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An iterative scheme for numerical solution of Volterra integral equations using collocation method and Chebyshev polynomials

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Abstract

When we discretize nonlinear Volterra integral equations using some numerical, such as collocation methods, the arising algebraic systems are nonlinear. Applying quasilinear technique to the nonlinear Volterra integral equations gives raise to linear Volterra integral equations. The solutions of these equations yield a functional sequence quadratically convergent to the solution. Then, we use collocation method based on Chebyshev polynomials and a modified Clenshaw-Curtis quadrature and obtain a numerical solution. Error analysis has been performed, and the method has been applied on three numerical examples.

Keywords: Volterra integral equations, Quasilinear technique, Collocation method, Chebyshev polynomials

Introduction

The method of quasilinearization has been started by Bellman and Kalaba [1], generalized by Lakshmikantham [2,3], and applied to a variety of problems by [4-7]. Based on the idea of quasilinearization [1], the generalized quasi-linear technique in [8] and later extended in [2] offers two monotonic sequences of linear iterations, uniformly and quadratically convergent to the unique solution of the initial value problem:

$$u'(t) = f(t, u), \quad u(t_0) = 0.$$

Consider the nonlinear Volterra integral equation:

$$u(t) = y(t) + \int_0^t k(t, s, u(s))ds. \tag{1}$$

Applying iterative processes to solve this equation, when $k(t, s, u)$ is nondecreasing with respect to u and satisfies a Lipschitz condition, the successive approximations method [9] yields a monotonic sequence, uniformly convergent to the solution of Equation (1). But the iterations that define the sequences are all nonlinear, and the rate of convergence is linear. The iterations employed in the

monotone iterative technique [10,11] and their convergence rate are linear. The method of the quasilinearization is developed by [12] to solve Equation (1) by the iterative scheme:

$$u_p(t) = y(t) + \int_0^t \big(k(t, s, u_{p-1}(s)) + k_u(t, s, u_{p-1}(s)) \\ \times \big(u_p(s) - u_{p-1}(s) \big) \big) ds, \tag{2}$$

for $p = 1, 2, \cdots$, where $u_0(t)$ is the lower solution of Equation (1), defined in the following. This scheme is linear; under nondecreasing monotonicity and convexity conditions on $k(t, s, u)$, it is quadratically convergent to the unique solution of Equation (1) and then rapidly in comparison with the successive processes in [9-11]. The purpose of this paper is to employ numerical methods to approximate the solution of the linear integral equations (2) in a piecewise continuous polynomial space and then generate a sequence of approximation solutions where converge to the unique solution of the nonlinear integral equation (1) under some conditions on $k(t, s, u)$ as mentioned above. For numerical integration and quadrature formulae, we employ Chebyshev polynomials and a modified Clenshaw-Curtis quadrature and its error analysis.

This paper has been organized as follows: section "Volterra integral inequalities and generalized quasilinearization" contains a general framework for the idea of quasilinearization used to solve the integral equations.

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Section "Piecewise polynomials space and collocation method" is devoted to derivation of step by step collocation method in a piecewise polynomials space to approximate the solution of the integral equations. In section "Fully discretizing using quadrature formulae and their errors", fully discretizing of the algebraic system is obtained by using a modified Clenshaw-Curtis quadrature and its error analysis is given. In section "Convergence of the fully discretized solution", the convergence of the method is discussed and the numerical examples are given in section "Numerical examples".

Volterra integral inequalities and generalized quasilinearization

For $T \in \mathbb{R}$ and $T > 0$, let $J = [0, T]$ and $D = \{(t, s) \in J \times J : s \leq t\}$, consider

$$u(t) = y(t) + \int_0^t k(t, s, u(s)) ds, \qquad (3)$$

where $y \in C[J, \mathbb{R}]$ and $k \in C[D \times \mathbb{R}, \mathbb{R}]$.

Definition 1. A function $v \in C[J, \mathbb{R}]$ is called a lower solution of Equation (3) on J if

$$v(t) \leq y(t) + \int_0^t k(t, s, v(s)) ds, \quad t \in J,$$

and an upper solution if the reversed inequality holds.

It is shown in [12] that if $v_0(t)$ and $w_0(t)$ in $C[J, \mathbb{R}]$ are lower and upper solutions of Equation (3), respectively, then $v_0(t) \leq w_0(t)$ holds on J, when $k(t, s, u)$ is nondecreasing in u for each fixed $(t, s) \in D$ and satisfies one-sided Lipschitz condition

$$k(t, s, \alpha) - k(t, s, \beta) \leq L(\alpha - \beta), \quad \alpha \leq \beta, L > 0,$$

provided $v_0(0) \leq w_0(0)$. If this holds, there exists a solution $u(t)$ of Equation (3) such that

$$v_0(t) \leq u(t) \leq w_0(t),$$

and this solution is unique.

Now, for $v_0, w_0 \in C[J, \mathbb{R}]$ and $v_0(t) \leq u \leq w_0(t)$ on J, let

$$\Omega = \{(t, s, u) \in D \times \mathbb{R} ; v_0(t) \leq u \leq w_0(t), t \in J\},$$

and $\|u\| = max_{t \in J}|u(t)|$. By defining two iterative schemes as two linear integral equations

$$v_p(t) = y(t) + \int_0^t \big(k(t, s, v_{p-1}(s)) + k_u(t, s, v_{p-1}(s)) \\ \times \big(v_p(s) - v_{p-1}(s)\big)\big) ds, \qquad (4)$$

and

$$w_p(t) = y(t) + \int_0^t \big(k(t, s, w_{p-1}(s)) + k_u(t, s, w_{p-1}(s)) \\ \times \big(w_p(s) - w_{p-1}(s)\big)\big) ds, \qquad (5)$$

for $p = 1, 2, \cdots$, and $v_0(t), w_0(t) \in C[J, \mathbb{R}]$, lower and upper solutions of Equation (3) respectively, the following theorem in [12] shows the quadratically convergence of two sequences $\{v_p(t)\}$ and $\{w_p(t)\}$ derived from Equations (4) and (5) to the unique solution of Equation (3).

Theorem 1. *Assume that* (H_1) $v_0, w_0 \in C[J, \mathbb{R}]$, $v_0(t) \leq w_0(t)$ *on J are lower and upper solutions of Equation (3) on J, respectively.*

(H_2) $k \in C^2[\Omega, \mathbb{R}]$, $k_u(t, s, u) \geq 0$, $k_{uu}(t, s, u) \geq 0$ *for* $(t, s, u) \in \Omega$.

Then, the two iterative schemes (4) and (5) define a nondecreasing sequence $\{v_p(t)\}$ and a nonincreasing sequence $\{w_p(t)\}$ in $C[J, \mathbb{R}]$ such that $v_p \longrightarrow u$ and $w_p \longrightarrow u$ uniformly on J, and the following quadratic convergent estimates hold:

$$\|u - v_p\| \leq A\|u - v_{p-1}\|^2,$$

$$\|w_p - u\| \leq B\|w_{p-1} - u\|^2 + A\|u - v_{p-1}\|^2,$$

$$p = 1, 2, \cdots, \ A > 0, B > 0.$$

Also, these two sequences satisfy the relation

$$v_0 \leq v_1 \leq \cdots \leq v_p \leq w_p \leq \cdots \leq w_1 \leq w_0.$$

The following two lemmas in [13] and [14] will be used during this work. In the whole of the work, we refer to $\|.\|$ as the maximum norm of the functions or matrices.

Lemma 1. *Suppose that A is a matrix such that $\|A\| < 1$. Then, the matrix $(I - A)$ is nonsingular and*

$$\|(I - A)^{-1}\| \leq (1 - \|A\|)^{-1}.$$

Lemma 2. *(Discrete Gronwall Inequality) Let for all $i \in \mathbb{N}(a) = \{a, a + 1, \ldots\}$ the following inequality be satisfied:*

$$u_i \leq p_i + q_i \sum_{j=a}^{i-1} f_j u_j, \quad a \in \mathbb{N}.$$

Then, for all $i \in \mathbb{N}(a)$

$$u_i \leq p_i + q_i \sum_{j=a}^{i-1} p_j f_j \prod_{k=j+1}^{i-1} (1 + q_k f_k).$$

Piecewise polynomials space and collocation method

We set the partition $\{0 = \tau_0 < \tau_1 < \cdots < \tau_N = T\}$ on J and put $h_n = \tau_n - \tau_{n-1}$ with $h = \max_n\{h_n\}$ and indicate the above partition by J_h. In the theory of interpolation, it is well known that the minimum value of the remainder term is obtained when the polynomial interpolation is carried out on the zeros of Chebyshev polynomials [15].

The first-kind Chebyshev polynomials are defined by the relation

$$T_m(z) = \cos(m\theta), \quad z = \cos(\theta), \quad m = 0, 1, 2, \dots,$$

and the zeros of these polynomials in an increasing arrangement are as

$$z_{m-k+1} = \cos\left(\frac{2k-1}{2m}\pi\right), \quad k = 1, \dots, m.$$

We define the mapping $\delta_n : [-1, 1] \longmapsto [\tau_{n-1}, \tau_n]$ with the relation

$$\delta_n(z) = \frac{\tau_n - \tau_{n-1}}{2} z + \frac{\tau_n + \tau_{n-1}}{2}.$$

Now, consider the linear Volterra integral equation (4). It may be shown in the form of

$$v_p(t) = y(t) + \int_0^t H_p(t, s) ds + \int_0^t k_p(t, s) v_p(s) ds,$$
$$p = 1, 2, \dots, \tag{6}$$

where

$$H_p(t, s) = k(t, s, v_{p-1}(s)) - k_u(t, s, v_{p-1}(s)) v_{p-1}(s), \tag{7}$$

and

$$k_p(t, s) = k_u(t, s, v_{p-1}(s)), \quad (t, s) \in D. \tag{8}$$

Definition 2. Suppose that J_h is a given partition on J. The piecewise polynomial space $S_\mu^{(d)}(J_h)$ with $\mu \geq 0$, $-1 \leq d \leq \mu$ is defined by

$$S_\mu^{(d)}(J_h) = \{q(t) \in C^d[J, \mathbb{R}] : q|_{\sigma_n} \in \pi_\mu ; 1 \leq n \leq N\},$$

where $\sigma_n = (\tau_{n-1}, \tau_n]$ and π_μ denotes the space of the polynomials of degree not exceeding μ, and it is easy to see that $S_\mu^{(d)}(J_h)$ is a linear vector space.

With this definition, we select the Lagrange polynomials constructed on the zeros of the Chebyshev polynomial $T_m(z)$ as a basis for π_{m-1} on the subinterval σ_n and approximate the solution of the integral equation (6) in the piecewise polynomial space $S_{m-1}^{(d)}(J_h)$ using collocation method. Then, by letting $t_{n,k} = \delta_n(z_k)$ and $Y_h = \{t_{n,k} : n = 1, \dots, N, \ k = 1, \dots, m\}$, the collocation points, this collocation solution $\hat{v}_p(t) \in S_{m-1}^{(d)}(J_h)$ in the subinterval σ_n may be written as

$$\hat{v}_p(t) = \hat{v}_p(\delta_n(z)) = \hat{v}_{p,n}(z) = \sum_{k=1}^m \hat{V}_{n,k}^p L_k(z), \ z \in (-1, 1],$$
$$n = 1, \dots, N, \tag{9}$$

for $p = 1, 2, \dots$, where $\hat{V}_{n,k}^P = \hat{v}_p(t_{n,k})$ and $L_k(z)$ are the Lagrange polynomials defined by the relation

$$L_k(z) = \prod_{\substack{j=1 \\ j \neq k}}^m \frac{z - z_j}{z_k - z_j}, \quad k = 1, \dots, m.$$

We observe that this collocation solution is not necessary to be continuous on J. Then, we must let $d = -1$ and $\hat{v}_p(t) \in S_{m-1}^{(-1)}(J_h)$. For $t \in \sigma_i$, Equation (6) may be written as

$$v_p(t) = y(t) + \int_0^{\tau_{i-1}} H_p(t, s) ds + \int_{\tau_{i-1}}^t H_p(t, s) ds$$
$$+ \int_0^{\tau_{i-1}} k_p(t, s) v_p(s) ds + \int_{\tau_{i-1}}^t k_p(t, s) v_p(s) ds$$
$$= y(t) + \sum_{j=1}^{i-1} \int_{\tau_{j-1}}^{\tau_j} \left(H_p(t, s) + k_p(t, s) v_p(s)\right) ds$$
$$+ \int_{\tau_{i-1}}^t H_p(t, s) ds + \int_{\tau_{i-1}}^t k_p(t, s) v_p(s) ds$$
$$= y(t) + \sum_{j=1}^{i-1} \frac{h_j}{2} \int_{-1}^1 \left(H_p(t, \delta_j(z)) + k_p(t, \delta_j(z)) v_{p,j}(z)\right) dz$$
$$+ \frac{h_i}{2} \int_{-1}^{\delta_i^{-1}(t)} H_p(t, \delta_i(z)) dz$$
$$+ \frac{h_i}{2} \int_{-1}^{\delta_i^{-1}(t)} k_p(t, \delta_i(z)) v_{p,i}(z) dz, \tag{10}$$

where $v_{p,i}(z) = v_p(\delta_i(z))$.

The collocation equation is defined by replacing the exact solution $v_p(t)$ with the collocation solution $\hat{v}_p(t)$ in Equation (6) on the collocation points Y_h.

Then, using Equation (10) for $t_{i,k} \in Y_h$, the collocation equation has the form

$$\hat{V}_{i,k}^p = \hat{v}_p(t_{i,k})$$
$$= y(t_{i,k})$$
$$+ \sum_{j=1}^{i-1} \frac{h_j}{2} \int_{-1}^1 \left(H_p(t_{i,k}, \delta_j(z)) + k_p(t_{i,k}, \delta_j(z)) \hat{v}_{p,j}(z)\right) dz$$
$$+ \frac{h_i}{2} \int_{-1}^{z_k} H_p(t_{i,k}, \delta_i(z)) dz + \frac{h_i}{2} \int_{-1}^{z_k} k_p(t_{i,k}, \delta_i(z)) \hat{v}_{p,j}(z) dz$$
$$= y(t_{i,k}) + \sum_{j=1}^{i-1} \frac{h_j}{2} \left(\int_{-1}^1 H_p(t_{i,k}, \delta_j(z)) ds \right.$$
$$\left. + \sum_{q=1}^m \hat{V}_{j,q}^p \int_{-1}^1 k_p(t_{i,k}, \delta_j(z)) L_q(z) dz \right)$$
$$+ \frac{h_i}{2} \int_{-1}^{z_k} H_p(t_{i,k}, \delta_i(z)) dz$$
$$+ \frac{h_i}{2} \sum_{q=1}^m \hat{V}_{i,q}^p \int_{-1}^{z_k} k_p(t_{i,k}, \delta_i(z)) L_q(z) dz, \tag{11}$$

for $k = 1, \ldots, m$, where we have used Equation (9) and by denoting the following vectors and matrices forms,

$$\hat{V}_i^p = \left[\hat{V}_{i,1}^p, \ldots, \hat{V}_{i,m}^p \right]^T, y_i = \left[y(t_{i,1}), \ldots, y(t_{i,m}) \right]^T,$$

$$i = 1, \ldots, N,$$

$$H_i^{p,j} = \left[h_i^{p,j}(1), \ldots, h_i^{p,j}(m) \right]^T$$

$$= \left[\int_{-1}^{1} H_p(t_{i,1}, \delta_j(z)) dz, \ldots, \int_{-1}^{1} H_p(t_{i,m}, \delta_j(z)) dz \right]^T,$$

$$H_i^p = \left[h_i^p(1), \ldots, h_i^p(m) \right]^T$$

$$= \left[\int_{-1}^{z_1} H_p(t_{i,1}, \delta_i(z)) dz, \ldots, \int_{-1}^{z_m} H_p(t_{i,m}, \delta_i(z)) dz \right]^T,$$

$$B_i^{p,j} = \left[b_i^{p,j}(k, q) \right] = \left[\int_{-1}^{1} k_p(t_{i,k}, \delta_j(z)) L_q(z) dz \right],$$

$$k, q = 1, \ldots, m,$$

$$B_i^p = \left[b_i^p(k, q) \right] = \left[\int_{-1}^{z_k} k_p(t_{i,k}, \delta_i(z)) L_q(z) dz \right],$$

$$k, q = 1, \ldots, m, \ j = 1, \ldots, i-1, \quad (12)$$

Equation (11) is transformed to the linear system

$$\left(I_m - \frac{h_i}{2} B_i^p \right) \hat{V}_i^p = y_i + \frac{h_i}{2} H_i^p + \sum_{j=1}^{i-1} \frac{h_j}{2} \left(H_i^{p,j} + B_i^{p,j} \hat{V}_j^p \right),$$

$$i = 1, \ldots, N, \quad (13)$$

where I_m is the identity matrix with dimension m.

Fully discretizing using quadrature formulae and their errors

The fundamental theorem of orthonormal Fourier expansions in the space $C[-1, 1]$ implies that for any $f \in C[-1, 1]$, there exists an expansion of Chebyshev polynomials as follows:

$$f(z) = \sum_{r=0}^{\infty}{}' a_r T_r(z), \quad a_r = \frac{2}{\pi} \int_{-1}^{1} \frac{f(z) T_r(z)}{\sqrt{1 - z^2}} \, dz,$$

where the prime denotes that the first term is halved. Integration of both sides of this expansion gives:

$$\mathcal{I}f = \int_{-1}^{1} f(z) \, dz = \sum_{r=0}^{\infty}{}' \frac{2a_{2r}}{1 - 4r^2}. \quad (14)$$

Then, we approximate the integral (14) by $M + 1$ first terms of the series:

$$\int_{-1}^{1} f(z) \, dz \approx \sum_{r=0}^{M}{}' \frac{2a_{2r}}{1 - 4r^2},$$

and the coefficients a_i also are approximated by the closed Gauss-Chebyshev rule. This is the Clenshaw-Curtis quadrature for estimating the integral (14), but we modify this rule and use it to the components of the matrices $H_i^{p,j}$, H_i^p, $B_i^{p,j}$, and B_i^p with the open Gauss-Chebyshev because this rule does not use the end points of the intervals and uses one node less than close Gauss-Chebyshev and same accuracy. M nodes open Gauss-Chebyshev rule for approximating a_r has the form

$$a_r \approx \tilde{a}_r^M = \frac{2}{M} \sum_{l=1}^{M} f\left(\xi_l^{(M)} \right) \cos\left(r \frac{2l-1}{2M} \pi \right), \quad (15)$$

$$\xi_l^{(M)} = \cos\left(\frac{2l-1}{2M} \pi \right),$$

and we use the following equation as an approximation for the integral (14):

$$\mathcal{I}f \approx \mathcal{Q}f = \sum_{r=0}^{M}{}' \frac{2\tilde{a}_{2r}^M}{1 - 4r^2}.$$

For the components of the matrices H_i^p and B_i^p, we use change of variable $z = \theta_k(w) : [-1, 1] \longmapsto [-1, z_k]$ defined by

$$\theta_k(w) = \frac{z_k + 1}{2} w + \frac{z_k - 1}{2}, \quad k = 1, \ldots, m.$$

For the error analysis of this quadrature formula using the equality (14), we have

$$\mathcal{I}f - \mathcal{Q}f = \sum_{r=0}^{M}{}' \frac{2\left(a_{2r} - \tilde{a}_{2r}^M \right)}{1 - 4r^2} + \sum_{r=M+1}^{\infty} \frac{2a_{2r}}{1 - 4r^2}. \quad (16)$$

To compute $(a_r - \tilde{a}_r^M)$, we utilize conclusions in [16] which presents the equality

$$a_r - \tilde{a}_r^M = -2 \sum_{l=1}^{\infty} (-1)^l c_{r,2Ml}, \quad (17)$$

$$c_{r,l} = \frac{2}{\pi} \int_{-1}^{1} \frac{f(z) T_r(z) T_l(z)}{\sqrt{1 - z^2}} \, dz.$$

Moreover, the coefficients $c_{r,l}$ depend on the properties of the function $f(z) T_r(z)$ using the following lemma which follows from results in [17].

Lemma 3. *Let $f(z) \in C^p[-1, 1]$, and suppose $f^{(p+1)}(z)$ exists and is continuous in $[-1, 1]$ except possibly for a finite number of finite discontinuities, then for some C_f and all $i > 0$ the fourier coefficients a_i satisfies in the relation*

$$|a_i| \leq C_f \, i^{-p-1}.$$

Substituting this result into (17), we find

$$|a_r - \tilde{a}_r^M| \leq \frac{2C_{f'}}{(2M)^{p+1}} \sum_{l=1}^{\infty} \frac{1}{l^{p+1}} \leq \frac{2(p+1)C_{f'}}{p(2M)^{p+1}}, \quad (18)$$

where $C_{f'}$ is related to the function $f(z)T_r(z)$, and we have used the following relation in [18]

$$\sum_{l=R+1}^{\infty} (l-i)^{-r} \leq (R+1-i)^{-r+1} \frac{r}{r-1}, \quad r > 1.$$

Applying Equations (18) to (16) and reusing lemma (3) for a_r^M, we obtain the bound

$$|\mathcal{I}f - \mathcal{Q}f| \leq \frac{4(p+1)C_{f'}}{p(2M)^{p+1}} + \frac{C_f}{(2M+1)(2M+2)^{p+1}} = \mathcal{M}_M$$

$$= O\left((2M)^{-p-1}\right), \quad (19)$$

where the constant C_f is related to the function $f(z)$.

According to this numerical quadrature, we define fully discretized matrices related to the matrices (12) as

$$\tilde{H}_i^{p,j} = \left[\tilde{h}_i^{p,j}(1), \dots, \tilde{h}_i^{p,j}(m)\right]^T,$$

$$\tilde{H}_i^p = \left[\tilde{h}_i^p(1), \dots, \tilde{h}_i^p(m)\right]^T,$$

$$\tilde{B}_i^{p,j} = \left[\tilde{b}_i^{p,j}(k,q)\right], \quad k,q = 1, \dots, m,$$

$$\tilde{B}_i^p = \left[\tilde{b}_i^p(k,q)\right], \quad k,q = 1, \dots, m,$$

and refresh the linear system (13) for them

$$\left(I_m - \frac{h_i}{2}\tilde{B}_i^p\right)\tilde{V}_i^p = y_i + \frac{h_i}{2}\tilde{H}_i^p + \sum_{j=1}^{i-1} \frac{h_j}{2}\left(\tilde{H}_i^{p,j} + \tilde{B}_i^{p,j}\tilde{V}_j^p\right),$$

$$i = 1, \dots, N, \quad (20)$$

where $\tilde{V}_i^p = \left[\tilde{V}_{i,1}^p, \dots, \tilde{V}_{i,m}^p\right]^T$. Then, we define the fully discretized solution $\tilde{v}(t) \in S_{m-1}^{(-1)}(J_h)$ on σ_i by the relation

$$\tilde{v}_p(t) = \tilde{v}_p(\delta_i(z)) = \tilde{v}_{p,i}(z) = \sum_{k=1}^{m} \tilde{V}_{i,k}^p L_k(z), \quad z \in (-1,1].$$

Theorem 2. *There exists an $\bar{h} > 0$ such that for any partition J_h with $0 < h < \bar{h}$, the linear system (20) has a unique solution $\tilde{V}_i^p$ for $1 \leq i \leq N$ and $p = 1, 2, \dots$.*

Proof. We first give a bound for the $\|\tilde{B}_i^p\|$ then use the lemma (1). The matrix $\tilde{B}_i^p$ has the components

$$\tilde{b}_i^p(k,q) = \sum_{r=0}^{M}{}' \frac{2\tilde{a}_{2r}^M}{1-4r^2}, \quad k,q = 1, \dots, m,$$

$$\tilde{a}_r^M = \frac{1+z_k}{M} \sum_{l=1}^{M} k_p\left(t_{i,k}, \delta_j\left(\theta_k\left(\xi_l^{(M)}\right)\right)\right) L_q\left(\theta_k\left(\xi_l^{(M)}\right)\right)$$

$$\times \cos\left(r\frac{2l-1}{2M}\pi\right).$$

$\square$

Then,

$$|\tilde{a}_r^M| < 2K_u\varphi,$$

$$\|\tilde{B}_i^p\| < \sum_{r=0}^{M}{}' \frac{4K_u\varphi}{|1-4r^2|} \leq 4K_u\varphi,$$

$$K_u = \max_{\Omega} k_u(t,s,u), \quad \varphi = \max_{q=1,\dots,m} \|L_q(z)\|.$$

Now using lemma (1), the matrix $\left(I_m - \frac{h_i}{2}\tilde{B}_i^p\right)$ is non-singular if we get

$$0 < h < \frac{1}{2K_u\varphi} = \bar{h},$$

and for this selection

$$\left\| \left(I_m - \frac{h_i}{2}\tilde{B}_i^p\right)^{-1} \right\| \leq \frac{1}{1 - \frac{h_i}{2}\|\tilde{B}_i^p\|} \leq \frac{1}{1 - 2hK_u\varphi},$$

and the linear system (20) has the unique solution

$$\tilde{V}_i^p = \left(I_m - \frac{h_i}{2}\tilde{B}_i^p\right)^{-1}\left(y_i + \frac{h_i}{2}\tilde{H}_i^p + \sum_{j=1}^{i-1} \frac{h_j}{2}\left(\tilde{H}_i^{p,j} + \tilde{B}_i^{p,j}\tilde{V}_j^p\right)\right)$$

$$i = 1, \dots, N. \quad (21)$$

It is easy to see that with this choice of h, the linear system (13) has a unique solution because of

$$\|B_i^p\| \leq 2K_u\varphi.$$

For the complete error analysis, we need the following bounds in the next section which we can conclude them until now:

$$\|B_i^p\| \le 2K_u\varphi,$$

$$\left\| \left(I_m - \frac{h_i}{2} B_i^p \right)^{-1} \right\| \le \frac{1}{1 - \frac{h_i}{2}\|B_i^p\|} \le \frac{1}{1 - hK_u\varphi} = \alpha_h,$$

$$\|B_i^{p,j}\| \le 2K_u\varphi, \quad \|\tilde{B}_i^{p,j}\| \le 4K_u\varphi, \quad \|\tilde{B}_i^p\| \le 4K_u\varphi,$$

$$\|H_i^p\| \le 2(K+K_u\|u\|), \quad \|H_i^{p,j}\| \le 2(K+K_u\|u\|),$$

$$K = \max_\Omega |k(t,s,u)|,$$

$$\|\tilde{H}_i^p\| \le 4(K+K_u\|u\|), \quad \|\tilde{H}_i^{p,j}\| \le 4(K+K_u\|u\|),$$

$$\left\| \left(I_m - \frac{h_i}{2} \tilde{B}_i^p \right)^{-1} \right\| \le \frac{1}{1 - \frac{h_i}{2}\|\tilde{B}_i^p\|} \le \frac{1}{1 - 2hK_u\varphi} = \beta_h,$$

$$\|EH_i^p\| = \|H_i^p - \tilde{H}_i^p\| \le \mathcal{L}_M,$$

$$\|EH_i^{p,j}\| = \|H_i^{p,j} - \tilde{H}_i^{p,j}\| \le \mathcal{L}_M,$$

$$\|EB_i^p\| = \|B_i^p - \tilde{B}_i^p\| \le \mathcal{L}_M,$$

$$\|EB_i^{p,j}\| = \|B_i^{p,j} - \tilde{B}_i^{p,j}\| \le \mathcal{L}_M,$$

$$\mathcal{L}_M = O\left((2M)^{-p-1} \right).$$

(22)

The last error bounds are obtained by applying Equation (19) to the entries of the matrices $H_i^p - \tilde{H}_i^p$, $H_i^{p,j} - \tilde{H}_i^{p,j}$, $B_i^p - \tilde{B}_i^p$, and $B_i^{p,j} - \tilde{B}_i^{p,j}$.

Convergence of the fully discretized solution

To analyze the error of the fully descretized solution, we employ the Cauchy remainder for polynomial interpolation in σ_i

$$v_p(t) = v_{p,i}(z) = \mathcal{P}_{m-1}v_p(t) + \frac{\prod_{k=1}^m (t - t_{i,k})}{m!} v_p^{(m)}(\xi)$$

$$= \sum_{k=1}^m V_{i,k}^p L_k(z) + \frac{2T_m(z)}{m!} \left(\frac{h_i}{4} \right)^m v_p^{(m)}(\xi)$$

$$= \sum_{k=1}^m V_{i,k}^p L_k(z) + R_{m,i}(z;\xi), \quad z \in (-1,1],$$

(23)

where $V_{i,k}^p = v_p(t_{i,k})$. The error function $\epsilon_p(t) = v_p(t) - \tilde{v}_p(t)$ on σ_i satisfies in the equation

$$\epsilon_p(t) = \epsilon_{p,i}(z) = \sum_{k=1}^m \varepsilon_{i,k}^p L_k(z) + R_{m,i}(z;\xi),$$

(24)

with $\varepsilon_{i,k}^p = v_{i,k}^p - \tilde{v}_{i,k}^p$. Then,

$$\|\epsilon_{p,i}(z)\| \le m\varphi\|\varepsilon_i^p\| + \frac{2}{m!} \left(\frac{h}{4} \right)^m \|v_p^{(m)}\|, \quad i=1,\ldots,N, \quad (25)$$

where $\varepsilon_i^p = [\varepsilon_{i,1}^p, \ldots, \varepsilon_{i,m}^p]^T$. Now, we need a bound for the $\|\varepsilon_i^p\|$. Letting Equation (23) in Equation (10) for $t = t_{i,k}$ and applying notation (12), we obtain

$$V_{i,k}^p = v_p(t_{i,k}) = y(t_{i,k}) + \sum_{j=1}^{i-1} \frac{h_j}{2} \left(h_i^{p,j}(k) + \sum_{q=1}^m V_{i,q}^p b_i^{p,j}(k,q) \right)$$

$$+ \frac{h_i}{2} h_i^p(k) + \frac{h_i}{2} \sum_{q=1}^m V_{i,q}^p b_i^p(k,q) + r_{i,k}^p,$$

$$r_{i,k}^p = \sum_{j=1}^{i-1} \frac{h_j}{2} \int_{-1}^1 k_p(t_{i,k}, \delta_j(z)) R_{m,j}(z;\xi) dz$$

$$+ \frac{h_i}{2} \int_{-1}^{z_k} k_p(t_{i,k}, \delta_i(z)) R_{m,i}(z;\xi) dz$$

(26)

for $k = 1, \ldots, m$.
The fully discretized version of this equation is

$$\tilde{V}_{i,k}^p = y(t_{i,k}) + \sum_{j=1}^{i-1} \frac{h_j}{2} \left(\tilde{h}_i^{p,j}(k) + \sum_{q=1}^m \tilde{V}_{i,q}^p \tilde{b}_i^{p,j}(k,q) \right)$$

$$+ \frac{h_i}{2} \tilde{h}_i^p(k) + \frac{h_i}{2} \sum_{q=1}^m \tilde{V}_{i,q}^p \tilde{b}_i^p(k,q).$$

From these two equations, we obtain

$$\varepsilon_{i,k}^p = \sum_{j=1}^{i-1} \frac{h_j}{2} \sum_{q=1}^m \left(v_{j,q}^p b_i^{p,j}(k,q) - \tilde{v}_{j,q}^p \tilde{b}_i^{p,j}(k,q) \right)$$

$$+ \frac{h_i}{2} \sum_{q=1}^m \left(v_{i,q}^p b_i^p(k,q) - \tilde{v}_{i,q}^p \tilde{b}_i^p(k,q) \right)$$

$$+ \sum_{j=1}^{i-1} \frac{h_j}{2} Eh_i^{p,j}(k) + \frac{h_i}{2} Eh_i^p(k) + r_{i,k}^p$$

$$= \sum_{j=1}^{i-1} \frac{h_j}{2} \sum_{q=1}^m b_i^{p,j}(k,q) \varepsilon_{j,q}^p + \sum_{j=1}^{i-1} \frac{h_j}{2} \sum_{q=1}^m \tilde{v}_{j,q}^p Eb_i^{p,j}(k,q)$$

$$+ \frac{h_i}{2} \sum_{q=1}^m b_i^p(k,q) \varepsilon_{i,q}^p + \frac{h_i}{2} \sum_{q=1}^m \tilde{v}_{i,q}^p Eb_i^p(k,q)$$

$$+ \sum_{j=1}^{i-1} \frac{h_j}{2} Eh_i^{p,j}(k) + \frac{h_i}{2} Eh_i^p(k) + r_{i,k}^p,$$

$$k = 1, \ldots, m,$$

where $Eh_i^{p,j}(k) = h_i^{p,j}(k) - \tilde{h}_i^{p,j}(k)$ and $Eh_i^p(k) = h_i^p(k) - \tilde{h}_i^p(k)$. This is equivalent to the linear system

$$\left(I_m - \frac{h_i}{2}B_i^p\right)\varepsilon_i^p = \sum_{j=1}^{i-1}\frac{h_j}{2}\left(EH_i^{p,j} + B_i^{p,j}\varepsilon_j^p + EB_i^{p,j}\tilde{V}_j^p\right) + \frac{h_i}{2}EH_i^p + \frac{h_i}{2}EB_i^p\tilde{V}_i^p + r_i^p,$$

and $r_i^p = [r_{i,1}^p, \ldots, r_{i,m}^p]^T$. With the selection $0 < h < \bar{h}$, this system has the unique solution

$$\varepsilon_i^p = \left(I_m - \frac{h_i}{2}B_i^p\right)^{-1}\left(\sum_{j=1}^{i-1}\frac{h_j}{2}\left(EH_i^{p,j} + B_i^{p,j}\varepsilon_j^p + EB_i^{p,j}\tilde{V}_j^p\right) + \frac{h_i}{2}EH_i^p + \frac{h_i}{2}EB_i^p\tilde{V}_i^p + r_i^p\right).$$

We use this equation and find a bound for the ε_i^p. Taking norm of both sides yields

$$\|\varepsilon_i^p\| \leq \left\|\left(I_m - \frac{h_i}{2}B_i^p\right)^{-1}\right\|$$
$$\times\left(\sum_{j=1}^{i-1}\frac{h_j}{2}\left(\|EH_i^{p,j}\| + \|B_i^{p,j}\|\|\varepsilon_j^p\| + \|EB_i^{p,j}\|\|\tilde{V}_j^p\|\right)\right.$$
$$\left. + \frac{h_i}{2}\|EH_i^p\| + \frac{h_i}{2}\|EB_i^p\|\|\tilde{V}_i^p\| + \|r_i^p\|\right),$$

and employing the bounds (22), we get

$$\|\varepsilon_i^p\| \leq \alpha_h\left(hK_u\varphi\sum_{j=1}^{i-1}\|\varepsilon_j^p\| + \frac{h}{2}\mathcal{L}_M\sum_{j=1}^{i}\|\tilde{V}_j^p\|\right.$$
$$\left. + \frac{ih}{2}\mathcal{L}_M + \|r_i^p\|\right), \tag{27}$$

but from Equation (21), $\|\tilde{V}_i^p\|$ satisfies in the inequality

$$\|\tilde{V}_i^p\| \leq \beta_h\left(\|y\| + \frac{h_i}{2}\|\tilde{H}_i^p\| + \sum_{j=1}^{i-1}\frac{h_j}{2}\left(\|\tilde{H}_i^{p,j}\| + \|\tilde{B}_i^{p,j}\|\|\tilde{V}_j^p\|\right)\right)$$
$$\leq \beta_h\left(\|y\| + ih(K + K_u\|u\|) + 2hK_u\varphi\sum_{j=1}^{i-1}\|\tilde{V}_j^p\|\right)$$
$$\leq \beta_h\|y\| + i\beta_hh(K + K_u\|u\|) + (\beta_h - 1)\sum_{j=1}^{i-1}\|\tilde{V}_j^p\|;$$

then, we can apply lemma (2) to this inequality and after simplification deduce

$$\|\tilde{V}_i^p\| \leq \beta_h\|y\| + 2i\beta_hh(K + K_u\|u\|) + (\beta_h - 1)$$
$$\sum_{j=1}^{i-1}\left(\beta_h\|y\| + 2j\beta_hh(K + K_u\|u\|)\right)\prod_{k=j+1}^{i-1}\beta_h$$
$$= \|y\|\beta_h^i + 2h\frac{\beta_h}{\beta_h - 1}(K + K_u\|u\|)(\beta_h^i - 1).$$

From this relation, we get

$$\sum_{j=1}^{i}\|\tilde{V}_j^p\| \leq \|y\|\frac{\beta_h^i - 1}{2hK_u\varphi} + \frac{K + K_u\|u\|}{K_u\varphi}\left(\frac{\beta_h^i - 1}{2hK_u\varphi} - i\right),$$

and a bound for $\|r_i^p\|$ is obtained using Equation (26) as follows

$$\|r_i^p\| = \max_k|r_{i,k}^p| \leq \frac{h}{2}\left((i-1)\frac{4K_u}{m!}\left(\frac{h}{4}\right)^m\|v_p^{(m)}\|\right.$$
$$\left. + \frac{2(z_m + 1)K_u}{m!}\left(\frac{h}{4}\right)^m\|v_p^{(m)}\|\right)$$
$$\leq \frac{2hK_u}{m!}\left(\frac{h}{4}\right)^m\|v_p^{(m)}\|i = \mathcal{C}_mi.$$

Taking these two results into Equation (27) after simplification and noticing that $hK_u\varphi\alpha_h = \alpha_h - 1$, we have

$$\|\varepsilon_i^p\| \leq \left(\frac{\|y\|}{4K_u\varphi} + \frac{K + K_u\|u\|}{(2K_u\varphi)^2}\right)\alpha_h\left(\beta_h^i - 1\right)\mathcal{L}_M$$
$$+ \alpha_h\left(\mathcal{C}_m + \frac{h}{2}\mathcal{L}_M - \frac{h}{2}\frac{K + K_u\|u\|}{K_u\varphi}\mathcal{L}_M\right)i$$
$$+ (\alpha_h - 1)\sum_{j=1}^{i-1}\|\varepsilon_j^p\|.$$

We utilize lemma (2) again to this inequality and conclude

$$\|\varepsilon_i^p\| \leq \left(\frac{\|y\|}{4K_u\varphi} + \frac{K + K_u\|u\|}{(2K_u\varphi)^2}\right)\alpha_h\left(\beta_h^i - 1\right)\mathcal{L}_M$$
$$+ \alpha_h\left(\mathcal{C}_m + \frac{h}{2}\mathcal{L}_M - \frac{h}{2}\frac{K + K_u\|u\|}{K_u\varphi}\mathcal{L}_M\right)i$$
$$+ (\alpha_h - 1)\alpha_h^{i-1}\left[\left(\frac{\|y\|}{4K_u\varphi} + \frac{K + K_u\|u\|}{(2K_u\varphi)^2}\right)\alpha_h\mathcal{L}_M\right.$$
$$\times\left(\sum_{j=1}^{i-1}\left(\frac{\beta_h}{\alpha_h}\right)^j - \sum_{j=1}^{i-1}\frac{1}{\alpha_h^j}\right)\left(\alpha_h\mathcal{C}_m + \frac{\alpha_hh}{2}\mathcal{L}_M\right.$$
$$\left.\left. - \frac{\alpha_hh}{2}\frac{K + K_u\|u\|}{K_u\varphi}\mathcal{L}_M\right)\sum_{j=1}^{i-1}\frac{j}{\alpha_h^j}\right]. \tag{28}$$

Calculating the sums separately

$$\sum_{j=1}^{i-1} \frac{j}{\alpha_h^j} = \sum_{j=1}^{i-1}\sum_{k=j}^{i-1} \frac{1}{\alpha_h^k} = \frac{1}{\alpha_h - 1}\left(\frac{\alpha_h - \frac{1}{\alpha_h^{i-2}}}{\alpha_h - 1} - \frac{i-1}{\alpha_h^{i-1}}\right),$$

$$\sum_{j=1}^{i-1}\left(\frac{\beta_h}{\alpha_h}\right)^j = \frac{\frac{\beta_h^i}{\alpha_h^{i-1}} - \beta_h}{\beta_h - \alpha_h},$$

$$\sum_{j=1}^{i-1}\frac{1}{\alpha_h^j} = \frac{1 - \frac{1}{\alpha_h^{i-1}}}{\alpha_h - 1}$$

and setting in Equation (19), we get

$$\|\varepsilon_i^p\| \le \left(\frac{\|y\|}{4K_u\varphi} + \frac{K + K_u\|u\|}{(2K_u\varphi)^2}\right)\alpha_h \frac{\beta_h - 1}{\beta_h - \alpha_h}\left(\beta_h^i - \alpha_h^i\right)\mathcal{L}_M$$
$$+ \left(\mathcal{C}_m + \frac{h}{2}\mathcal{L}_M - \left(\frac{h}{2}\right)\frac{K + K_u\|u\|}{K_u\varphi}\mathcal{L}_M\right)\frac{\alpha_h}{\alpha_h - 1}\left(\alpha_h^i - 1\right),$$

and using the following relations

$$\frac{\alpha_h}{\alpha_h - 1} = \frac{1}{hK_u\varphi},$$

$$\alpha_h\frac{\beta_h - 1}{\beta_h - \alpha_h} = 2.$$

By recalling the value of $\mathcal{C}_m$, yields to

$$\|\varepsilon_i^p\| \le \frac{2\left(\alpha_h^i - 1\right)}{\varphi\, m!}\left(\frac{h}{4}\right)^m \|x^{(m)}\| + \frac{\|y\|\left(\beta_h^i - \alpha_h^i\right)}{2K\varphi}\mathcal{L}_M.$$

Setting this bound in Equation (25), we obtain the main result as follows

$$\|\epsilon_{p,i}(z)\| \le \left(m\left(\alpha_h^i - 1\right) + 1\right)\frac{2}{m!}\left(\frac{h}{4}\right)^m \|v_p^{(m)}\|$$
$$+ \frac{m}{2K_u}\left[\left(\|y\| + \frac{K + K_u\|u\|}{K_u\varphi}\right)\left(\beta_h^i - \alpha_h^i\right)\right.$$
$$\left. + \left(1 - \frac{K + K_u\|u\|}{K_u\varphi}\left(\alpha_h^i - 1\right)\right)\right]\mathcal{L}_M.$$

In this error bound, the behavior of the α_h^i and β_h^i when N is increasing must be specified. To do this for simplicity,

we assume that the partition J_h is uniform, that is $h = \frac{T}{N}$. With this selection for large N

$$\alpha_h^i = \frac{1}{\left(1 - \frac{TK_u\varphi}{N}\right)^i} \le \frac{1}{\left(1 - \frac{TK_u\varphi}{N}\right)^N} \sim e^{TK_u\varphi},$$

$$\beta_h^i - \alpha_h^i = \alpha_h^i\left(\left(\frac{\beta_h}{\alpha_h}\right)^i - 1\right) \le \frac{1}{\left(1 - \frac{TK_u\varphi}{N}\right)^N}$$
$$\times \left(\left(1 + \frac{TK_u\varphi}{N - 2TK_u\varphi}\right)^N - 1\right)$$
$$\sim e^{TK_u\varphi}\left(e^{TK_u\varphi} - 1\right),$$

and when N increases, we deduce that the error bound is asymptotically equal to

$$\|\epsilon_p(z)\| \le \left(m\left(e^{TK_u\varphi} - 1\right) + 1\right)\frac{2}{m!}\left(\frac{h}{4}\right)^m \|v_p^{(m)}\| + \frac{m}{2K_u}$$
$$\times \left[\left(\|y\| + \frac{K + K_u\|u\|}{K_u\varphi}\right)e^{TK_u\varphi}\left(e^{TK_u\varphi} - 1\right)\right.$$
$$\left. + \left(1 - \frac{K + K_u\|u\|}{K_u\varphi}\left(e^{TK_u\varphi} - 1\right)\right)\right]\mathcal{L}_M.$$

After deriving fully discretized solution $\tilde{v}_p(t)$, an approximation to the solution $v_p(t)$ of linear integral equations (6), we can write general error bound as follows:

$$\|u(t) - \tilde{v}_p(t)\| \le \|u(t) - v_p(t)\| + \|v_p(t) - \tilde{v}_p(t)\|$$
$$\le \|u(t) - v_p(t)\| + \left(m\left(e^{TK_u\varphi} - 1\right) + 1\right)$$
$$\times \frac{2}{m!}\left(\frac{h}{4}\right)^m \|v_p^{(m)}\| + \frac{m}{2K_u}$$
$$\times \left[\left(\|y\| + \frac{K + K_u\|u\|}{K_u\varphi}\right)e^{TK_u\varphi}\left(e^{TK_u\varphi} - 1\right)\right.$$
$$\left. + \left(1 - \frac{K + K_u\|u\|}{K_u\varphi}\left(e^{TK_u\varphi} - 1\right)\right)\right]\mathcal{L}_M.$$

The first part of this general error bound is due to the quasilinearization, and it has been proven in Theorem (1)

Table 1 Absolute errors for Example 1: $N = 4$, $m = 3$ **and** $M = 4$

| t_i	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$
-1.00	8.33 E - 03	2.83 E - 04	4.02 E - 07	3.31 E - 10	6.14 E - 16
-0.75	2.21 E - 01	6.48 E - 02	3.84 E - 05	3.27 E - 09	3.33 E - 15
-0.50	3.51 E - 01	9.25 E - 02	3.24 E - 05	2.38 E - 08	2.22 E - 15
-0.25	4.25 E - 01	1.27 E - 01	6.13 E - 04	4.53 E - 08	1.11 E - 15
0.00	4.86 E - 01	1.66 E - 01	4.78 E - 04	2.18 E - 07	2.61 E - 14
0.25	5.41 E - 01	1.89 E - 01	7.45 E - 03	1.26 E - 07	1.11 E - 14
0.50	5.37 E - 01	1.69 E - 01	6.36 E - 03	1.14 E - 07	1.11 E - 14
0.75	5.54 E - 01	1.77 E - 01	3.20 E - 03	2.74 E - 06	4.44 E - 13
1.00	1.07 E - 00	1.75 E - 01	1.66 E - 02	1.40 E - 06	1.92 E - 13
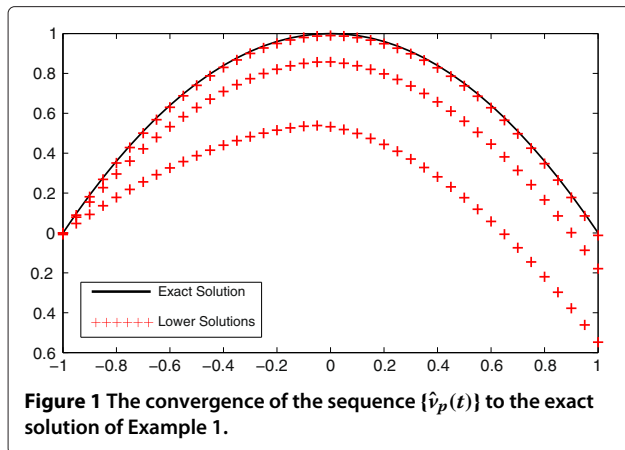
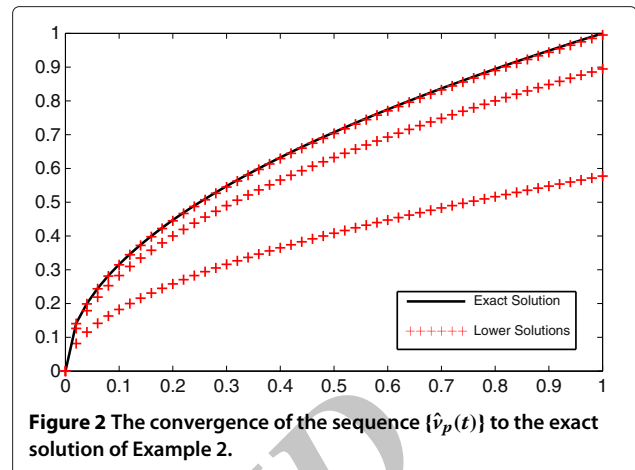**Figure 1** The convergence of the sequence $\{\hat{v}_p(t)\}$ to the exact solution of Example 1.

Figure 2 The convergence of the sequence $\{\hat{v}_p(t)\}$ to the exact solution of Example 2.

that is quadratically convergent. The second term is due to collocation which has a convergence order $O(h^m)$, and the third part of the above error is due to the quadrature formula which is convergent with the order $O\left((2M)^{-p-1}\right)$.

Numerical examples

In what follows, the method presented in this paper is applied to solve three numerical examples of the integral equation (3).

Example 1. The following integral equation is discussed in [19]:

$$u(t) = \frac{1}{15}\left(-2t^6 + 5t^4 - 15t^2 + 8t + 20\right) + \int_{-1}^{t}(2s - t)\,u^2(s)\,ds, \qquad (29)$$

where $-1 \leq t \leq 1$, and $k(t, s, u(s)) = (t - 2s)u^2(s)$ is convex and nondecreasing with respect to u for $u \geq 0$.

Table 2 Absolute errors for Examples 2 and 3: $N = 4$

t_i	$m = 3$ and $M = 5$		$m = 5$ and $M = 5$	
	$p = 3$	$p = 4$	$p = 4$	$p = 5$
0.0	0.00	0.00	0.00	0.00
0.1	1.33 E - 05	1.2416 E - 08	8.0286 E - 08	1.3721 E - 11
0.2	1.89 E - 05	3.4211 E - 08	6.0680 E - 07	1.8541 E - 10
0.3	2.31 E - 04	3.5308 E - 07	1.9592 E - 06	1.0478 E - 10
0.4	2.67 E - 04	2.3369 E - 06	4.4503 E - 06	1.0371 E - 09
0.5	2.98 E - 04	5.3270 E - 06	8.3357 E - 05	5.9125 E - 09
0.6	3.27 E - 03	1.2519 E - 05	1.3822 E - 04	9.3062 E - 08
0.7	3.53 E - 03	2.4433 E - 05	2.1077 E - 04	1.0544 E - 08
0.8	3.78 E - 03	2.5173 E - 05	3.0230 E - 03	1.5937 E - 08
0.9	4.01 E - 03	9.9403 E - 04	4.1387 E - 03	6.2534 E - 07
1.0	4.22 E - 03	2.5673 E - 04	5.4624 E - 03	1.5917 E - 07

The exact solution is $u(t) = 1 - t^2$, and $v_0(t) = 0$ is a lower solution. The iterative scheme is

$$v_p(t) = \left(\frac{1}{15}\left(-2t^6 + 5t^4 - 15t^2 + 8t + 20\right) - \int_{-1}^{t}(2s - t)\left[v_{p-1}(s)\right]^2 ds\right) + 2\int_{-1}^{t}(2s - t)v_{p-1}(s)v_p(s)ds.$$

We employed the presented method for $N = 4$, $m = 3$, and $M = 4$. We obtained the absolute values of the errors for Equation (29), which are shown in Table 1 and Figure 1; these results verify the convergence of the sequence $\{\hat{v}_p(t)\}$ to the exact solution.

Example 2. The second example,

$$u(t) = \sqrt{t}\,e^{(-\sqrt{t})} + \frac{1}{2}\int_{0}^{t}\sqrt{\frac{t}{s}}\,e^{(u(s) - \sqrt{t})}ds,$$
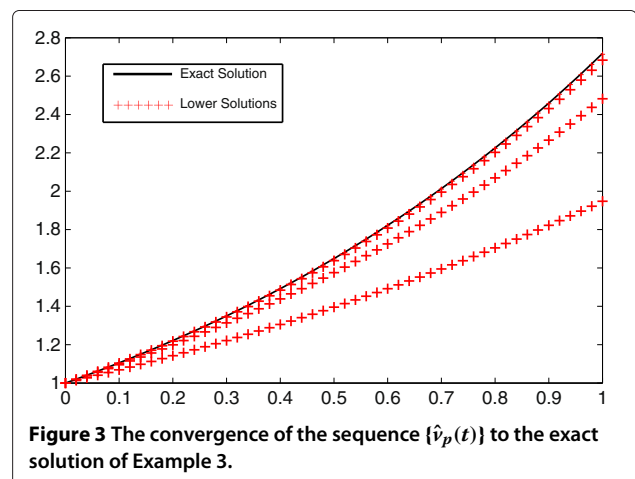

Figure 3 The convergence of the sequence $\{\hat{v}_p(t)\}$ to the exact solution of Example 3.

has the exact solution $u(t) = \sqrt{t}$ and the lower solution $v_0(t) = \sqrt{\frac{t}{2}}$. The absolute values of the errors are presented in Table 2 and Figure 2.

Example 3. The third example, is the following integral equation:

$$u(t) = e^t \sin(e^t) - \int_0^t e^{s+t} \cos(u(s)) ds, \quad 0 \le t \le 0.74.$$

This equation has the exact solution $u(t) = e^t$ and the lower solution $v_0(t) = 1 + t$. Table 2 shows the absolute values of the errors, and the convergence of the sequence $\{\hat{v}_p(t)\}$ to the exact solution is shown in Figure 3.

Conclusions

In this paper, we computed an error bound for the numerical solution of nonlinear Volterra integral equations that contain the quasilinearization, collocation, and the numerical quadrature errors separately and showed the effects of the collocation approximation and numerical quadratures on the accuracy of the approximated solution. The provided numerical examples confirm our results.

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