Sarma *Mathematical Sciences* 2012, **6**:53
http://www.iaumath.com/content/6/1/53

Mathematical Sciences
a SpringerOpen Journal

ORIGINAL RESEARCH **Open Access**

I-convergent sequences of fuzzy real numbers defined by Orlicz function

Bipul Sarma

Abstract

In this article we introduce some *I*-convergent sequence spaces of fuzzy real numbers defined by Orlicz function and study their different properties such as solidity and symmetricity. The notion of *I*-convergence generalizes the notion of some particular type of convergence of sequences.

Keywords: Fuzzy real numbers, *I*-convergence, Solid space, Symmetric space, Orlicz function
AMS classification: 40A05, 40D25, 46A45, 46E30

Introduction

The notion of I-convergence of real-valued sequence was studied at the initial stage by Kostyrko et al. [1] which generalizes and unifies different notions of convergence of sequences. The notion was further studied by Šalát et al. [2].

The notion of fuzzy sets was introduced by Zadeh [3]. After that, many authors have studied and generalized this notion in many ways due to the potential of the introduced notion. Also, it has a wide range of applications in almost all of the branches studied, particularly in the field of science where mathematics is used. It attracted many workers to introduce different types of fuzzy sequence spaces.

Bounded and convergent sequences of fuzzy numbers were studied by Matloka [4]. Later on, sequences of fuzzy numbers have been studied by Kaleva and Seikkala [5], Tripathy and Sarma [6,7], and many others.

An Orlicz function is a function M: $[0, \infty) \to [0, \infty)$, which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If the convexity of M is replaced by the following:

$$M(x+y) \leq M(x) + M(y)$$

then this function is called the *modulus function*.

Remark 1. It is well known that if M is an Orlicz function, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri [8] used the idea of Orlicz function to construct the following sequence space:

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) < \infty, \text{for some } r > 0 \right\}$$

The space ℓ_M becomes a Banach space, with the following norm:

$$\|x\| = \inf\left\{ r > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r}\right) \leq 1 \right\}.$$

Definitions and background

Let X be a nonempty set, then a non-void class $I \subseteq 2^X$ (power set of X) which is called an *ideal* if I is an hereditary (i.e., $A \in I$ and $B \subseteq A \Rightarrow B \in I$) and additive (i.e. $A, B \in I \Rightarrow A \cup B \in I$). An ideal $I \subseteq 2^X$ is said to be nontrivial if $I \neq 2^X$. A nontrivial ideal I is said to be *admissible* if I contains every finite subset of N. A nontrivial ideal I is said to be *maximal* if there does not exist any nontrivial ideal $J \neq I$ containing I as a subset.

Let X be a nonempty set, then a non-void class $F \subseteq 2^X$ is said to be a *filter* in X if $\phi \notin F$; $A, B \in F \Rightarrow A \cap B \in F$ and $A \in F, A \subseteq B \Rightarrow B \in F$. For any ideal I, there is a filter $\Psi(I)$ corresponding to I, given by the following:

$$\Psi(I) = \{K \subseteq N : N \ K \in \mathrm{I}\}.$$

Correspondence: drbsar@yahoo.co.in
MC College, Gauhati University, Barpeta, Assam, India

Example.

(a) Let $I = I_f$, which is the class of all finite subsets of N, then I_f is a nontrivial admissible ideal.

(b) Let $A \subset N$. If $\delta(A) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_A(k)$ exists, then the class I_δ of all $A \subset N$ with $\delta(A) = 0$ forms a nontrivial admissible ideal.

(c) Let $A \subset N$ and $s_n = \sum_{k=1}^{n} \frac{1}{k}$, for all $n \in N$. If $d(A) = \lim_{n\to\infty} \frac{1}{s_n}\sum_{k=1}^{n} \frac{\chi_A(k)}{k}$ exists, then the class I_d of all $A \subset N$ with $d(A) = 0$ forms a nontrivial admissible ideal.

(d) The *uniform density* of a set $A \subset N$ is defined as follows. For integers $t \geq 0$ and $s \geq 1$, let $A(t + 1, t + s) = \text{card}\,\{n \in A{:}t + 1 \leq n \leq t + s\}$. Put $\beta_s = \liminf_{t\to\infty} A(t+1, t+s)$, $\beta^s = \limsup_{t\to\infty} A(t+1, t+s)$. If $\lim_{s\to\infty} \frac{\beta_s}{s}$ and $\lim_{s\to\infty} \frac{\beta^s}{s}$ both exist and $\lim_{s\to\infty} \frac{\beta_s}{s} = \lim_{s\to\infty} \frac{\beta^s}{s}$ $(= u(A)$, say), then $u(A)$ is called the uniform density of A. The class I_u of all $A \subset N$ with $u(A) = 0$ forms a nontrivial ideal.

Let D denote the set of all closed and bounded intervals $X = [a_1, b_1]$ on the real line R. For $X = [a_1, b_1] \in D$ and $Y = [a_2, b_2] \in D$, define $d(X, Y)$ by the following:

$$d(X, Y) = \max(|a_1 - b_1|, |a_2 - b_2|).$$

It is known that (D, d) is a complete metric space.

A fuzzy real number X is a fuzzy set on R, i.e., a mapping $X{:}R \to L(= [0, 1])$ associating each real number t with its grade of membership $X(t)$.

The α-level set $[X]^\alpha$ set of a fuzzy real number X for $0 < \alpha \leq 1$, and is defined as follows:

$$X^a = \{t\in \mathrm{R} : \mathrm{X(t)} \geq \alpha\}.$$

A fuzzy real number X is called *convex*, if $X(t) \geq X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

If there exists $t_0 \in R$ such that $X(t_0) = 1$, then the fuzzy real number X is called *normal*.

A fuzzy real number X is said to be *upper semi-continuous* if for each $\varepsilon > 0$, $X^{-1}([0, a + \varepsilon])$, for all $a \in L$ is open in the usual topology of R.

The set of all upper semi-continuous, normal, convex fuzzy number is denoted by $L(R)$.

The absolute value $|X|$ of $X \in L(R)$ is defined as follows (see for instance Kaleva and Seikkala [5]):

$$\begin{aligned} |X|(t) &= \max\{X(t), X(-t)\}, \text{if } t > 0 \\ &= 0 \qquad\qquad\qquad\quad , \text{if } t > 0. \end{aligned}$$

Let $\overline{d} : L(R) \times L(R) \to \mathrm{R}$ be defined by the following:

$$\overline{d}(X, Y) = \sup_{0\leq\alpha\leq 1} d(X^\alpha, Y^\alpha).$$

then $\overline{d}$ defines a metric on $L(R)$.

A sequence (X_k) of fuzzy real numbers is said to be convergent to the fuzzy real number X_0 if for every $\varepsilon > 0$, there exists $n_0 \in N$ such that $\overline{d}\,(X_k, X_0) < \varepsilon$ for all $k \geq k_0$.

A fuzzy real-valued sequence space E^F is said to be *solid* if $(Y_k) \in E^F$ whenever $(X_k) \in E^F$ and $|Y_k| \leq |X_k|$, for all $k \in N$.

Let $K = \{k_1 < k_2 < k_3 \ldots\} \subseteq N$ and E^F be a sequence space. A *K-step space* of E^F is a sequence space $\lambda_K^{E^F} = \{(X_{k_n}) \in \mathrm{w}^{\mathrm{F}} : (\mathrm{X_k}) \in \mathrm{E^F}\}$.

A *canonical pre-image of a sequence* $(X_{k_n}) \in \lambda_{\mathrm{K}}^{\mathrm{E^F}}$ is a sequence $(Y_k) \in w^F$ which is defined as follows:

$$Y_k = \begin{cases} X_k, & \text{if } k \in \mathrm{K}, \\ \overline{0}, & \text{otherwise}. \end{cases}$$

A *canonical pre-image of a step space* $\lambda_K^{E^F}$ is a set of canonical pre-images of all elements in $\lambda_K^{E^F}$, *i.e.*, Y is in canonical pre-image $\lambda_K^{E^F}$ if and only if Y is canonical pre-image of some $X \in \lambda_K^{E^F}$.

A sequence space E^F is said to be *monotone* if E^F contains the canonical pre-images of all its step spaces.

A sequence space E^F is said to be *symmetric* if $(X_{\pi(k)}) \in E^F$, whenever $(X_k) \in E^F$, π is a permutation on N.

A sequence $X = (X_k)$ of fuzzy numbers is said to be *I*-convergent if there exists a fuzzy number X_0 such that for all $\varepsilon > 0$, the set $\{n \in N{:}\ \overline{d}\,(X_k, X_0) \geq \varepsilon\} \in I$. We write $I\text{-}\lim X_k = X_0$.

A sequence (X_k) of fuzzy numbers is said to be I^*-convergent to X_0 (I^*-lim $X_k = X_0$) if there is a set $\{k_1 < k_2 < - - - - -\} \in \Psi(I)$ such that $\lim_{i\to\infty} X_{k_i} = X_0$.

A sequence (X_k) of fuzzy numbers is said to be *I*-bounded if there exists a real number μ such that the set $\{k \in N{:}\ \overline{d}\,(X_k, \overline{0}) > \mu\} \in I$.

If $I = I_f$, then I_f convergence coincides with the usual convergence of fuzzy sequences. If $I = I_d(I_\delta)$, then $I_d(I_\delta)$ convergence coincides with statistical convergence (logarithmic convergence) of fuzzy sequences. If $I = I_u$, I_u convergence is said to be a uniform convergence of fuzzy sequences.

Throughout $c^{I(F)}$, $c_0^{I(F)}$ and $\ell_\infty^{I(F)}$ denote the spaces of fuzzy real-valued *I-convergent*, *I-null* and *I-bounded* sequences, respectively.

It is clear from the definitions that $c_0^{I(F)} \subset c^{\mathrm{I(F)}} \subset \ell_\infty^{\mathrm{I(F)}}$, and the inclusions are proper.

It can be easily shown that $\ell_\infty^{I(F)}$ is complete with respect to the metric ρ defined by $f(X, Y) = \sup_n \overline{d}(X_k, Y_k)$, where $X = (X_k)$, $Y = (Y_k) \in \ell_\infty^{I(F)}$.

Lemma 1. *A sequence space E^F is solid, implying that E^F is monotone.*

For the crisp set case, one may refer to Kamthan and Gupta [9].

Lemma 2. *If $I \subseteq 2^N$ is a maximal ideal, then for each $A \subset N$, we have either $A \in I$ or $N \setminus A \in I$.* (see for instance lemma 5.1 of the work of Kostyrko et al. [1])

The existing sequence space $\ell_\infty^F(M)$ is defined as follows:

$$\ell_\infty^F(M) = \left\{ (X_k) \in \mathrm{w_F} : \sup_k M\left(\frac{\overline{d}(X_k, \overline{0})}{r}\right) < \infty, \text{ for some } r > 0 \right\}$$

which is a complete metric space.

We define the following sequence spaces:

$$\left(c^I\right)^F(M) = \left\{ (X_k) : \left\{ k : M\left(\frac{\overline{d}(X_k, L)}{r}\right) \geq \mathrm{K}, \text{for some } r > 0 \text{ and } L \in R(\mathrm{I}) \right\} \in \mathrm{I} \right\}$$

For $L = \overline{0}$, the above space is denoted by $(c_0^I)^F(M)$

$$\left(\ell_\infty^I\right)^F(M) = \left\{ (X_k) : \left\{ \mathrm{K} : M\left(\frac{\overline{d}(X_k, \overline{0})}{r}\right) \geq \varepsilon, \text{ for some } r > 0 \right\} \in \mathrm{I} \right\}$$

Also, we define $(m^I)^F(M) = (c^I)^F(M) \cap \ell_\infty^F(M)$, $(m_0^I)^F(M) = (c_0^I)^F(M) \cap \ell_\infty^F(M)$.

Results and discussion

Theorem 1. *The classes $(c^I)^F(M)$, $(c_0^I)^F(M)$, $(m^I)^F(M)$, and $(m_0^I)^F(M)$ are complete metric spaces with respect to the metric given by the following:*

$$f(X, Y) = \inf\left\{ r > 0 : \sup_k M\left(\frac{\overline{d}(X_k, Y_k)}{r}\right) \leq 1 \right\}$$

Proof. Let (X^n) be a Cauchy sequence in $(m^I)^F(M)$ such that $X^n \to X$ in $\ell_\infty^F(M)$, where $(X^n) = (X_k^n)$ and $X = (X_k)$.

Let $\varepsilon > 0$ be given. For a fixed $x_0 > 0$, choose $r > 0$ such that $M\left(\frac{rx_0}{3}\right) \geq 1$ and $m_0 \in N$ such that

$$f(X^n, X^m) < \frac{\varepsilon}{rx_0} \text{ for all } n, m \geq m_0.$$

By definition of f, we have the following:

$$M\left(\frac{\overline{d}(X_k^m, X_k^n)}{f(X^m, X^n)}\right) \leq 1 \leq M\left(\frac{rx_0}{3}\right)$$

$$\Rightarrow \overline{d}(X_k^m, X_k^n) < \frac{\varepsilon}{3} \text{ for all } n, m \geq m_0.$$

Since $X^m, X^n \in (m^I)^F(M)$, so there exist fuzzy numbers Y_m and Y_n such that

$$A = \left\{ k \in N : M\left(\frac{\overline{d}(X_k^n, Y_n)}{r}\right) < M\left(\frac{\varepsilon}{3r}\right) \right\} \in \psi(I)$$

$$= \left\{ k \in N : \overline{d}(X_k^n, Y_m) < \frac{\varepsilon}{3} \right\} \in \psi(I).$$

$$B = \left\{ k \in N : \overline{d}(X_k^m, Y_m) < \frac{\varepsilon}{3} \right\} \in \psi(I).$$

Now, $A \cap B \in \psi(I)$, and let $k \in A \cap B$, and then

$$\overline{d}(Y_n, Y_m) \leq \overline{d}(Y_n, X_k^n) + \overline{d}(X_k^n, X_k^m) + \overline{d}(X_k^m, Y_m)$$
$$< \mathrm{e} \text{ for all } n, m \geq m_0.$$

Thus, (Y_n) is a Cauchy sequence of fuzzy real numbers. Thus, there exists a fuzzy real number Y such that $\lim Y_n = Y$. To show that I-lim $X_k = Y$. Let $\eta > 0$. Since $X^n \to X$, so there exists $s \in N$ such that

$$f(X^s, X) < \frac{\eta}{3}.$$

The number s can be chosen in such a way that

$$d(Y_s, Y) < \frac{\eta}{3}$$

Since $I - \lim X_k^{(s)} = Y_t$, thus we have the following:

$$C = \left\{ k \in N : \overline{d}\left(X_k^{(s)}, Y_t\right) < \frac{\eta}{3} \right\} \in \psi(I).$$

Hence, for each $k \in C$,

$$\overline{d}(X_k, Y) \leq \overline{d}(X_k, X_k^s) + \overline{d}(X_k^s, Y_s) + \overline{d}(Y_s, Y)$$
$$< \eta.$$

Thus I-lim $X_k = Y$.

Property 1. *The sequence spaces $(c^I)^F(M)$, $(c_0^I)^F(M)$, $(m^I)^F(M)$ and $(m_0^I)^F(M)$ are not symmetric.*

For this result consider the following example.

Example 1. Let $I = I_\delta$. Let the sequence (X_k) be defined as follows:

$$\text{For } k = i^2, i \in N, X_k(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq 1, \\ -tk^{-1} + 1 + k^{-1}, & \text{for } 1 \leq t \leq k+1, \\ 0, & \text{otherwise.} \end{cases}$$

and for $k \neq i^2$, $i \in N$, $X_k = \overline{1}$.

Let (Y_k) be a rearrangement of (X_k) defined by the following:

$$(Y_k) = (X_1, X_2, X_4, X_3, X_9, X_5, X_{16}, X_6, \ \ ....)$$

The sequence (X_k) is statistically convergent, but (Y_k) is not statistically convergent.

Theorem 2. *The sequence spaces* $(c_0^I)^F(M)$, $(m^I)^F(M)$, *and* $(m_0^I)^F(M)$ *are solid.*

Proof. We prove the result for $(c_0^I)^F(M)$. For the other spaces, the result can be proven similarly. Let $(X_k) \in (c_0^I)^F(M)$ and (Y_k) be such that $|Y_k| \leq |X_k|$, for all $k \in N$, and for the given $\varepsilon > 0$, $A = \left\{ k \in N : M\left(\frac{\overline{d}(X_k, \overline{0})}{r}\right) \geq \varepsilon, \ \text{for some } r > 0 \right\} \in I.$

Since M is increasing, $B = \left\{ k \in N : M\left(\frac{\overline{d}(Y_k, \overline{0})}{r}\right) \geq \varepsilon \text{ for some } r > 0 \right\} \subseteq A.$

Thus, $B \in I$ and so $<Y_k> \in (c_0^I)^F(M)$. Hence, $(c_0^I)^F(M)$ is solid.

Property 2. *The sequence space* $(c^I)^F(M)$ *is not monotone.*

For this result, consider the following example.

Example 2. Let $I = I_u$. Let the sequence (X_k) be defined as follows:

$$\text{For all } k \in N, \ \ X_k(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq 1, \\ -t+2, & \text{for } 1 \leq t \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Then $(X_k) \in (c^I)^F(M)$. Let $J = \{k \in N$ and k even$\}$.

Let the sequence (Y_k) be defined by the following:

$$\begin{aligned} Y_k &= X_k, \ \text{if } k \in J. \\ &= \overline{0}, \ \ \text{otherwise.} \end{aligned}$$

Also, (Y_k) belongs to the canonical pre-image of the J-step space of $(c^I)^F(M)$, but $(Y_k) \notin (c^I)^F(M)$. Thus, $(c^I)^F(M)$ is not monotone and hence not solid.

Property 3. *The sequence spaces* $(c^I)^F(M)$, $(c_0^I)^F(M)$, $(m^I)^F(M)$, *and* $(m_0^I)^F(M)$ *are not convergence free.*

For this result, consider the following example.

Example 3. Consider the sequence space $(c^I)^F(M)$, and let $I = I_\delta$.

Let the sequence (X_k) be defined as follows:

$$X_k = \overline{0}, \text{for } k = i^2, i \in \mathrm{N},$$

$$\text{and for other values}, X_k(t) = \begin{cases} t+1, & \text{for } 0 \leq t \leq 1, \\ -t+1, & \text{for } 1 \leq t \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let the sequence (Y_k) be defined by the following:

$$Y_k = \overline{0}, \text{for } k = i^2, i \in \mathrm{N},$$

$$\text{and for other values}, Y_k(t) = \begin{cases} 1, & \text{for } 0 \leq t \leq 1, \\ (k-t)(k-1)^{-1}, & \text{for } 1 \leq t \leq k, \\ 0, & \text{otherwise.} \end{cases}$$

Also, (X_k) is statistically convergent, but (Y_k) is not statistically convergent. Hence, $(c^I)^F(M)$ is not convergence free. Similarly, the other spaces are also not convergence free.

Theorem 3. (i) $Z(M_1) \subseteq Z(M_2 o M_1)$ *for* $Z = (c^I)^F$, $(c_0^I)^F$, $(\ell_\infty^I)^F$.

(ii) $Z(M_1, q) \cap Z(M_2, q) \subseteq Z(M_1 + M_2, q)$ *for* $Z = (c^I)^F$ $(c_0^I)^F$, $(\ell_\infty^I)^F$.

Proof. (i) Let $Z = (c^I)^F$ and $(X_k) \in (c^I)^F(M)$; then,

$$\left\{ k : M\left(\frac{\overline{d}(X_k, L)}{r}\right) \geq \varepsilon, \ \text{for some } r > 0 \right\} \in \mathrm{I}.$$

Since M_2 is continuous, so for $\varepsilon > 0$, there exists $\eta > 0$ such that $M_2(\varepsilon) = \eta$. The result follows from

$$M_2\left(M_1\left(\frac{\overline{d}(X_k, L)}{r}\right)\right) \geq M_2(\varepsilon) = \eta.$$

(ii) The proof is easy, so it is omitted.

Theorem 4. (i) $Z(M) \subseteq \left(\ell_\infty^I\right)^F(M)$ *for* $Z = (c^I)^F$, $(c_0^I)^F$. *The inclusion is proper.*

Proof. The first part of the result is obvious. For the inclusion to be proper, consider the following example.

Example 4. Let the sequence (X_k) be defined by the following:

$$\text{For } k \text{ odd}, X_k(t) = \begin{cases} t, & \text{for } 0 \leq t \leq 1, \\ -t+2, & \text{for } 1 \leq t \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{and for } k \text{ even}, \ \ X_k(t) = \begin{cases} t, & \text{for } 0 \leq t \leq 1, \\ (-t+3)2^{-1}, & \text{for } 1 \leq t \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

The sequence (X_k) is bounded on a set of logarithmic density 1, but it is not logarithmically convergent.

Conclusions

Generalizing the notion of convergence of sequences, we have studied the notion of I-convergence of sequences through this paper. A few works have been done in the direction of I-convergence of sequences. We have studied some important properties of I-convergent sequence spaces introduced with Orlicz function. The notion of I-convergence generalizes the notion of some particular type of convergence of sequences.

Competing interests
The author declares that there are no competing interests.

Acknowledgements
The author thanks the referees for their valuable comments.

Received: 15 August 2012 Accepted: 12 September 2012
Published: 17 October 2012

References
1. Kostyrko, P, Šalat, T, Wilczyński, W: *I*-convergence. Real Anal Exchange **26**, 669–686 (2000–2001)
2. Šalát, T, Tripathy, BC, Ziman, M: On some properties of *I*-convergence. Tatra Mountain Publ Math **28**, 279–286 (2004)
3. Zadeh, LA: Fuzzy sets. Inform and Control **8**, 338–353 (1965)
4. Matloka, M: Sequences of fuzzy numbers. BUSEFAL **28**, 28–37 (1986)
5. Kaleva, O, Seikkala, S: On fuzzy metric spaces. Fuzzy Set Syst. **12**, 215–229 (1984)
6. Tripathy, BC, Sarma, B: Statistically convergent double sequence spaces defined by Orlicz function. Soochow Journal of Mathematics **32**(2), 211–221 (2006)
7. Tripathy, BC, Sarma, B: Some double sequence spaces of fuzzy numbers defined by Orlicz functions. Acta Math Scientia **31**(1, Ser B), 134–140 (2011)
8. Lindenstrauss, J, Tzafriri, L: On Orlicz sequence spaces. Israel J Math **101**, 379–390 (1971)
9. Kamthan, PK, Gupta, M: Sequence Spaces and Series. Marcel Dekkar, New York (1980)

doi:10.1186/2251-7456-6-53
Cite this article as: Sarma: *I*-convergent sequences of fuzzy real numbers defined by Orlicz function. *Mathematical Sciences* 2012 **6**:53.