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The oblique derivative problem for nonlinear elliptic equations of second order in unbounded domains

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Abstract

This article deals with the general oblique derivative boundary value problem for nonlinear elliptic equations of second order in an unbounded multiply connected domain. The problem includes the Dirichlet problem, the Neumann problem and the third boundary value problem as its spacial cases. We first provide the formulation of the above boundary value problem and obtain the representation theorem for the problem. Then, we give *a priori* estimates of solutions for the boundary value problem by using the reduction to absurdity and the uniqueness of solutions. Finally, by the above estimates of solutions and the Leray-Schauder theorem, the existence of solutions of the above problem for the nonlinear elliptic equations of second order can be proved.

Keywords: Oblique derivative problem, Nonlinear elliptic equations, Unbounded domains

MSC: 35J65; 35J60; 35J45

Introduction

Formulation of oblique derivative problems of second-order elliptic equations in unbounded domains

Let D be an $(N+1)$-connected domain including the infinite point with the boundary $\Gamma = \cup_{j=0}^{N}\Gamma_j$ in $\mathbb{C}$, where $\Gamma \in C^2_\mu\,(0<\mu<1)$. Without loss of generality, we assume that D is a circular domain in $|z|>1$, where the boundary consists of $N+1$ circles $\Gamma_0 = \Gamma_{N+1} = \{|z|=1\}$, $\Gamma_j = \{|z-z_j| = r_j\}, j=1,\ldots,N$ and $z=\infty \in D$. In this article, the notations are the same as those in [1-6]. We consider the second-order nonlinear elliptic equation in the complex form

$$\begin{cases} u_{z\bar z} = F(z,u,u_z,u_{zz}),\ F = \mathrm{Re}\,[Qu_{zz} + A_1u_z] + \hat{A}_2u + A_3, \\ Q = Q(z,u,u_z,u_{zz}), A_j = A_j(z,u,u_z), j=1,2,3, \hat{A}_2 = A_2 + |u|^\sigma, \end{cases} \tag{1}$$

satisfying the following conditions.

Condition C

1. $Q(z,u,w,U)$ and $A_j(z,u,w)(j=1,2,3)$ are continuous in D for $u\in\mathbb{R}$, $w\in\mathbb{C}$ for almost every point $z\in D$, $U\in\mathbb{C}$, and $Q=0$ and $A_j = 0\,(j=1,2,3)$ for $z\notin D$, where σ is a positive number.
2. The above functions are measurable in D for all continuous functions $u(z)$, $w(z)$ in $\overline{D}$ and satisfy

$$\begin{aligned} &L_{p,2}\left[A_j(z,u(z),w(z)),\overline{D}\right] \le k_0, j=1,2,\\ &L_{p,2}\left[A_3(z,u(z),w(z)),\overline{D}\right] \le k_1, \end{aligned} \tag{2}$$

   in which $p_0, p\ (2<p_0\le p), k_0$ and k_1 are nonnegative constants.
3. Equation 1 satisfies the uniform ellipticity condition

$$|F(z,u,w,U_1) - F(z,u,w,U_2)| \le q_0\,|U_1-U_2|\,,\ A_2 \ge 0 \ \text{in}\ D, \tag{3}$$

   for almost every point $z\in D$ and any number $u\in\mathbb{R}$, $w, U_1, U_2 \in \mathbb{C}$, where $q_0(<1)$ is a nonnegative constant.

Problem O

In the domain D, find a solution $u(z)$ of Equation 1, which is continuously differential in $\overline{D}$ and satisfies the boundary condition

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$$\frac{1}{2}\frac{\partial u}{\partial \nu}+c_1(z)u(z)=c_2(z)+h(z),\ z\in\Gamma,\ u(a_0)=b_0, \tag{4}$$

where $\nu(=\ \nu_1+i\nu_2)$ can be arbitrary provided that $\cos(\nu,n)\geq 0$ on Γ, n is the outward normal vector on Γ and $a_0=1$. It is easy to see that the boundary condition (4) can be rewritten in the complex form

$$\mathrm{Re}\left[\overline{\lambda(z)}u_z\right]+c_1(z)u=c_2(z)+h(z),z\in\Gamma,\ u(a_0)=b_0, \tag{5}$$

for $\lambda(z)=\cos(\nu,x)+i\sin(\nu,x)=e^{i(\nu,x)}$, where (ν,x) is the angle between ν and the x-axis. Suppose that $\lambda(z),c_1(z),c_2(z)$ and $h(z)$ satisfy the conditions

$$\begin{aligned}&C_\alpha\left[\lambda(z),\Gamma\right]\leq k_0,\ C_\alpha[\,c_1(z),\Gamma]\leq k_0,\\&C_\alpha\left[c_2(z),\Gamma\right]\leq k_2,\ |b_0|\leq k_2,\ c_1(z)\geq 0\ \text{ on }\Gamma,\end{aligned} \tag{6}$$

in which $\alpha\,(1/2<\alpha<1)$and k_2 are nonnegative constants. If $\cos(\nu,n)=0$ and $c_1(z)\equiv 0$ on Γ_j, we assume that

$$u(a_j)=b_j,\ 1\leq j\leq N_0, \tag{7}$$

without loss of generality, we suppose that $\cos(\nu,n)=0$ and $c_1(z)=0$ on $\Gamma_*=\Gamma_1\cup\ldots\cup\Gamma_{N_0}$ $(N_0\leq N)$, but not on Γ_0 and $\Gamma_j(N_0<j\leq N)$, and $a_j\in\Gamma_j$, b_j is a real constant, $|b_j|\leq k_2, 1\leq j\leq N_0$. Set $\Gamma_{**}=\Gamma\backslash\{\Gamma_*\}$ and

$$h(z)=\begin{cases}h_0,\ z\in\Gamma_0,\\0,\ z\in\ \Gamma\backslash\Gamma_0,\end{cases} \tag{8}$$

where h_0 is a real constant to be determined appropriately.

It is clear that if $\cos(\nu,n)>0$ on Γ, Problem O is the third boundary value problem (Problem III). If $\cos(\nu,n)=1$ and $c_1(z)=0$ on Γ, then Problem O is the Neumann boundary problem (Problem II). If $\cos(\nu,n)=0$ and $c_1(z)=0$ on Γ, Problem O is equivalent to the first boundary value problem (Problem I). It is not difficult to see that for Problems I, II and III, $K_0=\arg_{\Gamma_0}\lambda(z)/\pi=-2, K_j=\arg_{\Gamma_j}\lambda(z)/\pi=2, j=1,\ldots,N$ and the index of $\lambda(z)$ on Γ is $K=[K_0+K_1+\ldots+K_N]/2=N-1$. Hence, Problem O is a general boundary value problem.

Methods

A priori estimates of solutions of oblique derivative problems for elliptic equations of second order

First of all, we prove the uniqueness of solutions for Problem O of Equation 1.

Theorem 1. *Suppose that Equation 1 satisfies Condition C. Then, Problem O for Equation 1 with the condition $A_3=0$ in D, $c_2=0$ on Γ and $b_0=0$ has only the trivial solution.*

Proof. Let $u(z)$ be any solution of Problem O for Equation 1. From Condition C, it is easily seen that $u(z)$ is a solution of the following uniformly elliptic equation:

$$\begin{aligned}u_{z\bar z}=&\mathrm{Re}[\,Qu_{zz}+A_1u_z]+(A_2+|u|^\sigma)u,\\&|Q|\leq q_0<1, A_2\geq 0\ \text{ in } D,\end{aligned} \tag{9}$$

and satisfies the boundary condition

$$\frac{1}{2}\frac{\partial u}{\partial \nu}+c_1(z)u(z)=h(z),\ z\in\Gamma,\ u(a_j)=0, j=0,1,\ldots,N_0. \tag{10}$$

Substituting the solution $u(z)$ into the coefficients of Equation 9, we can find a solution $\Psi(z)$ of Equation 1 satisfying the condition

$$\Psi(z)=1\ \text{ on }\ \Gamma; \tag{11}$$

thus, the function $U(z)=u(z)/\Psi(z)$ is a solution of the equation

$$\begin{aligned}U_{z\bar z}&=\mathrm{Re}[\,QU_{zz}+A_0U_z], A_0\\&=-2(\log\Psi)_{\bar z}+2Q(\log\Psi)_z+A_1,\end{aligned} \tag{12}$$

satisfying the boundary conditions

$$\frac{1}{2}\frac{\partial U}{\partial \nu}+c_1^*(z)U(z)=h(z)\ \text{ on }\Gamma,\ U(a_j)=0, j=0,1,\ldots,N_0, \tag{13}$$

where $c_1^*(z)=c_1(z)+(\partial\Psi/\partial\nu)/\Psi(z)\geq 0$ on Γ. If $U(z)\not\equiv 0$ in D, then there exists a point $z^*\in\Gamma$ such that $M=U(z^*)=\max_{\overline{D}}U(z)>0$. When $z^*\in\Gamma_*$, noting that $\cos(\nu,n)=0$, $c_1(z)=0$ and $\partial\Psi(z)/\partial\nu=0$ on Γ_*, we have $\partial U/\partial\nu=0$ and $U(z)=M$ on L_*, which is impossible. When $z^*\in\Gamma_{**}$, if $\cos(\nu,n)>0$ at z^*, in the same way as in the proof of Theorem 2.3.1 (see Chapter 2 in [5]), we have $\partial U/\partial\nu>0$ at z^*, which contradicts the formula (10) on Γ_{**}. If $\cos(\nu,n)=0$ at z^*, denoting the longest curve of Γ including the point z^* by Γ' so that $\cos(\nu,n)=0$ and $u(z)=M$ on Γ', then there exists a point $z'\in\Gamma_{**}\backslash\Gamma'$, such that at z', $\cos(\nu,n)>0, \partial u/\partial n>0, \cos(\nu,s)>0\,(<0)$ and $\partial u/\partial s\geq 0\,(\leq 0)$; hence,

$$\frac{\partial u}{\partial \nu}=\cos(\nu,n)\frac{\partial u}{\partial n}+\cos(\nu,s)\frac{\partial u}{\partial s}>0\ \text{ at }z' \tag{14}$$

holds, where s is the tangent vector of at $z'\in\Gamma_{**}$. Again, that is impossible, which shows $z^*\in\Gamma_0$. Thus, $u(z)$ attains its maximum M at a point z^*. Similarly, we can prove that $u(z)$ attains its minimum m at a point z_*; hence, $h(z)=0$ on Γ_0. By a similar method, we can verify $M=m=0$; thus, $U(z)=0$ in $\overline{D}$. □

Next, we consider the nonlinear elliptic equations of second order

$$u_{z\bar z}-\mathrm{Re}\left[Qu_{zz}+A_1u_z\right]-\hat A_2u=A_3, \tag{15}$$

where $\hat A_2=A_2+|u|^\sigma$, σ is a positive number, and assume that the above equation satisfies the above Condition C.

Theorem 2. *Let Equation 15 satisfy Condition* C. *Then, any solution of Problem* O *for Equation 15 satisfies the estimates*

$$\hat{C}_\beta\left[u,\overline{D}\right]=C^1_\beta\left[|u|^{\sigma+1},\overline{D}\right]\le M_1,\ ||u||_{W^2_{p_0,2}(D)}\le M_1,$$
$$\hat{C}_\beta\left[u,\overline{D}\right]\le M_2(k_1+k_2),\ ||u||_{W^2_{p_0,2}(D)}\le M_2(k_1+k_2), \tag{16}$$

in which $k=(k_0,k_1,k_2)$, $\beta\ (0<\beta\le\alpha)$, $M_1=M_1(q_0,p_0,\beta,k,D)$ *and* $M_2=M_2(q_0,p_0,\ \beta,k_0,p,D)$ *are nonnegative constants.*

Proof. By using the reduction to absurdity, we shall prove that any solution $u(z)$ of Problem O satisfies the estimate of boundedness

$$\hat{C}\left[u,\overline{D}\right]=C\left[|u|^{\sigma+1},\overline{D}\right]+C\left[u_z,\overline{D}\right]\le M_3, \tag{17}$$

where $M_3=M_3(q_0,p_0,\alpha,k,p,D)$ is a nonnegative constant. Suppose that Equation 17 is not true, then there exist sequences of coefficients $\{A_j^{(m)}\}\ (j=1,2,3)$, $\{Q^{(m)}\}$, $\{\lambda^{(m)}(z)\}, \{c_j^{(m)}\}(j=1,2)$ and $\{b_j^{(m)}\}(j=0,1,\ldots,N_0)$, which satisfy the same conditions of Condition C and Equations 6 to 8, such that $\{A_j^{(m)}\}\ (j=1,2,3)$, $\{Q^{(m)}\}$, $\{\lambda^{(m)}(z)\}$, $\{c_j^{(m)}\}\ (j=1,2)$ and $\{b_j^{(m)}\}\ (j=0,1,...,N_0)$ in $\overline{D},\Gamma$ weakly converge or uniformly converge to $A_j^{(0)}\ (j=1,2,3)$, $Q^{(0)}$, $\lambda^{(0)}(z)$, $c_j^{(0)}(j=1,2)$, $b_j^{(0)}\ (j=0,1,\ldots,N_0)$, and the corresponding boundary value problems

$$u_{z\bar{z}}-\mathrm{Re}\left[Q^{(m)}u_{zz}+A_1^{(m)}u_z\right]-\hat{A}_2^{(m)}u=A_3^{(m)},\ \hat{A}_2^{(m)}=A_2^{(m)}+|u|^\sigma \tag{18}$$

and

$$\frac{1}{2}\frac{\partial u}{\partial \nu}+c_1^{(m)}(z)u=c_2^{(m)}(z)+h(z)\ \text{ on } \Gamma,\ u(a_j)=b_j,\ j=0,1,\ldots,N_0 \tag{19}$$

have the solutions $\{u^{(m)}(z)\}$, where $\hat{C}\left[u^{(m)}(z),\overline{D}\right]\ (m=1,2,\ldots)$ are unbounded. Hence, we can choose a subsequence of $\{u^{(m)}(z)\}$ denoted by $\{u^{(m)}(z)\}$ again, such that $h_m=\hat{C}\left[u^{(m)}(z),\overline{D}\right]\to\infty$ as $m\to\infty$. We can assume $h_m\ge\max[k_1,k_2,1]$. It is obvious that $\tilde{u}^{(m)}(z)=u^{(m)}(z)/h_m(m=1,2,\ldots)$ are solutions of the boundary value problems

$$\tilde{u}_{z\bar{z}}-\mathrm{Re}\left[Q^{(m)}\tilde{u}_{zz}+A_1^{(m)}\tilde{u}_z\right]-\hat{A}_2^{(m)}\tilde{u}=A_3^{(m)}/h_m \tag{20}$$

and

$$\frac{1}{2}\frac{\partial\tilde{u}}{\partial\nu}+c_1^{(m)}(z)\tilde{u}=c_2^{(m)}(z)/h_m+h(z)\ \text{ on } \Gamma,\ \tilde{u}(a_j)=b_j^{(m)},\ j=0,1,\ldots,N_0. \tag{21}$$

We can see that the functions in the above equation and boundary conditions satisfy Condition C, Equations 6 to 8 and

$$|u|^{\sigma+1}/h_m\le 1,\ \ L_{p,2}\left[A_3^{(m)}/h_m,\overline{D}\right]\le 1,$$
$$\left|c_2^{(m)}/h_m\right|\le 1,\ \left|b_j^{(m)}/h_m\right|\le 1,\ j=0,1,\ldots,N_0; \tag{22}$$

hence, we can obtain the estimate

$$\hat{C}_\beta\left[\tilde{u}^{(m)}(z),\overline{D}\right]\le M_4,\ \left\|\tilde{u}^{(m)}(z)\right\|_{W^2_{p_0,2}(D)}\le M_4,$$

in which $M_4=M_4(q_0,p_0,\beta,k,D)$ is a nonnegative constant. Thus, from the sequence of functions $\left\{\tilde{u}^{(m)}(z)\right\}$, we can choose the subsequence denoted by $\left\{\tilde{u}^{(m)}(z)\right\}$, which uniformly converges to $\tilde{u}^{(0)}(z)$ in $\overline{D}$ and whose partial derivatives $\tilde{u}_x^{(m)}$ and $\tilde{u}_y^{(m)}$ in $\overline{D}$ are uniformly convergent and $\tilde{u}_{xx}^{(m)}, \tilde{u}_{yy}^{(m)}$ and $\tilde{u}_{xy}^{(m)}$ in $\overline{D}$ are weakly convergent. This shows that $\tilde{u}_0(z)$ is a solution of the boundary value problem

$$\tilde{u}_{0z\bar{z}}-\mathrm{Re}\left[Q^{(0)}\tilde{u}_{0zz}+A_1^{(0)}\tilde{u}_{0z}\right]-\hat{A}_2^{(0)}\tilde{u}_0=0 \tag{23}$$

and

$$\frac{1}{2}\frac{\partial\tilde{u}_0}{\partial\nu}+c_1^{(0)}(z)\tilde{u}_0=h(z)\ \text{ on } \Gamma,\ u_0(a_j)=0,\ j=0,1,\ldots,N_0. \tag{24}$$

We see that the above Equation 23 possesses the condition $A_3^{(0)}=0$ and Equation 24 is the homogeneous boundary condition. On the basis of Theorem 1, the solution $\tilde{u}_0(z)=0$. However, from $\hat{C}\left[\tilde{u}^{(m)}(z),\overline{D}\right]=1$, we can derive that there exists a point $z^*\in\overline{D}$, such that $\left[|\tilde{u}_0(z)|^{\sigma+1}+|\tilde{u}_{0z}|\right]_{z=z^*}\ne 0$, which is impossible. This shows that the first two estimates in Equation 16 are true. Moreover, it is not difficult to verify the third estimate in Equation 16. □

Results and discussion

Solvability of oblique derivative problem for nonlinear elliptic equations of second order

By the above estimates and the Leray-Schauder theorem, we can prove the existence of solutions of Problem O for Equation 1. We first introduce the nonlinear elliptic equation of second order

$$u_{z\bar{z}}=f_m\left(z,u,u_z,u_{zz}\right),\ f_m\left(z,u,u_z,u_{zz}\right)=\mathrm{Re}\left[Q_m u_{zz}+A_{1m}u_z\right]+\hat{A}_{2m}u+A_3\ \text{ in } D, \tag{25}$$

with the coefficients

$$Q_m=\begin{cases}Q,\\ 0,\end{cases}\ A_{jm}=\begin{cases}A_j,\\ 0,\end{cases}\ j=1,3,\ \hat{A}_{2m}=\begin{cases}\hat{A}_2 & \text{in } D_m,\\ 0 & \text{in } \mathbb{C}\backslash D_m,\end{cases}$$

where $D_m = \{z \in D | \mathrm{dist}(z, \Gamma \cup \{\infty\}) \geq 1/m\}$ and m is a positive integer.

Theorem 3. *If Equation 25 satisfies Condition* C *and $u(z)$ is any solution of Problem* O *for Equation 25, then $u(z)$ can be expressed in the form*

$$\begin{cases} u(z) = U(z) + \tilde{v}(z) = U(z) + \hat{v}(z) + v(z), \\ v(z) = Hf_m = \dfrac{2}{\pi}\displaystyle\iint_{D_0} \frac{f_m(1/\zeta)}{|\zeta|^4} \ln\left|\frac{1-\zeta z}{\zeta}\right| d\sigma_\zeta. \end{cases}$$

Here, $\tilde{v}(z) = \hat{v}(z) + v(z)$ is a solution of Equation 25 with the homogeneous Dirichlet boundary condition

$$\tilde{v}(z) = 0 \ \ on \ \ \partial D_0 = \{|z| = 1\},$$

in which D_0 is the image domain of $\{|\zeta| < 1\}$ under the mapping $z = 1/\zeta$; the above boundary value problem is called Problem D, in which $U(z)$ is a solution of the boundary value problems (32) and (33) in the succeeding paragraph and $U(z)$ and $\tilde{v}(z)$ satisfy the estimates

$$\begin{aligned} &\hat{C}^1_\beta\left[U, \overline{D}\right] + \|U\|_{W^2_{p_0,2}(D)} \leq M_5, \\ &\hat{C}^1_\beta\left[\tilde{v}, \overline{D_0}\right] + \|\tilde{v}\|_{W^2_{p_0,2}(D_0)} \leq M_6, \end{aligned} \qquad (26)$$

where $\beta(> 0)$ and $M_j = M_j(q_0, p_0, \beta, k, D_m)(j = 5, 6)$ are nonnegative constants.

Proof. It is clear that the solution $u(z)$ can be expressed as before. On the basis of Theorem 2, it is easy to see that $\tilde{v}$ satisfies the second estimate in Equation 26, and then we know that $U(z)$ satisfies the first estimate of Equation 26. □

Theorem 4. *If Equation 1 satisfies Condition* C*, then Problem* O *for Equation 1 has a solution.*

Proof. In order to prove the existence of solutions of Problem O for Equation 25 by using the Leray-Schauder theorem, we introduce the equation with the parameter $t \in [0, 1]$:

$$V_{z\bar{z}} = tf_m\left(z, u, u_z, (U + V)_{zz}\right) \ \text{ in } D. \qquad (27)$$

Denote by B_M a bounded open set in the Banach space $B = \hat{W}^2_{p_0,2}(D_0) = \hat{C}^1_\beta(\overline{D_0}) \cap W^2_{p_0,2}(D_0)(0 < \beta \leq \alpha)$, the elements of which are real functions $V(z)$ satisfying the inequalities

$$\hat{C}^1_\beta\left[V(z), \overline{D_0}\right] + ||V||_{W^2_{p_0,2}(D_0)} < M_7 = M_6 + 1, \qquad (28)$$

in which M_6 is a nonnegative constant as stated in Equation 26. We choose any function $V(z) \in \overline{B_M}$ and make an integral $v(z) = H\rho$ as follows:

$$v(z) = H\rho = \frac{2}{\pi}\iint_{D_0} \frac{\rho(1/\zeta)}{|\zeta|^4} \ln\left|\frac{1-\zeta z}{\zeta}\right| d\sigma_\zeta, \qquad (29)$$

where $\rho(z) = V_{z\bar{z}}$. Next, we find a solution $\hat{v}(z)$ of the boundary value problem in D_0:

$$\hat{v}_{z\bar{z}} = 0 \ \text{ in } \ D_0, \qquad (30)$$

$$\hat{v}(z) = -v(z) \ \text{ on } \ \partial D_0. \qquad (31)$$

Denote $\tilde{v}(z) = \hat{v}(z) + v(z)$. Moreover, we find a solution $U(z)$ of the boundary value problem in D:

$$U_{z\bar{z}} = 0 \ \text{ in } \ D, \qquad (32)$$

$$\begin{aligned} \frac{1}{2}\frac{\partial U}{\partial \nu} + c_1(z)U = c_2(z) - \frac{\partial \tilde{v}}{\partial \nu} - c_1(z)\tilde{v} + h(z) \ \text{ on } \Gamma \\ U(a_j) = b_j, \ j = 0, 1, \ldots, N_0. \end{aligned} \qquad (33)$$

Now, we discuss the equation

$$\tilde{V}_{z\bar{z}} = tf_m(z, u, u_z, U_{zz} + \tilde{v}_{zz}), \ 0 \leq t \leq 1, \qquad (34)$$

where $u(z) = U(z) + \tilde{v}(z)$. By using Condition C and the principle of contraction mapping, the boundary value problem, i.e. Problem D for Equation 34 in D_0 has a unique solution $\tilde{V}(z)$ with the boundary condition

$$\tilde{V}(z) = 0 \ \text{ on } \ \partial D_0.$$

Denote by $\tilde{V} = S(V, t)(0 \leq t \leq 1)$ the mapping from V onto $\tilde{V}$. Furthermore, if $u(z)$ is a solution of Problem O in D for the equation

$$u_{z\bar{z}} = tf_m(z, u, u_z, u_{zz}), \ 0 \leq t \leq 1, \qquad (35)$$

then from Theorem 2, the solution $u(z)$ of Problem O for Equation 35 satisfies Equation 16; consequently, $\tilde{V}(z) = u(z) - U(z) \in B_M$. Set $B_0 = B_M \times [0, 1]$. In the following, we shall verify that the mapping $\tilde{V} = S(V, t)$ satisfies the three conditions of the Leray-Schauder theorem:

1. For every $t \in [0, 1]$, $\tilde{V} = S(V, t)$ continuously maps the Banach space B into itself and is completely continuous in B_M. Besides, for every function $V(z) \in \overline{B_M}$, $S(V, t)$ is uniformly continuous with respect to $t \in [0, 1]$.
   In fact, we arbitrarily choose $V_n(z) \in \overline{B_M}$, $n = 1, 2, \ldots$. It is clear that from $\{V_n(z)\}$, there exists a subsequence $\{V_{n_k}(z)\}$, such that $\{V_{n_k}(z)\}$, $\{V_{n_k z}(z)\}$ and corresponding functions $\{U_{n_k}(z)\}$ and $\{U_{n_k z}(z)\}$ uniformly converge to $V_0(z)$, $V_{0z}(z)$, $U_0(z)$ and $U_{0z}(z)$ in $\overline{D}$, respectively. We can find a solution $\tilde{V}_0(z)$ of Problem D for the equation
   $$\tilde{V}_{0z\bar{z}} = hf_m(z, u_0, u_{0z}, U_{0zz} + \tilde{v}_{0zz}), \ 0 \leq t \leq 1.$$
   Noting that $u_{n_k z\bar{z}} = U_{n_k z\bar{z}} + \tilde{v}_{n_k z\bar{z}}$, from $\tilde{V}_{n_k} = S(V_{n_k}, t)$ and $\tilde{V}_0 = S(V_0, t)$, we have
   $$\begin{aligned} (\tilde{V}_{n_k} - \tilde{V}_0)_{z\bar{z}} = h[f_m(z, u_{n_k}, u_{n_k z}, U_{n_k zz} + \tilde{v}_{n_k zz}) \\ -f_m(z, u_{n_k}, u_{n_k z}, U_{n_k zz} + \tilde{v}_{0zz}) + C_{n_k}(z)], \ 0 \leq t \leq 1, \end{aligned}$$

where

$$C_{n_k} = f_m(z, u_{n_k}, u_{n_k z}, U_{n_k zz} + \tilde{v}_{0zz}) \\ -f_m(z, u_0, u_0, U_{0zz} + \tilde{v}_{0zz}),\ z \in D_0.$$

Similar to (2.4.18) (see Chapter 2 in [6]), we can derive

$$L_{p_0,2}\left[C_{n_k}, \overline{D_0}\right] \to 0 \text{ as } k \to \infty.$$

Similar to Equations 16 to 24, we can derive that

$$\left\| \tilde{V}_{n_k} - \tilde{V}_0 \right\|_{\hat{W}^2_{p_0,2}(D_0)} \le L_{p_0,2}\left[C_{n_k}, \overline{D_0}\right] / [1 - q_0], \tag{36}$$

where $q_0 < 1$. It is easy to show that $\left\| \tilde{V}_{n_k} - \tilde{V}_0 \right\|_{\hat{W}^2_{p_0,2}(D)} \to 0$ as $k \to \infty$. Moreover, from Theorem 2, we can verify that from $\{\tilde{V}_{n_k}(z) - \tilde{V}_0(z)\}$, there exists a subsequence, denoted by $\{\tilde{V}_{n_k}(z) - \tilde{V}_0(z)\}$ again, such that $C^1_\beta\left[\tilde{V}_{n_k} - \tilde{V}_0, \overline{D_0}\right] \to 0$ as $k \to \infty$. This shows that the complete continuity of $\tilde{V} = S(V, t)(0 \le t \le 1)$ in $\overline{B_M}$. By using a similar method, we can prove that $\tilde{V} = S(V, t)(0 \le t \le 1)$ continuously maps $\overline{B_M}$ into B, and $\tilde{V} = S(V, t)$ is uniformly continuous with respect to $t \in [0, 1]$ for $V \in \overline{B_M}$.

2. For $t = 0$, from Theorem 2 and Equation 28, it is clear that $\tilde{V}(z) = S(V, 0) \in B_M$.
3. From Theorem 2 and Equation 28, we see that $\tilde{V} = S(V, t)(0 \le t \le 1)$ does not have a solution $\tilde{V}(z)$ on the boundary $\partial B_M = \overline{B_M} \backslash B_M$.
   Hence, by the Leray-Schauder theorem, we know that Problem O for Equation 27 with $t = 1$, namely Equation 25, has a solution $u(z) = U(z) + \tilde{v}(z) = U(z) + \hat{v}(z) + v(z) \in B_M$.

□

Theorem 5. *Under the same conditions in Theorem 3, Problem* O *for Equation 1 has a solution.*

Proof. By Theorems 2 and 4, Problem O for Equation 25 possesses a solution $u_m(z)$, and the solution $u_m(z)$ of Problem O for Equation 25 satisfies the estimate (Equation 16), where $m = 1, 2, \ldots$. Thus, we can choose a subsequence $\{u_{m_k}(z)\}$, such that $\{u_{m_k}(z)\}$ and $\{u_{m_k z}(z)\}$ in $\overline{D}$ uniformly converge to $u_0(z)$ and $u_{0z}(z)$, respectively. Obviously, $u_0(z)$ satisfies the boundary conditions of Problem O for Equation 1. □

Solvability of oblique derivative problems for general nonlinear elliptic equations of second order

In this section, we consider the general nonlinear elliptic equations

$$\begin{cases} u_{z\bar{z}} = F(z, u, u_z, u_{zz}) + G(z, u, u_z), \\ F = \mathrm{Re}\,[\,Qu_{zz} + A_1 u_z] + \hat{A}_2 u + A_3, \\ G = G(z, u, u_z),\ Q = Q(z, u, u_z, u_{zz}), \\ A_j = A_j(z, u, u_z), j = 1, 2, 3,\ \hat{A}_2 = A_2 + |u|^\sigma \end{cases} \tag{37}$$

and assume that Equation 37 satisfies Condition C′, in which Condition C, as stated in the ‘Formulation of oblique derivative problems of second-order elliptic equations in unbounded domains’ section, is met and the function $G(z, u, u_z)$ possesses the form

$$G(z, u, u_z) = \mathrm{Re} B_1 u_z + B_2 |u|^\tau \ \text{ in } D, \tag{38}$$

where $0 < \tau < \infty$ and $L_{p,2}\left[B_j, \overline{D}\right] \le k_0\,(< \infty, j = 1, 2,\ 2 < p_0 \le p)$ with a positive constant k_0; the above conditions will be called Condition C′.

Theorem 6. *Let the complex Equation 37 satisfy Condition* C′.

1. *When* $0 < \tau < 1$, *Problem* O *for*
$$\begin{cases} w_{\bar{z}} = F(z, u, w, w_z) + G(z, u, w), \\ F = Re\,[\,Qw_z + A_1 w] + \hat{A}_2 u + A_3, \\ G = G(z, u, w),\ Q = Q(z, u, w, w_z), \\ A_j = A_j(z, u, w),\ j = 1, 2, 3,\ w = u_z \end{cases} \tag{39}$$
has a solution $[\,w(z), u(z)]$, *where* $w(z), u(z) \in W^1_{p_0,2}(D)$ *and* p_0 $(2 < p_0 \le p)$ *is a constant as stated before.*
2. *When* $\tau > 1$, *Problem* O *for Equation 39 has a solution* $[\,w(z), u(z)]$, *where* $w(z) \in W^1_{p_0,2}(D)$, *provided that*
$$M_8 = L_{p_0,2}\left[A_3, \overline{D}\right] + C_\alpha\,[c_2, \Gamma] + \sum_{j=0}^{N_0} |b_j| \tag{40}$$
is sufficiently small.

Proof. 1. In this case, the algebraic equation for t is as follows:

$$M_2 \left\{ L_{p_0,2}\left[A_3, \overline{D}\right] + L_{p_0,2}\left[B_2, \overline{D}\right] t^\tau + L_\alpha[\,c_2, \Gamma] + \sum_{j=0}^{N_0} |b_j| \right\} = t, \tag{41}$$

where M_2 is a constant as stated in Equation 16. Because $0 < \tau < 1$, Equation 41 has a unique solution $t = M_9 > 0$. Now, we introduce a bounded, closed and convex subset B^* of the Banach space

$C(\overline{D}) \times C(\overline{D})$, whose elements are of the form $[w(z), u(z)]$ satisfying the condition

$$w(z), u(z) \in C(\overline{D}),\ C[w(z), \overline{D}] + C[|u(z)|^{\sigma+1}, \overline{D}] \le M_9. \tag{42}$$

We choose a pair of functions $[\tilde{w}(z), \tilde{u}(z)] \in B^*$ and substitute it into the appropriate positions of $F(z, u, w, w_z)$ and $G(z, u, w)$ in Equation 39 and the boundary conditions (5) and (7) and obtain

$$w_{\bar{z}} = \tilde{F}(z, u, w, \tilde{u}, \tilde{w}, w_z) + G(z, \tilde{u}, \tilde{w}), \tag{43}$$

$$\begin{aligned} \operatorname{Re}\left[\overline{\lambda(z)} w(z)\right] &= -c_1(z)\tilde{u} + c_2(z) + h(z),\ z \in \Gamma, \\ u(a_j) &= b_j,\ 0 \le j \le N_0, \end{aligned} \tag{44}$$

where

$$\begin{aligned} \tilde{F}(z, u, w, \tilde{u}, \tilde{w}, w_z) = &\operatorname{Re}\ [Q(z, \tilde{u}, \tilde{w}, w_z) w_z \\ &+ A_1(z, \tilde{u}, \tilde{w}) w] \\ &+ \hat{A}_2(z, \tilde{u}, \tilde{w}) u + A_3(z, \tilde{u}, \tilde{w}). \end{aligned}$$

In accordance with the method in the proof of Theorem 5, we can prove that the boundary value problems (43) and (44) have a unique solution $[w(z), u(z)]$. Denote by $[w, u] = T[\tilde{w}(z), \tilde{u}(z)]$ the mapping from $[\tilde{w}(z), \tilde{u}(z)]$ to $[w(z), u(z)]$. Noting that Condition C′ and other conditions, as stated in the 'Formulation of oblique derivative problems of second order elliptic equations in unbounded domains' section, are similar to the proof of Theorem 2, we can obtain

$$\begin{aligned} &C\left[w(z), \overline{D}\right] + C\left[|u(z)|^{\sigma+1}, \overline{D}\right] \\ &\le M_2 \left\{ L_{p_0,2}\left[A_3, \overline{D}\right] + C_\alpha[c_2, \Gamma] + \sum_{j=0}^{N} |b_j| + L_{p,2}[G, \overline{D}] \right\} \\ &\le M_2 \left\{ M_9 + L_{p_0,2}\left[B_2, \overline{D}\right] C\left[\tilde{u}, \overline{D}\right]^\tau \right\} \\ &\le M_2 \left\{ M_9 + L_{p_0,2}\left[B_2, \overline{D}\right] M_{10}^\tau \right\} = M_{10}. \end{aligned} \tag{45}$$

This shows that T maps B^* onto a compact subset in B^*. Next, we verify that T in B^* is a continuous operator. In fact, we arbitrarily select a sequence $\{\tilde{w}_n(z), \tilde{u}_n(z)\}$ in B^*, such that

$$C(\tilde{w}_n - \tilde{w}_0, \overline{D}) + C(|\tilde{u}_n - \tilde{u}_0|^{\sigma+1}, \overline{D}) \to 0 \text{ as } n \to \infty. \tag{46}$$

We can see that

$$\begin{aligned} L_{p_0,2}&\left[A_j(z, \tilde{u}_n, \tilde{w}_n) - A_j(z, \tilde{u}_0, \tilde{w}_0), \overline{D}\right] \\ &\to 0\ (j = 1, 2, 3) \text{ as } n \to \infty. \end{aligned} \tag{47}$$

Moreover, from $[w_n, u_n] = T[\tilde{w}_n, \tilde{u}_n]$ and $[w_0, u_0] = T[\tilde{w}_0, \tilde{u}_0]$, it is clear that $[w_n - w_0, u_n - u_0]$ is a solution of Problem O for the following equation:

$$\begin{aligned} (w_n - w_0)_{\bar{z}} = &\tilde{F}(z, u_n, w_n, \tilde{u}_n, \tilde{w}_n, w_{nz}) \\ &- \tilde{F}(z, u_0, w_0, \tilde{u}_0, \tilde{w}_0, w_{0z}) + G(z, \tilde{u}_n, \tilde{w}_n) \\ &- G(z, \tilde{u}_0, \tilde{w}_0) \text{ in } D, \end{aligned} \tag{48}$$

$$\operatorname{Re}\left[\overline{\lambda(z)}(w_n - w_0)\right] = -c_1(z)(\tilde{u}_n - \tilde{u}_0) + h(z) \text{ on } \Gamma, \tag{49}$$

$$u_n(a_j) - u_0(a_j) = 0,\ j = 1, ..., N_0,\ u_n(1) - u_0(1) = 0. \tag{50}$$

In accordance with the method in the proof of Theorem 2, we can obtain the estimate

$$\begin{aligned} &C\left[w_n(z) - w_0(z), \overline{D}\right] + C\left[|u_n(z) - u_0(z)|^{\sigma+1}, \overline{D}\right] \\ &\le M_{10} \left\{ L_{p_0,2}[\hat{A}_2(z, \tilde{u}_n, \tilde{w}_n)\tilde{u}_n - \hat{A}_2(z, \tilde{u}_0, \tilde{w}_0)\tilde{u}_0, \overline{D}] \right. \\ &+ L_{p_0,2}\left[A_3(z, \tilde{u}_n, \tilde{w}_n) - A_3(z, \tilde{u}_0, \tilde{w}_0), \overline{D}\right] \\ &+ L_{p_0,2}\left[G(z, \tilde{u}_n, \tilde{w}_n) - G(z, \tilde{u}_0, \tilde{w}_0), \overline{D}\right] \\ &\left. + C_\alpha\left[c_1(z)(\tilde{u}_n - \tilde{u}_0), \Gamma\right] \right\}, \end{aligned} \tag{51}$$

in which $M_{10} = M_{10}(q_0, p_0, k_0, \alpha, K, D)$. From Equations 46 and 47 and the above estimate, we obtain $C\left[w_n - w_0, \overline{D}\right] + C\left[u_n - u_0, \overline{D}\right] \to 0$ as $n \to \infty$. On the basis of the Schauder fixed-point theorem, there exists a function $[w(z), u(z)]$ $(w(z), u(z) \in C(\overline{D}))$ such that $[w(z), u(z)] = T[w(z), u(z)]$, and from Theorem 2, it is easy to see that $w(z), u(z) \in W^1_{p_0,2}(D)$ and $[w(z), u(z)]$ is a solution of Problem O for Equation 37 and $w(z) = u_z$ with $0 < \tau < 1$.

2. When $\tau > 1$, in this case, Equation 41 has the solution $t = M_9$ provided that M_8 in Equation 39 is small enough. Now ,we consider a closed and convex subset B_* in the Banach space $C(\overline{D}) \times C(\overline{D})$, i.e.

$$B_* = \left\{ w(z), u(z) \in C(\overline{D}),\ C\left[w, \overline{D}\right] + C\left[|u|^{\sigma+1}, \overline{D}\right] \le M_9 \right\}. \tag{52}$$

Applying a method similar as before, we can verify that there exists a solution $[w(z), u(z)] \in W^1_{p_0,2}(D) \times W^1_{p_0,2}(D)$ of Problem O for Equation 39 for the constant $\tau > 1$, and then $u(z)$ is a solution of Problem O for Equation 37.

□

Conclusions

Some solvability results of the general oblique derivative boundary value problem for nonlinear elliptic equations of second order in an unbounded multiply connected domain are obtained.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

GW and ZX together discussed and proposed the problem and solved the above boundary value problem for the nonlinear elliptic equations of second order. Both authors are members of a study organization and developed the cooperative investigation for a long time. Both authors read and approved the final manuscript.

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