the principal part mainly determines the properties of the solution. Throughout this paper, we also use the following notations

$$|a_{mn}| = \sum_{\beta=1}^{n}\sum_{\alpha=1}^{m} |a_{\alpha\beta}|, \ |b_{mn}| = \sum_{\zeta=1}^{n}\sum_{\eta=1}^{m} |b_{\zeta\eta}|, \text{ and } |c_{mn}| = \sum_{l=1}^{n}\sum_{k=1}^{m} |c_{kl}|.$$

Now, in order to generate new PDEs, we convolute Equation 4 by a polynomial with single convolution as $p(x)*^x$ where $p(x) = \sum_{i=1}^{m} p_i x^i$, then Equation 4 becomes

$$A_1(x,y)u_{xx} + B_1(x,y)u_{xy} + C_1(x,y)u_{yy} + F(x,y,u,u_x,u_y) = 0 \quad (5)$$

where the coefficients in Equation 5 are given by $A_1(x,y) = p(x) *^x a(x,y)$, $B_1(x,y) = p(x) *^x b(x,y)$, and $C_1(x,y) = p(x) *^x c(x,y)$ and the symbol $*^x$ indicates single convolution with respect to x. Then, we shall classify Equation 5 instead of Equation 4 by considering and examining the function

$$D(x,y) = B_1(x,y)^2 - C_1(x,y)A_1(x,y). \quad (6)$$

From Equation 6, one can see that if D is positive, then Equation 5 is called hyperbolic, and if D is negative, then Equation 5 is called elliptic, otherwise it is parabolic.

First of all, we compute and examine the coefficients of the principal part of Equation 5 as follows:

$$A_1(x,y) = p(x) *^x a(x,y) = \sum_{i=1}^{m} p_i x^i *^x \sum_{\beta=1}^{n}\sum_{\alpha=1}^{m} a_{\alpha\beta}x^{\alpha}y^{\beta}$$

by using the single convolution definition and integration by parts; thus we obtain the first coefficient of Equation 5 in the form of

$$A_1(x,y = \sum_{\beta=1}^{n}\sum_{i=1}^{m}\sum_{\alpha=1}^{m} \frac{p_i a_{\alpha\beta} i!\, x^{\alpha+i+1} y^{\beta}}{((\alpha+1)((\alpha+2)...(\alpha+i+1))}. \tag{7}$$

Similarly, for the coefficients of the second part in Equation 5, we have

$$B_1(x,y) = \sum_{j=1}^{n}\sum_{\zeta=1}^{m}\sum_{i=1}^{m} \frac{p_i b_{\zeta\eta} i!\, x^{\zeta+i+1} y^{\eta}}{((\zeta+1)((\zeta+2)...(\zeta+i+1))}. \tag{8}$$

Also, the last coefficient of Equation 5 given by

$$C_1(x,y) = \sum_{l=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{m} \frac{p_i c_{kl} i!\, x^{k+i+1} y^{l}}{((k+1)((k+2)...(k+i+1))}; \tag{9}$$

then, one can easily set up

$$D_1(x,y) = B_1^2(x,y) - A_1(x,y)C_1(x,y). \tag{10}$$

Now, we have the following several cases:

(1) Suppose that i, $\zeta+\eta$, $\alpha+\beta$ and $k+l$ are odd, for each $a_{\alpha\beta} > 0$, $b_{\zeta\eta} > 0$, $c_{kl} > 0$ and $p_i < 0$; Equation 5 is to be a hyperbolic equation under the condition that the power $\zeta = \frac{\alpha+k}{2}$ and $\eta = \frac{\beta+l}{2}$ and the power of x and y in polynomials $a(x,y)$, $c(x,y)$ are either even or odd. Now, we are going to study the classification of Equation 5. If we look at the power of x, y in Equation 10, we see that the power of $B(x,y)^2 =$ the power of $A_1(x,y)C_1(x,y)$ and the coefficient of $A_1(x,y)C_1(x,y) > 1$. Thus, the power of Equation 10 is even, and thus for all point (x_0, y_0) in the domain $\mathbb{R}^2$, Equation 5 is a hyperbolic equation. In particular, if we consider the simple example of the non-constant equation in the form

$$\begin{aligned}\left(-3x^3 * 2x^2y^3\right) u_{xx} + \left(-3x^3 * 4x^3y^4\right) u_{xy} \\ + \left(-3x^3 * 4x^4y^5\right) u_{yy} \\ = \sin(x+y) * *e^{x+y},\end{aligned} \tag{11}$$

and then consider the coefficients by using Equations 7, 8, and 9; we obtain

$$A_1(x,y) = -\frac{1}{10}x^6y^3,$$

$$B_1(x,y) = -\frac{12}{35}x^7y^4,$$

and

$$C_1(x,y) = -\frac{3}{70}x^8y^5.$$

Then,

$$D_1(x,y) = B_1^2 - A_1(x,y)C_1(x,y) = \frac{111}{980}x^{14}y^8. \tag{12}$$

We can easily see from Equation 12 that Equation 11 is hyperbolic for all $(x_0, y_0) \in \mathbb{R}^2$.

(2) Suppose that i is even and $\zeta+\eta$, $\alpha+\beta$, and $k+l$ are odd, for each $a_{\alpha\beta} < 0$, $b_{\zeta\eta} > 0$, $c_{kl} < 0$, and $p_i > 0$; Equation 5 is to be an elliptic equation under the condition that the power $\zeta = \frac{\alpha+k}{2}$ and $\eta = \frac{\beta+l}{2}$ and the power of x and y in polynomials $a(x,y)$, $c(x,y)$ are either even or odd. Now, we are going to study the classification of Equation 5. If we look at the power of x, y in Equation 10, we see that the power of $B(x,y)^2 =$ the power of $A(x,y)C(x,y)$ and the coefficient of $A_1(x,y)C_1(x,y) > B_1(x,y)^2$ under the condition $\min\left\{\left|a_{\alpha\beta}\right|, \left|c_{kl}\right|\right\} \geq \left|b_{\zeta\eta}\right|$. Thus, the power of Equation 10 is even and the coefficients are negative; thus, for all point (x_0, y_0) in the domain $\mathbb{R}^2$, Equation 5 is an elliptic equation. The coefficient of $A_1(x,y)C_1(x,y) < B_1(x,y)^2$ under the condition $\max\left\{\left|a_{\alpha\beta}\right|, \left|c_{kl}\right|\right\} < \left|b_{\zeta\eta}\right|$ and the coefficients are positive; thus, for all point (x_0, y_0) in the domain $\mathbb{R}^2$, Equation 5 is a hyperbolic equation. In particular, if we consider a non-constant equation of the form

$$\begin{aligned}\left(4x^2 *^x -3x^5y\right) u_{xx} + \left(4x^2 *^x 2x^3y^2\right) u_{xy} \\ + \left(4x^2 *^x -5xy^3\right) u_{yy} = f(x,y).\end{aligned} \tag{13}$$

Now, if we look at the min $\{|-3|, |-5|\} \geq |2|$ in a similar way, we obtain

$$D_1(x,y) = -\frac{319}{3150}y^4x^{12}. \tag{14}$$

Then, it is easy to see that Equation 14 is negative for all $(x_0, y_0) \in \mathbb{R}^2$; thus, Equation 13 is an elliptic equation. If we consider the coefficient $b_{\zeta\eta}$ in Equation 13 given by the condition $\max\left\{\left|a_{\alpha\beta}\right|, \left|c_{kl}\right|\right\} < \left|b_{\zeta\eta}\right|$, then Equation 13 is a hyperbolic equation. In particular, we consider the following example:

$$\begin{aligned}\left(4x^2 * -3x^5y\right) u_{xx} + \left(4x^2 * 6x^3y^2\right) u_{xy} \\ + \left(4x^2 * -5xy^3\right) u_{yy} = f(x,y).\end{aligned} \tag{15}$$

Now, if we look at the max $\{|-3|, |-5|\} < |6|$, by using Equations 7, 8, and 9, we have

$$D_1(x,y) = \frac{43}{1050}x^{12}y^4. \tag{16}$$

From Equation 16, we see that Equation 15 is a hyperbolic equation.

(3) Suppose that i, $\zeta+\eta$, $\alpha+\beta$, and $k+l$ are odd, for each $a_{\alpha\beta}>0$, $b_{\zeta\eta}<0$, $c_{kl}>0$, and $p_i>0$; Equation 5 is to be a hyperbolic equation under the condition that the power $\zeta=\frac{\alpha+k}{2}$ and $\eta=\frac{\beta+l}{2}$ and the power of x and y in polynomials $a(x,y)$, $c(x,y)$ are either even or odd. Now, we are going to study the classification of Equation 5. If we look at the power of x, y in Equation 10, we see that the power of $B(x,y)^2=$ the power of $A_1(x,y)C_1(x,y)$ and the coefficient of $A_1(x,y)C_1(x,y)<B_1(x,y)^2$ under the condition $\max\left\{\left|a_{\alpha\beta}\right|,\left|c_{kl}\right|\right\}\le\left|b_{\zeta\eta}\right|$. Thus, the power of Equation 10 is even and $D_1>0$; thus, Equation 5 is a hyperbolic equation. In this case, the coefficient of $A_1(x,y)C_1(x,y)>B_1(x,y)^2$ under the condition $\min\left\{\left|a_{\alpha\beta}\right|,\left|c_{kl}\right|\right\}>\left|b_{\zeta\eta}\right|$. Thus, the power of Equation 10 is even and $D_1<0$; thus, Equation 5 is an elliptic equation. In particular, let us consider the simple example of the non-constant equation in the form

$$\left(3x^5*2x^2y^3\right)u_{xx}+\left(3x^5*-5x^3y^4\right)u_{xy}+\left(3x^5*4x^4y^5\right)u_{xx}=f(x,y), \tag{17}$$

and we compute the coefficients of Equation 17 by using Equations 7, 8, and 9; we obtain

$$D_1=\frac{11}{20160}y^8x^{18}. \tag{18}$$

Then, it is easy to see that Equation 18 is always positive under the condition $\max\left\{\left|a_{\alpha\beta}\right|,\left|c_{kl}\right|\right\}\le\left|b_{\zeta\eta}\right|$, and thus, Equation 17 is a hyperbolic equation.

(4) Suppose that i, $\zeta+\eta$, $\alpha+\beta$, and $k+l$ are even, for each $a_{\alpha\beta}>0$, $b_{\zeta\eta}>0$, $c_{kl}>0$, and $p_i<0$; Equation 5 is to be either a hyperbolic or elliptic equation under condition that the power $\zeta=\frac{\alpha+k}{2}$ and $\eta=\frac{\beta+l}{2}$ and the power of x and y in polynomials $a(x,y)$, $c(x,y)$ are either even or odd. Now, let us study the classification of Equation 5. Under the following conditions:

(a) If the $\max\left\{\left|a_{\alpha\beta}\right|,\left|c_{kl}\right|\right\}\le\left|b_{\zeta\eta}\right|$ then $A_1(x,y)C_1(x,y)<B_1(x,y)^2$, thus $D_1>0$, and Equation 5 is a hyperbolic equation. In particular, we consider the following example

$$(-3x^4*2x^2y^{12})u_{xx}+(-3x^4*5x^4y^{10})u_{xy}+(-3x^4*5x^6y^8)u_{yy}=f(x,y). \tag{19}$$

By using Equations 7, 8, and 9, we compute the coefficients of Equation 19; thus, we have

$$D_1=\frac{19}{97020}y^{20}x^{18}. \tag{20}$$

Thus, Equation 20 is positive under the condition $\left\{\left|a_{\alpha\beta}\right|,\left|c_{kl}\right|\right\}\le\max\left|b_{\zeta\eta}\right|$; thus, Equation 19 is hyperbolic.

(b) If $\max\left\{\left|a_{\alpha\beta}\right|,\left|c_{kl}\right|\right\}>\left|b_{\zeta\eta}\right|$ and $\frac{a_{\alpha\beta}+c_{kl}}{2}$ then $A(x,y)C(x,y)>B(x,y)^2$, thus, $D_1<0$, Equation 5 is an elliptic equation.

(5) Suppose that i is odd and $\zeta+\eta$, $\alpha+\beta$ and $k+l$ are even , $a_{\alpha\beta}<0$, $b_{\zeta\eta}>0$, $c_{kl}<0$ and $p_i>0$; Equation 5 is to be an elliptic or hyperbolic equation under condition that the power $\zeta=\frac{\alpha+k}{2}$ and $\eta=\frac{\beta+l}{2}$ and the power of x and y in polynomials $a(x,y)$, $c(x,y)$ are either even or odd. Now, let us study the following conditions: (1) If the $\max\left\{\left|a_{\alpha\beta}\right|,\left|c_{kl}\right|\right\}\le\left|b_{\zeta\eta}\right|$ and $p_i\ge b_{\zeta\eta}$ then $A(x,y)C(x,y)>B(x,y)^2$, thus $D_1<0$, and Equation 5 is an elliptic equation. (2) If the $\max\left\{\left|a_{\alpha\beta}\right|,\left|c_{kl}\right|\right\}\le\left|b_{\zeta\eta}\right|$ and $p_i<b_{\zeta\eta}$ then $A(x,y)C(x,y)<B(x,y)^2$, thus $D_1>0$, and Equation 5 is a hyperbolic equation. One can easily check that the method will work when we consider the following equation:

$$A_2(x,y)u_{xx}+B_2(x,y)u_{xy}+C_2(x,y)u_{yy}+F(x,y,u,u_x,u_y)=0 \tag{21}$$

where the coefficients of Equation 21 are given by $A_2(x,y)=p(y)*^y a(x,y)$, $B_2(x,y)=p(y)*^y b(x,y)$, and $C_2(x,y)=p(y)*^y c(x,y)$. Similar results can be obtained. Now, let us extend the above results from a single convolution to a double convolution as follows: If we multiply Equation 4 by a polynomial by using the double convolution as $p(x,y)**$ where $p(x,y)=\sum_{i=1}^{m}p_{ij}x^iy^j$, then Equation 4 becomes

$$A(x,y)u_{xx}+B(x,y)u_{xy}+C(x,y)u_{yy}+F(x,y,u,u_x,u_y)=0 \tag{22}$$

where the coefficients in Equation 22 are given by $A(x,y)=p(x,y)**a(x,y)$, $B(x,y)=p(x,y)**b(x,y)$, and $C(x,y)=p(x,y)**c(x,y)$ of which the symbol $**$ indicates double convolution. Then, we shall classify Equation 22 as the more general form instead of Equation 4 by considering the function

$$D(x,y)=B(x,y)^2-A(x,y)C(x,y). \tag{23}$$

First of all, we compute the coefficients of Equations 22 by using the results that were given by

Kılıçman and Eltayeb in their previous works [3,8] as follows. The first coefficient of Equation 22 is given by

$$A(x,y)=\sum_{j=1}^{n}\sum_{\beta=1}^{n}\sum_{i=1}^{m}\sum_{\alpha=1}^{m}\frac{i!\,j!\,a_{\alpha\beta}p_{ij}x^{\alpha+i+1}y^{\beta+j+1}}{((\alpha+1)((\alpha+2)...(\alpha+i+1))\left((\beta+1)((\beta+2)...(\beta+j+1)\right)}. \tag{24}$$

Similarly, for the coefficients in the second part of Equation 22, we have

$$B(x,y)=\sum_{j=1}^{n}\sum_{\eta=1}^{n}\sum_{\zeta=1}^{m}\sum_{i=1}^{m}\frac{i!\,j!\,b_{\zeta\eta}p_{ij}x^{\zeta+i+1}y^{\eta+j+1}}{((\zeta+1)((\zeta+2)...(\zeta+i+1))\left((\eta+1)((\eta+2)...(\eta+j+1)\right)}; \tag{25}$$

also, the last coefficient of Equation 22 is given by

$$C(x,y)=\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{m}\frac{i!\,j!\,c_{kl}p_{ij}x^{k+i+1}y^{l+j+1}}{((k+1)((k+2)...(k+i+1))\left((l+1)((l+2)...(l+j+1)\right)}. \tag{26}$$

Then, one can easily set up

$$D(x,y)=B^2(x,y)-A(x,y)C(x,y). \tag{27}$$

We assume that all the coefficients $A(x,y), B(x,y)$, and $C(x,y)$ are convergent. Now, we can consider some particular cases:

(a) Suppose that $i+j$, $\zeta+\eta$, $\alpha+\beta$ and $k+l$ are odd, for each $a_{\alpha\beta}>0$, $b_{\zeta\eta}>0$, $c_{kl}>0$ and $p_{ij}<0$; Equation 5 is to be a hyperbolic equation under condition that the power $\zeta=\frac{\alpha+k}{2}$ and $\eta=\frac{\beta+l}{2}$ and the power of x and y in polynomials $a(x,y)$, $c(x,y)$ are either even or odd. Now, let us classify Equation 22 under the following conditions:

(i) If the max $\left\{\left|a_{\alpha\beta}\right|,\left|c_{kl}\right|\right\}\le\left|b_{\zeta\eta}\right|$ then $A(x,y)C(x,y)<B(x,y)^2$, thus $D>0$, and Equation (22) is a hyperbolic equation. In particular, if we consider the following example:

$$\left(-2x^5y^6**5x^2y^5\right)u_{xx}+\left(-2x^5y^6**5x^3y^4\right)u_{xy}+\left(-2x^5y^6**3x^4y^3\right)u_{xx}=f(x,y) \tag{28}$$

and we compute the coefficients of Equation 28 by using Equations 24, 25, and 26, we obtain

$$D=\frac{1}{77454558720}x^{18}y^{22}. \tag{29}$$

One can easily see from Equation 29 that Equation 28 is a hyperbolic equation.

(ii) If the max $\left\{\left|a_{\alpha\beta}\right|,\left|c_{kl}\right|\right\}>\left|b_{\zeta\eta}\right|$ and $\frac{a_{\alpha\beta}+c_{kl}}{2}$ then $A(x,y)C(x,y)>B(x,y)^2$, thus $D<0$, and Equation 22 is an elliptic equation. In Equation 28, if we change the constant coefficient of the second term from 5 to 4, the equation becomes

$$\left(-2x^5y^6**5x^2y^5\right)u_{xx}+\left(-2x^5y^6**4x^3y^4\right)u_{xy}+\left(-2x^5y^6**3x^4y^3\right)u_{xx}=f(x,y). \tag{30}$$

Now, it is easy to check that Equation 29 is an elliptic equation.

(b) Suppose that $i+j$ is even and $\zeta+\eta$, $\alpha+\beta$ and $k+l$ are odd, for each $a_{\alpha\beta}<0$, $b_{\zeta\eta}>0$, $c_{kl}<0$ and $p_{ij}>0$; Equation 22 is to be an elliptic or hyperbolic equation under the condition that the power $\zeta=\frac{\alpha+k}{2}$ and $\eta=\frac{\beta+l}{2}$ and the power of x and y in polynomials $a(x,y)$, $c(x,y)$ are either even or odd. Similar as above, we have the following conditions:

(i) If the $\left|b_{\zeta\eta}\right|>\max\left\{\left|a_{\alpha\beta}\right|,\left|c_{kl}\right|\right\}$ then $A(x,y)C(x,y)<B(x,y)^2$, thus $D>0$, and Equation 22 is a hyperbolic equation.
In particular, we consider the following example:

$$\left(3x^3y^5**-4xy^2\right)u_{xx}+\left(3x^3y^5**6x^2y^3\right)u_{xy}+\left(3x^3y^5**-3x^3y^4\right)u_{yy}=f(x,y) \tag{31}$$

and by using Equations 24, 25, and 26, and substituting in Equation 27, we have

$$D = \frac{17}{98784000} x^{12} y^{18}. \tag{32}$$

From Equation 32, we see that Equation 31 is a hyperbolic equation for all (x, y) in $\mathbb{R}^2$.

(ii) If the $|b_{\zeta\eta}| \leq \max\{|a_{\alpha\beta}|, |c_{kl}|\}$ then $A(x,y)C(x,y) > B(x,y)^2$, thus $D < 0$, and Equation 22 is an elliptic equation. Now, if we make a simple change in Equation 31, since $|b_{\zeta\eta}| \leq \max\{|a_{\alpha\beta}|, |c_{kl}|\}$ will be different by replacing the constant coefficient 6 of the second term by 4, then Equation 31 becomes

$$\begin{aligned}\left(3x^3y^5 ** - 4xy^2\right) u_{xx} + \left(3x^3y^5 ** 4x^2y^3\right) u_{xy} \\ + \left(3x^3y^5 ** - 3x^3y^4\right) u_{yy} \\ = f(x, y).\end{aligned} \tag{33}$$

Similarly, as above, we obtain

$$D = -\frac{11}{444528000} x^{12} y^{18}. \tag{34}$$

From Equation 34, we see that Equation 33 is an elliptic equation.

Conclusions

Thus, the above examples lead us to make following statement: The classification of partial differential equations with polynomial coefficients depends very much on the signs of the coefficients. In this particular case, if we use continuously differential functions as in [9], we can solve some boundary value problems having singularity since the convolution regularizes the singularity.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Each author contributed equally in the development of the manuscript. All authors read and approved the final manuscript.

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