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On some n-normed sequence spaces

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Abstract

In this paper, we introduce the idea of constructing sequence spaces with elements in an n-norm space in comparison with the spaces c_0, c, ℓ_∞ and the Orlicz space ℓ_M and extend the notion of n-norm to such spaces. Further we state and define some statements about the n-best approximation in n-normed spaces.

Keywords: n-Norm, Locally convex space, n-orthogonality, Orlicz function, n-best approximation

Introduction

The concept of 2-normed spaces was initially developed by Gähler [1] in the middle of 1960s, while that of n-normed spaces can be found in Misiak [2]. Since then, many others have studied this and related concepts and obtained various results; see for instance Lewandowska [3-5], Cho et al. [6], Gunawan [7,8], Gunawan and Mashadi [9], Dutta [10] and Esi [11-13].

Let $n \in N$ and X be a real vector space of dimension d, where $n \leq d$. A real-valued function $\|., \ldots, .\|$ on X^n satisfying the following four conditions:

(1) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation,

(3) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in R$,

(4) $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$ is called an n-norm on X, and the pair $(X, \|., \ldots, .\|)$ is called an n-normed space.

Let $n \in N$ and X, a real vector space of dimension d, where $2 \leq n \leq d$. β_{n-1} be the collection of linearly independent sets B with $n-1$ elements. For $B \in \beta_{n-1}$, let us define

$$p_B(x_1) = \|x_1, x_2, \ldots, x_n\|, \; x_1 \in X, x_2, \ldots, x_n \in B.$$

Then p_B is a seminorm on X and the family $P = \{p_B : B \in \beta_{n-1}\}$ of seminorms generates a locally convex topology on X.

Let $(X, \|., \ldots, .\|)$ be an n-normed space and $W_1, W_2, \ldots, W_n$ be n subspaces of X. A map $f : W_1 \times W_2 \times \ldots \times W_n \to \mathbf{R}$ is called an n-functional on $W_1 \times W_2 \times \ldots, W_n$, whenever for all $x_1{}^1, x_2{}^1, \ldots, x_n{}^1 \in W_1$, $x_1{}^2, x_2{}^2, \ldots, x_n{}^2 \in W_2, \ldots, x_1{}^n, x_2{}^n, \ldots, x_n{}^n \in W_n$ and all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{R}$;

(i) $f(x_1{}^1 + x_2{}^1 + \ldots + x_n{}^1, x_1{}^2 + x_2{}^2 + \ldots + x_n{}^2, \ldots, x_1{}^n + x_2{}^n + \ldots + x_n{}^n) = \Sigma_{1 \leq i_1, i_2, \ldots, i_n \leq n} f(x_{i_1}{}^1, x_{i_2}{}^2, \ldots, x_{i_n}{}^n)$

(ii) $f(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n) = \lambda_1 \lambda_2 \ldots \lambda_n f(x_1, x_2, \ldots, x_n)$.

An n-functional $f : W_1 \times W_2 \times \ldots \times W_n \to \mathbf{R}$ is called bounded if there exists a non-negative real number M (called a Lipschitz constant for f) such that $|f(x_1, x_2, \ldots, x_n)| \leq M\|x_1, x_2, \ldots, x_n\|$ for all $x_1 \in W_1, x_2 \in W_2, \ldots, x_n \in W_n$. Also, the norm of an n-functional f is defined by

$$\|f\| = \inf\{M \geq 0 : \text{M is a Lipschitz constant for f}\}.$$

For an n-normed space $(X, \|., \ldots, .\|)$ and $0 \neq u_2, u_3, \ldots, u_n \in X$, we denote by X^*_{B} the Banach space of all bounded n-functionals on $X \times < u_2 > \times < u_3 > \times \ldots \times < u_n >$, where $< z >$ be the subspace of X generated by z and $B = \{u_2, \ldots, u_n\}$.

A sequence (x_k) in an n-normed space $(X, \|., \ldots, .\|)$ is said to converge to some $L \in X$ in the n-norm if

$$\lim_{k\to\infty} \|x_k - L, u_2, \ldots, u_n\| = 0, \text{ for every } u_2, \ldots, u_n \in X.$$

A sequence (x_k) in an n-normed space $(X, \|., \ldots, .\|)$ is said to be Cauchy with respect to the n-norm if

$$\lim_{k,l\to\infty} \|x_k - x_l, u_2, \ldots, u_n\| = 0, \text{ for every } u_2, \ldots, u_n \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Definition 2.1. Let $(X, \|., \ldots, .\|)$ be an n-normed space. We say that x is n-orthogonal to y if $\|x, u_2, u_3, \ldots, u_n\| \leq$

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$\|x+\alpha y, u_2, u_3, \ldots, u_n\|$, for all $u_2, u_3, \ldots, u_n \in X$, $\alpha \in \mathbf{R}$ and we write $x \perp^n y$.

Definition 2.2. Let $(X, \|., \ldots, .\|)$ be an n-normed space, M a nonempty subspace of X and $x \in X$, then $g_0 \in M$ is called an n-best approximation of $x \in X$ in M, if for every $g \in M$ and $u_2, u_3, \ldots, u_n \in X$,

$$\|x-g_0, u_2, u_3, \ldots, u_n\| \leq \|x-g, u_2, u_3, \ldots, u_n\|.$$

If for every $x \in X \backslash \bar{M}$ there exists at least one n-best approximation in M, then M is called n-proximinal subspace of X.

If for every $x \in X \backslash \bar{M}$ there exists a unique n-best approximation in M, then M is called an n-Chebyshev subspace of X.

For $x \in X$ we write,

$$P_M^n(x) = \{g_0 \in M : g_0 \text{ is an } n-\text{best approximation of } x\}.$$

Definition 2.3. A function $M : [0, \infty) \longrightarrow [0, \infty)$, which is continuous, non-decreasing and convex with $M(0)=0, M(x)>0$, for $x>0$ and $M(x) \to \infty$, as $x \to \infty$ is called an Orlicz function.

Let $(X, \|., \ldots, .\|)$ be a real linear n-normed space and $w(X)$ denotes X-valued sequence space. Then for an Orlicz function M, we define the following sequence spaces *for some* $\rho > 0$, L and *every* $z_2, \ldots, z_n \in X$:

$$(M, \|., \ldots, .\|)_1 = \{(x_k) \in w(X) : \lim_{k \to \infty} M(\|\frac{x_k - L}{\rho}, z_2, \ldots, z_n\|) = 0\},$$

$$(M, \|., \ldots, .\|)_0 = \{(x_k) \in w(X) : \lim_{k \to \infty} M(\|\frac{x_k}{\rho}, z_2, \ldots, z_n\|) = 0\},$$

and

$$(M, \|., \ldots, .\|)_\infty = \{(x_k) \in w(X) : \sup_k M(\|\frac{x_k}{\rho}, z_2, \ldots, z_n\|) < \infty\}.$$

When $X=C$, the complex field and $M(x)=|x|$, for all $x \in [0, \infty)$, the above spaces reduce to the spaces c, c_0, and ℓ_∞ respectively.

It is obvious that

$$(M, \|., \ldots, .\|)_0 \subseteq (M, \|., \ldots, .\|)_1 \subseteq (M, \|., \ldots, .\|)_\infty.$$

When $L=0$, we have $(M, \|., \ldots, .\|)_0 = (M, \|., \ldots, .\|)_1$.

Lemma 2.1. *The spaces* $(M, \|., \ldots, .\|)_0$, $(M, \|., \ldots, .\|)_1$ *and* $(M, \|., \ldots, .\|)_\infty$ *are linear spaces over the field of reals.*

Proof. The proof is a routine verification and so omitted. □

Methods

The 'Introduction' section recalls the notions of n-normed space, n-functional, Cauchy, and convergence sequences in n-normed spaces as well as defined the notions of n-orthogonality and n-best approximation and introduced three sequences spaces using an Orlicz function M with base space X, a real linear n-normed spaces in comparison with the classical spaces c_0, c, and ℓ_∞. In the 'Results and discussion' section, we prove some statements about the n-best approximation in n-normed spaces and investigate the introduced spaces for n-Banach spaces. The method applied for the main results is that first we give statement for each results and then each statement is supported with mathematical arguments as 'proof'.

Results and discussion

Now we state some statements about the n-best approximation in n-normed spaces and investigate the main results of this article involving the sequence spaces $(M, \|., \ldots, .\|)_0$ and $(M, \|., \ldots, .\|)_1$ and $(M, \|., \ldots, .\|)_\infty$.

Theorem 3.1. *Let* $(X, \|., \ldots, .\|)$ *be an n-normed linear space and* $0 \neq x, y \in X$. *Then the following statements are equivalent:*

(*i*) $x \perp^n y$.

(*ii*) *There exist* $u_2, \ldots, u_n \in X$ *and* $F \in X_B^*$ *such that* $\|F\| = 1$, $F(x, u_2, \ldots, u_n) = \|x, u_2, \ldots, u_n\|$, $F(y, u_2, \ldots, u_n) = 0$ *and* $B = \{u_2, \ldots, u_n\}$.

Corollary 3.2. *Let* $(X, \|., \ldots, .\|)$ *be an n-normed space, M a non-empty subspace of X,* $0 \neq x \in X$ *and* $g_0 \in M$. *Then the following statements are equivalent:*

(*i*) $g_0 \in P_M^n(x)$.

(*ii*) *There exist* $u_2, \ldots, u_n \in X$ *and* $F \in X_B^*$ *such that* $\|F\| = 1$, $F(x-g_0, u_2, \ldots, u_n) = \|x-g_0, u_2, \ldots, u_n\|$ *and* $F(g, u_2, \ldots, u_n) = 0$ *for all* $g \in M$ *and* $B = \{u_2, \ldots, u_n\}$.

Now we define an n-norm on the spaces $(M, \|., \ldots, .\|)_0$ then $(M, \|., \ldots, .\|)_1$ and $(M, \|., \ldots, .\|)_\infty$ and prove that they are n-Banach spaces.

Lemma 3.1. *Let Y be any one of the spaces* $(M, \|., \ldots, .\|)_0$ *then* $(M, \|., \ldots, .\|)_1$ *and* $(M, \|., \ldots, .\|)_\infty$. *We define the following function* $(\|., \ldots, .\|)_Y$ *on* $Y \times Y \times \ldots \times Y$ *(n factors) by* $\|x^1, \ldots, x^n\|_Y = 0$ *if* $x^1, \ldots, x^n$ *are linearly dependent, and* $\|x^1, \ldots, x^n\|_Y = \inf\{\rho > 0 : \sup_{k \geq 1,\, z_2, \ldots, z_n \in X} M(\|\frac{x_k{}^1}{\rho}, z_2, \ldots, z_n\|) < 1\}$, *if* $x^1, \ldots, x^n$ *are linearly independent.*

Then $\|., \ldots, .\|_Y$ *is an n-norm on Y.*

Proof. Proof is a routine verification and so omitted. □

Theorem 3.3. *If X is an n-Banach space then the spaces* $(M, \|.,\ldots,.\|)_0$ *and* $(M, \|.,\ldots,.\|)_1$ *and* $(M, \|.,\ldots,.\|)_\infty$ *are n-Banach spaces.*

Proof. Let Y be any one of the spaces $(M, \|.,\ldots,.\|)_0$ and $(M, \|.,\ldots,.\|)_1$ and $(M, \|.,\ldots,.\|)_\infty$. Let (x^i) be any Cauchy sequence in Y. Let $x_0 > 0$ be fixed and $t > 0$ be such that for a $0 < \epsilon < 1$ and $\frac{\epsilon}{x_0 t} > 0$ and $x_0 t \geq 1$. Then there exists a positive integer n_0 such that

$$\|x^i - x^j, u^2, \ldots, u^n\|_Y < \frac{\epsilon}{x_0 t}, \text{ for all } i, j \geq n_0 \text{ and for every } u^2, \ldots, u^n \in Y.$$

□

Using the definition of n-norm, we get

$$\inf\{\rho : \sup_{k\geq 1} M(\|\frac{x_k{}^i - x_k{}^j}{\rho}, z_2, \ldots, z_n\|) < 1, \} < \frac{\epsilon}{x_0 t} \text{ for } i, j \geq n_0.$$

Then for every $z_2, \ldots, z_n \in X$, we get

$$\sup_{k\geq 1} M(\|\frac{x_k{}^i - x_k{}^j}{\|x^i - x^j, u^2, \ldots, u^n\|_Y}, z_2, \ldots, z_n\|) \leq 1 \text{ for all } i, j \geq n_0.$$

It follows that for every $z_2, \ldots, z_n \in X$,

$$M(\|\frac{x_k{}^i - x_k{}^j}{\|x^i - x^j, u^2, \ldots, u^n\|_Y}, z_2, \ldots, z_n\|) \leq 1 \text{ for } k \geq 1 \text{ and for } i, j \geq n_0.$$

For $t > 0$ with $M(\frac{tx_0}{2}) \geq 1$, we have

$$M(\|\frac{x_k{}^i - x_k{}^j}{\|x^i - x^j, u^2, \ldots, u^n\|_Y}, z_2, \ldots, z_n\|) \leq M(\frac{tx_0}{2}).$$

Since an Orlicz function is non-decreasing, this implies that for every $z_2, \ldots, z_n \in X$,

$$\|x_k{}^i - x_k{}^j, z^2, \ldots, z^n\| \leq \frac{tx_0}{2}.\frac{\epsilon}{tx_0} = \frac{\epsilon}{2}, \text{ for all } i, j \geq n_0.$$

Hence, (x^i) is a Cauchy sequence in X for all $k \in N$ and so convergent in X for all $k \in N$, since X is an n-Banach space. Suppose $\lim_{i\to\infty} x_k{}^i = x_k$ (say) for each . Now, using the continuity of Orlicz function M and n-norm, we can have

$$\inf\{\rho : \sup_{k\geq 1} M(\|\frac{x_k{}^i - x_k}{\rho}, z_2, \ldots, z_n\|) < 1, z_2, \ldots, z_n \in X\} < \epsilon, \text{ for } i \geq n_0$$

and as $j \to \infty$. It follows that $(x^i - x) \in Y$.

Since $(x^i) \in Y$ and Y is a linear space, so we have $x = x^i - (x^i - x) \in Y$. This completes the proof of the theorem.

Example 3.1. Consider the space C_0 of real sequences with only finite number of non-zero terms. Let us define:

$$\|x_1, x_2, \ldots, x_n\| = 0, \text{ if } x_1, x_2, \ldots, x_n \text{ are linearly dependent,}$$
$$= \sum_{k=1}^{\infty}(|x_k{}^1||x_k{}^2| \ldots |x_k{}^n|), \text{ if } x_1, x_2, \ldots, x_n \text{ are independent.}$$

Then $\|.,\ldots,.\|$ is an n-norm on C_0. That is not an n-norm on c_0 and l_∞ consisting of real sequences.

Conclusion

After observing the investigations of this paper, we can comment that while studying the n-normed structure, the main issue should be the use of the meaning of n-norms. We also observe that if a term in the definition of n-norm represents the change of shape and the n-norm stands for the associated area or center of gravity of the term, we can think of some plausible applicable of the notion of n-norm. As an example, we can think of the use of the notion of n-norm for a process where for a particular output we need n-inputs but with one main input and other (n-1)-inputs as dummy inputs to complete the process.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

HD wrote the abstract and background. Both authors wrote the preliminaries. Results concerning n-best approximation are proposed by HD and verified by HM. Results concerning n-normed spaces and n-Banach spaces are proposed by HM and verified by HD. Both authors read and approved the final manuscript.

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