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Effective approximate methods for strongly nonlinear differential equations with oscillations

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Abstract

Purpose: This paper proposes the use of different analytical methods in obtaining approximate solutions for nonlinear differential equations with oscillations.

Methods: Three methods are considered in this paper: Lindstedt-Poincare method, the Krylov-Bogoliubov first approximate method, and the differential transform method.

Results: Figures that are given in this paper give a strong evidence that the proposed methods are effective in handling nonlinear differential equations with oscillations.

Conclusions: This study reveals that the differential transform method provides a remarkable precision compared with other perturbation methods.

Keywords: Lindstedt-Poincare method, Krylov-Bogoliubov method, Differential transform method, Nonlinear oscillations

Introduction

When a mathematical model is formulated for a physical problem, it is often represented by equations of a one-mass system with two degrees of freedom that are mostly described using a second-order differential equation, and these equations are not solvable exactly by analytic techniques. If in the differential equation, some small nonlinearities exist, they are introduced in the differential equation of motion as small nonlinear terms, and so the motion of the system is described by a second-order strongly nonlinear differential equation in the form of

$$y''(t) + y(t) + \epsilon f(y(t), y'(t)) = 0, \ \ y(0) = 1, \ \ y'(0) = 0, \tag{1.1}$$

where ϵ is a sufficiently small parameter, so the nonlinear term $\epsilon f(y(t), y'(t))$ is relatively small. As mentioned in [1], these terms arise because the physical process has small effects. For example, in a fluid flow problem the viscosity may be small, or in the problem of the motion of a projectile, the force due to air resistance may be small. These low order effects are represented by terms in the model equation that, when compared to the other terms, are small and represented by a coefficient parameter ϵ. Therefore, we must resort to approximation and numerical methods. Foremost, among approximation techniques are the so-called perturbation methods. Perturbation method [2-5] provides the most versatile tools available in nonlinear analysis of physical problems. These methods have their own limitations; for example, the solutions are valid, in most cases, only for small values of the parameter ϵ. Therefore, we cannot rely fully on the approximations, since there is no criterion on how small the parameters should be. In [6], the authors proved the existence of an approximate solution in the mean for general strongly nonlinear differential equations. They investigate the behavior of the class of solutions. In addition, they implement the homotopy perturbation method and find analytic solutions for strongly nonlinear differential equations of the form

$$x''(t) + ax'^2(t) + bx(t) = F(t) \tag{1.2}$$

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for some constants a, b. As mentioned in [6], numerous methods are developed for analytic solution of strongly nonlinear differential equations describing the vibrations of the oscillator. For example, the harmonic balance method [7-9] which leads to algebraic equations has been used. An analytical approximate approach for determining periodic solutions of nonlinear jerk equations involving third-order time-derivative is presented in [10,11]. In [12], the authors used the Adomian decomposition method to find the approximate solutions for general form of the second-order ordinary differential equations. In [13], a modified variational approach called global error minimization method is developed for obtaining an approximate closed-form analytical solution for nonlinear oscillator differential equations. In this paper, we propose the use of Lindstedt-Poincare method, the Krylov-Bogoliubov first approximate method, and the differential transform method (DTM) in obtaining the approximate solutions for nonlinear differential equations of the form in Equation 1.1. In order to illustrate the efficiency of these methods, we examine three numerical examples. Comparison is made based on the obtained results. The paper is concluded in the last section. In what follows, we survey briefly the DTM steps while we use directly the other two methods within the text.

Methods

The goal of this section is to recall notations, definitions, and some theorems of the DTM that will be used in this paper. These are discussed in [12,14,15]. Also, we will highlight the main steps of implementing the DTM in solving differential equations. The other two methods (Lindstedt-Poincare and the Krylov-Bogoliubov) are given within the section of Results and discussion.

Differential transform method

The differential transform of the *kth* derivative of a function $u(x)$ is defined to be

$$U(k) = \frac{1}{k!}\left(\frac{d^k}{dx^k}u(x)\right)\Big|_{x=x_0} \tag{2.1}$$

and the inverse transform of $U(k)$ is defined as

$$u(x) = \sum_{k=0}^{\infty} U(k)(x - x_0)^k \tag{2.2}$$

Equation 2.2 is known as the Taylor series expansion of $u(x)$ around $x = x_0$. Also, we need the following theorems. If $G(k)$ is the differential transform of $g(x)$ then the following theorems apply.

Theorem 1. *If* $f(x)=\frac{d^n g(x)}{dx^n}$*, then* $F(k)=\frac{(k+n)!}{k!}G(k+n)$.

Theorem 2. *If* $f(x)=g^2(x)$*, then* $F(k)=\sum_{i=0}^{k}G(i)G(k-i)$.

Theorem 3. *If* $f(x)= g^3(x)$*, then*
$$F(k)=\sum_{i=0}^{k}\sum_{j=0}^{k-i}G(i)G(j)G(k-i-j).$$

Results and discussion

This section is divided into two parts: weakly and strongly nonlinear oscillations.

Application I: weakly nonlinear oscillation

An example of a weak nonlinear oscillation is the Duffing equation. It is a nonlinear second-order differential equation used to model certain damped and driven oscillators. The equation is given by

$$y''(t) + \delta y'(t) + \alpha y(t) + \beta y^3(t) = 0 \tag{3.1}$$

where the (unknown) function $y = y(t)$ is the displacement at time t; y' is the first derivative of y with respect to time, i.e., velocity; and y'' is the second time-derivative of y, i.e., acceleration. The number δ controls the size of the damping (friction), α controls the size of the restoring force, and β controls the amount of nonlinearity in the restoring. The equation describes the motion of a damped oscillator with a more complicated potential than in simple harmonic motion (which corresponds to the case $\beta = \delta = 0$); in physical terms, it models, for example, a spring pendulum whose spring's stiffness does not exactly obey Hooke's law. The Duffing equation is an example of a dynamic system that exhibits chaotic behavior.

In this section, we are concerned about

$$y''(t) + y(t) + \epsilon y^3(t) = 0,\ \ y(0) = 1,\ \ y'(0) = 0. \tag{3.2}$$

First, by means of Lindstedt-Poincare method [16-18] we use the transformation $\tau = \omega t$ which transforms Equation 3.2 into

$$\omega^2 u''(\tau) + u(\tau) + \epsilon u^3(\tau) = 0, \tag{3.3}$$

where $u(\tau) = y(t)$. Let

$$\begin{aligned} \omega &= 1 + \epsilon\omega_1, \\ u &= u_0 + \epsilon u_1. \end{aligned} \tag{3.4}$$

If we substitute Equation 3.4 in Equation 3.3 and equate the coefficients of various powers of ϵ to 0, then we obtain

$$\begin{aligned} &O(1) : u_0'' + u_0 = 0;\ \ u_0(0) = 1,\ \ u_0'(0) = 0, \\ &O(\epsilon) : 2\omega_1 u_0'' + u_1'' + u_1 + u_0^3 = 0;\ \ u_1(0) = 0,\ \ u_1'(0) = 0. \end{aligned} \tag{3.5}$$

Solving the equations in Equation 3.5 and avoiding the occurrence of secular terms in the perturbation solutions yield

$$u_0(\tau) = \cos(\tau),$$
$$u_1(\tau) = \frac{1}{32}\left(\cos(\tau) - \cos(3\tau)\right). \quad (3.6)$$

Therefore, the solution of Equation 3.2 is

$$y(t) = \cos(\left(1+\frac{3\epsilon}{8}\right)t) + \frac{\epsilon}{32}\left(\cos(\left(1+\frac{3\epsilon}{8}\right)t) - \cos(3\left(1+\frac{3\epsilon}{8}\right)t)\right). \quad (3.7)$$

Second, by applying the differential transform to Equation 3.2, we get the recursive formula

$$F(k+2) = -\frac{F(k)}{(k+1)(k+2)} - \frac{\epsilon}{(k+1)(k+2)}\sum_{i=0}^{k}\sum_{j=0}^{k-i} F(i)F(j)F(k-i-j), \quad (3.8)$$

where $F(k)$ is the differential transform of $y(t)$. By the given initial conditions, we have $F(0)=1$ and $F(1) = 0$. Considering $N = 8$ in the DTM series solution, the approximate solution is

$$y_{\text{DTM}}(t) = \sum_{k=0}^{8} F(k)t^k = 1 - \frac{1}{2}(1+\epsilon)t^2 + \frac{1}{24}(1+\epsilon)(1-3\epsilon)t^4 - \ldots \quad (3.9)$$

Third, by means of Krylov-Bogoliubov first approximate method [19], the solution of

$$y''(t) + y(t) + \epsilon f(y(t), y'(t)) = 0 \quad (3.10)$$

is sought as

$$y(t) = a(t)\sin(t + \phi(t)), \quad (3.11)$$

where $a(t)$ and $\phi(t)$ are given by

$$a(t) = \frac{\epsilon}{2\pi}\int_0^{2\pi} f(c\sin(\theta), c\cos(\theta))\cos(\theta)d\theta,$$
$$\phi(t) = \frac{\epsilon}{2\pi c}\int_0^{2\pi} f(c\sin(\theta), c\cos(\theta))\sin(\theta)d\theta, \quad (3.12)$$

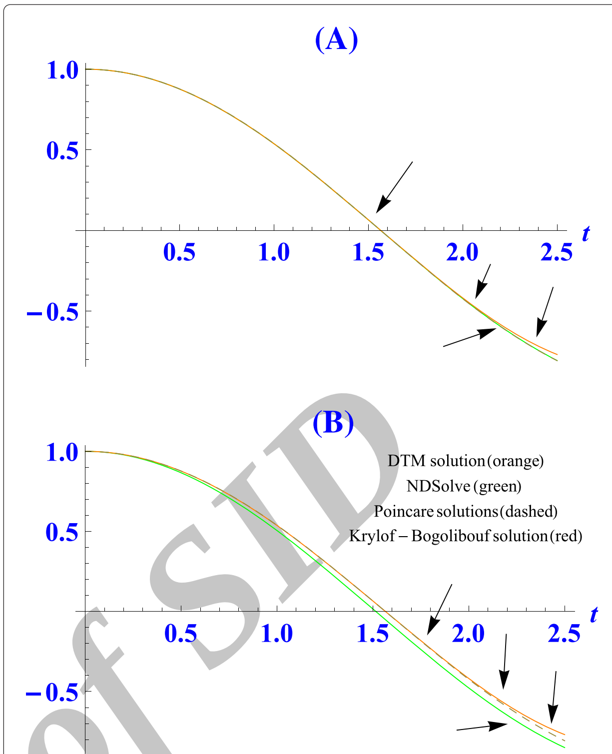

Figure 1 The obtained solutions of the Duffing equation for (A) ϵ=0.01 and (B) ϵ=0.1.

and where c is a free constant. Therefore, according to Equation 3.2, the solution is

$$y(t) = A\sin(\left(\frac{3\epsilon A^2}{8}+1\right)t + B), \quad (3.13)$$

where A and B are constants. Applying the initial conditions given in Equation 3.2 yields $A = -1$ and $B = -\frac{\pi}{2}$.

Figure 1 shows the plots of the obtained solutions using the proposed methods compared with Mathematica NDSolve tool.

Application II: strongly nonlinear oscillation

In this section, we apply the Lindstedt-Poincare method and the differential transform method on two strongly nonlinear oscillations.

Example 1. Consider the nonlinear oscillation

$$y''(t) + y(t) = \epsilon y(t)(1 - y'^2(t)), \;\; y(0) = 1, \;\; y'(0) = 0. \quad (4.1)$$

First, by means of Lindstedt-Poincare method, the transformation $\tau = \omega t$ converts Equation 4.1 to

$$\omega^2 u''(\tau) + u(\tau) - \epsilon u(\tau) + \epsilon\omega^2 u(\tau)u'(\tau))^2 = 0, \quad (4.2)$$

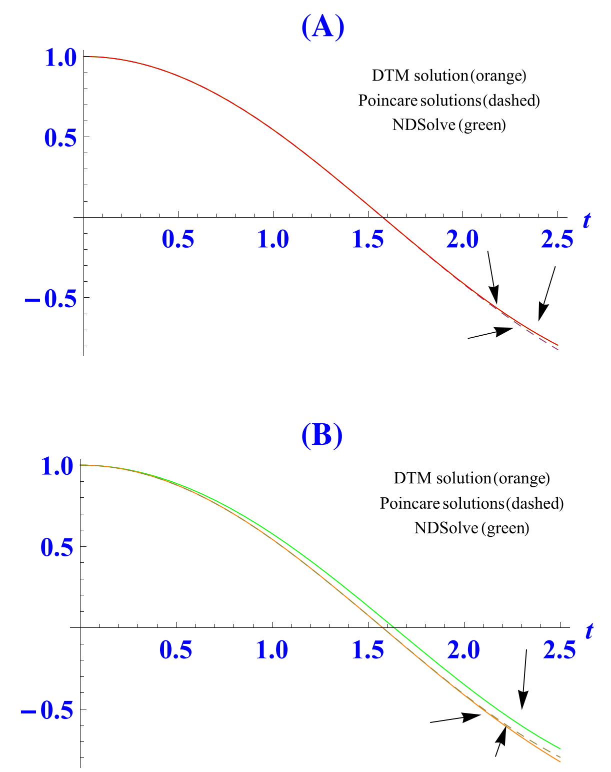

Figure 2 The obtained solutions of Equation 4.1 for (A) ϵ=0.01 and (B) ϵ=0.1.

where $u(\tau) = y(t)$. Let

$$\omega = 1 + \epsilon\omega_1, \quad u = u_0 + \epsilon u_1. \tag{4.3}$$

Through inserting Equation 4.3 in Equation 4.2 and equating the coefficients of various powers of ϵ to 0, we yield

$$O(1) : u_0'' + u_0 = 0; \quad u_0(0) = 1, \quad u_0'(0) = 0,$$
$$O(\epsilon) : 2\omega_1 u_0'' + u_1'' + u_1 - u_0 + u_0(u_0')^2 = 0; \quad u_1(0) = 0, \quad u_1'(0) = 0. \tag{4.4}$$

Solving the equations in Equation 4.4 and avoiding the occurrence of secular terms in the perturbation solutions yield

$$u_0(\tau) = \cos(\tau), \quad u_1(\tau) = -\frac{1}{32}\left(\cos(\tau) - \cos(3\tau)\right). \tag{4.5}$$

Therefore, the solution of Equation 4.1 is

$$y(t) = \cos(\left(1 - \frac{3\epsilon}{8}\right)t) - \frac{\epsilon}{32}\left(\cos(\left(1 + \frac{3\epsilon}{8}\right)t) - \cos(3\left(1 + \frac{3\epsilon}{8}\right)t)\right). \tag{4.6}$$

Second, by applying the differential transform to Equation 4.1, we get the recursive formula

$$F(k+2) = \frac{(\epsilon - 1)F(k)}{(k+1)(k+2)} - \frac{\epsilon}{(k+1)(k+2)} \times \sum_{i=0}^{k}\sum_{j=0}^{k-i}(i+1)(j+1)F(i)F(j)F(k-i-j), \tag{4.7}$$

where $F(k)$ is the differential transform of $y(t)$. By the given initial conditions, we have $F(0) = 1$ and $F(1) = 0$. Considering $N = 8$ in the DTM series solution, the approximate solution is

$$y_{\mathrm{DTM}}(t) = \sum_{k=0}^{8} F(k)t^k = 1 - \frac{1}{2}(1-\epsilon)t^2 + \frac{1}{24}(1-\epsilon)^2(1-2\epsilon)t^4 - \ldots \tag{4.8}$$

The solutions obtained by the proposed methods show an excellent agreement with the one obtained by Mathematica NDSolve tool if we choose ϵ to be small (see Figure 2).

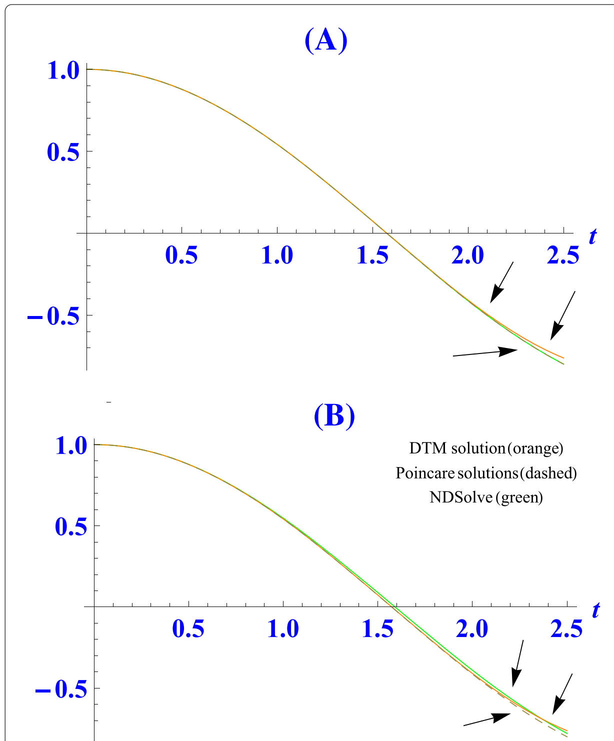

Figure 3 Obtained solutions of Equation 4.9 for (A) ϵ=0.01 and (B) ϵ=0.1.

Example 2. Consider the nonlinear oscillation

$$y''(t) + y(t) = \epsilon y(t) y'^2(t), \;\; y(0) = 1, \;\; y'(0) = 0. \quad (4.9)$$

First, by means of Lindstedt-Poincare method, the transformation $\tau = \omega t$ converts Equation 4.9 to

$$\omega^2 u''(\tau) + u(\tau) - \epsilon \omega^2 u(\tau) u'(\tau))^2 = 0, \quad (4.10)$$

where $u(\tau) = y(t)$. Let

$$\begin{aligned} \omega &= 1 + \epsilon \omega_1, \\ u &= u_0 + \epsilon u_1. \end{aligned} \quad (4.11)$$

Inserting Equation 4.11 in Equation 4.10 and equating the coefficients of various powers of ϵ to 0 yield

$$\begin{aligned} &O(1) : u_0'' + u_0 = 0; \;\; u_0(0) = 1, \;\; u_0'(0) = 0, \\ &O(\epsilon) : 2\omega_1 u_0'' + u_1'' + u_1 - u_0 (u_0')^2 = 0; \;\; u_1(0) = 0, \\ &\qquad u_1'(0) = 0. \end{aligned} \quad (4.12)$$

Moreover, solving the equations in Equation 4.12 and avoiding the occurrence of secular terms in the perturbation solutions yield

$$\begin{aligned} u_0(\tau) &= \cos(\tau), \\ u_1(\tau) &= -\frac{1}{32} \left(\cos(\tau) - \cos(3\tau)\right). \end{aligned} \quad (4.13)$$

Therefore, the solution of Equation 4.9 is

$$\begin{aligned} y(t) = &\cos(\left(1 - \frac{\epsilon}{8}\right) t) - \frac{\epsilon}{32} \left(\cos(\left(1 - \frac{\epsilon}{8}\right) t) \right. \\ &\left. - \cos(3 \left(1 - \frac{\epsilon}{8}\right) t) \right). \end{aligned} \quad (4.14)$$

Second, by applying the differential transform to Equation 4.9, we get the recursive formula

$$\begin{aligned} F(k+2) = &- \frac{F(k)}{(k+1)(k+2)} + \frac{\epsilon}{(k+1)(k+2)} \\ &\times \sum_{i=0}^{k} \sum_{j=0}^{k-i} (i+1)(j+1) F(i) F(j) F(k-i-j), \end{aligned} \quad (4.15)$$

where $F(k)$ is the differential transform of $y(t)$. By the given initial conditions, we have $F(0) = 1$ and $F(1) = 0$. Considering $N = 8$ in the DTM series solution, the approximate solution is

$$\begin{aligned} y_{\rm DTM}(t) &= \sum_{k=0}^{8} F(k) t^k \\ &= 1 - \frac{1}{2} t^2 + \frac{1}{24} (1 + 2\epsilon) t^4 - ... \end{aligned} \quad (4.16)$$

Figure 3 shows the plots of the obtained solutions using the proposed methods compared with Mathematica NDSolve tool.

Conclusions

In this paper, the nonlinear differential equations with oscillator are considered to study the validity of DTM compared with Lindstedt-Poincare method. The most significant features of DTM are its simplicity and its excellent accuracy for different values of the parameter ϵ. DTM is very effective and convenient for solving truly nonlinear oscillator equations.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

MA and KA contributed equally to this work. Both authors read and approved the final manuscript.

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