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Some estimates for multilinear commutators on the weighted Morrey spaces

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Abstract

Purpose: The purpose of this paper is to study the multilinear commutator in weighted Morrey space.

Methods: We divide the multilinear commutator into four parts and apply with the methods of weighted Lebesgue space and Pérez and Trujillo-González.

Results: Let $\vec{b} = (b_1, \cdots, b_m)$, $b_j \in \text{BMO}(R^n)$, $1 \leq j \leq m$; we show that $T_{\vec{b}}$ is bounded on weighted Morrey spaces $L^{p,\kappa}(w)$, $1 < p < \infty$, $0 < \kappa < 1$, $w \in A_p$. When $p = 1$, it is shown that $T_{\vec{b}}$ is weak type bounded on weighted Morrey space.

Conclusions: We extend the results of the commutator in weighted Morrey space. In particular, for $m = 1$, this result is also new in the classical commutator.

Keywords: Multilinear commutators, Weighted Morrey spaces, bounded mean oscillation, Orlicz maximal operator

Mathematics Subject Classification 2000: 42B20; 42B35

Introduction

Let $\vec{b} = (b_1, \cdots, b_m)$, b_j, and $1 \leq j \leq m$ be locally integrable functions when we consider multilinear commutators as defined by

$$T_{\vec{b}}f(x) = \int_{R^n} \prod_{j=1}^{m}(b_j(x) - b_j(y))K(x,y)f(y)dy,$$

where $K(x,y)$ is Calderón-Zygmund kernel. That is, for all distinct $x, y \in R^n$, and all z with $2|x-z| < |x-y|$, there exist positive constant C and γ such that

i) $|K(x,y)| \leq C|x-y|^{-n}$;
ii) $|K(x,y) - K(z,y)| \leq C\frac{|x-z|^{\gamma}}{|x-y|^{n+\gamma}}$; and
iii) $|K(y,x) - K(y,z)| \leq C\frac{|x-z|^{\gamma}}{|x-y|^{n+\gamma}}$

when $m = 1$, it is the classical commutator which was introduced by Coifman et al. in [1]. The commutators are useful in many nondivergence elliptic equations with discontinuous coefficients [2-4]. In the recent development of commutators, Pérez and Trujillo-González [5] generalized these multilinear commutators and proved the weighted Lebesgue estimates. The weighted Morrey spaces $L^{p,\kappa}(w)$ was introduced by Komori and Shirai [6]. Moreover, they showed that some classical integral operators and corresponding commutators are bounded in weighted Morrey spaces. Recently, Wang [7-9] obtained that some other kind of integral operators (e.g., Bochner-Riesz operator and Marcinkiewicz operators) and commutators are also bounded in weighted Morrey spaces. In his work, He [10] showed that multilinear operators are bounded on weighted Morrey spaces with the symbol of $b \in Lip(\beta)$. The main purpose of this paper is to discuss the boundedness of the multilinear commutators in weighted Morrey spaces $L^{p,\kappa}(w)$ for $1 < p < \infty$ and $0 < \kappa < 1$, where the symbol b belongs to bounded mean oscillation (BMO). Furthermore, we shall give the weighted weak type estimate of these operators in weighted Morrey spaces of $L^{p,\kappa}(w)$ for $p = 1$ and $0 < \kappa < 1$, which is also a new result even with $m = 1$. Our main results are stated as follows.

Theorem 1.1 *Let $1 < p < \infty, 0 < \kappa < 1$, if $b_j \in BMO(R^n)$, $1 \leq j \leq m, w \in A_p$, then there exists a constant $C > 0$ such that*

$$\|T_{\vec{b}}f\|_{L^{p,\kappa}(w)} \leq C\|\vec{b}\|\|f\|_{L^{p,\kappa}(w)},$$

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where $\|\vec{b}\| = \prod_{j=1}^{m} \|b_j\|_*$ *and* $\|.\|_*$ *is the norm of* $BMO(R^n)$.

Theorem 1.2 *Let* $0 < \kappa < 1$*, if* $b_j \in BMO(R^n)$*,* $1 \le j \le m$*,* $w \in A_1$*, then for any* $\lambda > 0$ *and cube Q, there exists a constant* $C > 0$ *such that*

$$\lambda w(x \in Q : |T_{\vec{b}} f(x)| > \lambda) \le C\|\vec{b}\|\|f\|_{L^{\Phi,\kappa}(w)},$$

where $\Phi(t) = t log^m(e + t)$ *and* $\|f\|_{L^{\Phi,\kappa}(w)} = \|\Phi(|f|)\|_{L^{1,\kappa}(w)}$.

In the section that follows, we denote by C positive constants which are independent of the main parameters, but it may vary from line to line.

Methods

In this section, we introduce the basic definitions and lemmas needed for the proof of the main results.

Definition 2.1. Let $1 < p < \infty$, for any locally integrable function w, if

$$(\frac{1}{|Q|}\int_Q w(x)dx)(\frac{1}{|Q|}\int_Q w(x)^{\frac{1}{1-p}}dx)^{p-1} < \infty$$

holds, then w belongs to the Muckenhoupt class A_p. We denote $A_\infty = \bigcup_{1<p<\infty} A_p$.

when $p = 1$, $w \in A_1$, if there exists $C > 1$ such that

$$Mw(x) \le Cw(x),$$

for almost every $x \in R^n$.

Remark 2.2. Given a weight function $w \in A_p$, $1 \le p \le \infty$, it also satisfies the doubling condition in Δ_2: for any cube Q, there exists a constant $C > 0$ such that $w(2Q) \le Cw(Q)$.

In fact, if $w \in \Delta_2$, we have the following inequality

Lemma A. [11] *Suppose* $w \in \Delta_2$*, then there exists a constant* $D > 1$ *such that*

$$w(2Q) \ge Dw(Q),$$

for any cube Q.

Definition 2.3. A locally integrable function b is said to be in BMO(R^n) if

$$\|b\|_* = \sup_Q \frac{1}{|Q|}\int_Q |b(x) - b_Q|dx < \infty,$$

where $b_Q = \frac{1}{|Q|}\int_Q b(x)dx$ and the supremum is taken over all cubes Q in R^n .

Lemma B. [12] *Suppose* $w \in A_\infty$*, then the norm of* $BMO(w)$ *is equivalent to the norm of* $BMO(R^n)$*, where*

$$BMO(w) = \{b : \|b\|_{*,w} = \sup_Q \frac{1}{w(Q)}\int_Q |b(x) - b_{Q,w}|w(x) \times dx < \infty\},$$

where $b_{Q,w} = \frac{1}{w(Q)}\int_Q b(x)w(x)dx$.

Lemma C. [13,14] *Let cube* $Q = Q(x_0, r)$ *centered at* x_0 *with side length of r, for any positive integer i,* 2^iQ *denote the cube centered at* x_0 *with side length of* 2^ir*, then we have the inequality*

$$|b_{2^iQ} - b_Q| \le Ci\|b\|_*.$$

where $b_{Q,w} = \frac{1}{|Q|}\int_Q b(x)w(x)dx$.

Definition 2.4. Let $\Phi(t) = t\log^m(t + e)$, and the Orlicz maximal operator M_Φ is given by

$$M_\Phi f(x) = \sup_{x\in Q}\|f\|_{\Phi,Q} = \sup_{x\in Q}\frac{1}{|Q|}\int_Q \Phi(|f|)(x)dx.$$

From above definition, observe that $Mf(x) \le M_\Phi f(x) \le M(\Phi(|f|))(x)$; this inequality will be relevant in our work.

Aside from the properties of A_p, weight function, and BMO function, we need some estimates of multilinear commutators. The following results are proved by Pérez and Trujillo-González [5].

Theorem D *Let* $1 < p < \infty$ *and* $w \in A_p$ *and suppose that* $b_j \in BMO(R^n)$ *and* $1 \le j \le m$*, then there exists a constant* $C > 0$ *such that*

$$\int_{R^n} |T_{\vec{b}}(f)|^p w(x)dx \le C\|\vec{b}\|^p \int_{R^n} |f|^p w(x)dx.$$

Although the commutators with BMO function are not of weak type (1, 1), they have the following inequality.

Lemma E. *Let* $w \in A_\infty$ *if there exists a constant* $C > 0$ *such that*

$$\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})} w(x \in R^n : |T_{\vec{b}}(f)(x)| > t) \le C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})} w(x \in R^n : M_\Phi(\|\vec{b}\|f)(x) > t)$$

where $\Phi(t) = t log^m(e + t)$.

By the above inequality, it has the following result.

Lemma F. *Let $w \in A_1$ if there exists a constant $C > 0$ such that for all $\lambda > 0$*

$$w(x \in R^n : |T_{\vec{b}}(f)(x)| > \lambda) \leq C \int_{R^n} \Phi(|f|)(x)w(x)dx.$$

where $\Phi(t) = tlog^m(e + t)$.

Results and discussion

In this section, we shall use the definition of a weighted Morrey space. Our method is similar to the method of Pérez and Trujillo-González [5].

Proof of Theorem 1.1 Let $1 < p < \infty$ and $0 < \kappa < 1$; for any cube Q, it only need to obtain the inequality

$$\int_Q |T_{\vec{b}}(f)(x)|^p w(x)dx \leq C\|\vec{b}\|^p w(Q)^\kappa \|f\|^p_{L^{p,\kappa}(w)}.$$

Fix the above cube $Q = Q(x_0, r)$ and decompose $f = f_1 + f_2$, where $f_1 = f\chi_{2Q}$, χ_{2Q} denotes the characteristic function of $2Q$; since $T_{\vec{b}}$ is a linear operator, we have

$$\begin{aligned}\int_Q |T_{\vec{b}}(f)(x)|^p w(x)dx &\leq C\{\int_Q |T_{\vec{b}}(f_1)(x)|^p w(x)dx \\ &\quad + \int_Q |T_{\vec{b}}(f_2)(x)|^p w(x)dx\} \\ &:= C\{I + II\}. \end{aligned} \tag{3.1}$$

It is easy to estimate the term I. Using Theorem D, we get

$$\begin{aligned} I &\leq \int_{R^n} |T_{\vec{b}}(f)(x)|^p w(x)dx \\ &\leq C\|\vec{b}\|^p \int_{R^n} |f_1(x)|^p w(x)dx \\ &\leq C\|\vec{b}\|^p w(Q)^\kappa \|f\|^p_{L^{p,\kappa}(w)}. \end{aligned} \tag{3.2}$$

For the term II, without loss of generality, we can assume $m = 2$. Thus, the operator $T_{\vec{b}}$ can be divided into four parts

$$\begin{aligned} T_{\vec{b}}f_2(x) &= (b_1(x) - \lambda_1)(b_2(x) - \lambda_2) \int_{R^n} K(x,y)f_2(y)dy \\ &\quad + \int_{R^n} K(x,y)(b_1(y) - \lambda_1)(b_2(y) - \lambda_2)f_2(y)dy \\ &\quad - (b_1(x) - \lambda_1) \int_{R^n} K(x,y)(b_2(y) - \lambda_2)f_2(y)dy \\ &\quad - (b_2(x) - \lambda_2) \int_{R^n} K(x,y)(b_1(y) - \lambda_1)f_2(y)dy \\ &= II_1(x) + II_2(x) + II_3(x) + II_4(x), \end{aligned} \tag{3.3}$$

where $\lambda_i = (b_i)_{Q,w} = \frac{1}{w(Q)} \int_Q b_i(x)w(x)dx$, and $i = 1, 2$. For the term $II_1(x)$, observe that $x \in Q$ and $y \in R^n \backslash 2Q$, we have $|x_0 - y| \leq C|x - y|$, thus we yield

$$\begin{aligned} &\int_Q |II_1(x)|^p w(x)dx \\ &\leq C \int_Q |(b_1(x) - (b_1)_{Q,w})(b_2(x) - (b_2)_{Q,w})|^p w(x)dx \\ &\quad \times (\int_{R^n\backslash 2Q} \frac{|f(y)|}{|x_0 - y|^n} dy)^p \\ &\leq Cw(Q)(\frac{1}{w(Q)} \int_Q |b_1(x) - (b_1)_{Q,w}|^{2p} w(x)dx)^{\frac{1}{2}} \\ &\quad \times (\frac{1}{w(Q)} \int_Q |b_2(x) - (b_2)_{Q,w}|^{2p} w(x)dx)^{\frac{1}{2}} \\ &\quad \times (\sum_{i=1}^\infty \int_{2^{i+1}Q\backslash 2^iQ} \frac{|f(y)|}{|x_0 - y|^n} dy)^p \\ &\leq C\|b_1\|^p_* \|b_2\|^p_* w(Q)(\sum_{i=1}^\infty \frac{1}{|2^iQ|} (\int_{2^{i+1}Q} |f(y)|^p w(y)dy)^{\frac{1}{p}} \\ &\quad \times (\int_{2^{j+1}Q} w(y)^{-\frac{p'}{p}} dx)^{\frac{1}{p'}})^p, \end{aligned}$$

since $w \in A_p$ and by the definition of weighted Morrey space, we get

$$\begin{aligned} &\int_Q |II_1(x)|^p w(x)dx \\ &\leq C\|\vec{b}\|^p w(Q)(\sum_{i=1}^\infty w(2^{i+1}Q)^{\frac{-1}{p}} (\int_{2^{i+1}Q} |f(y)|^p w(y)dy)^{\frac{1}{p}})^p \\ &\leq C\|\vec{b}\|^p w(Q)(\sum_{i=1}^\infty w(2^{i+1}Q)^{\frac{\kappa-1}{p}} \|f\|_{L^{p,\kappa}(w)})^p \\ &\leq C\|\vec{b}\|^p w(Q)(\sum_{i=1}^\infty D^{i\frac{\kappa-1}{p}} w(Q)^{\frac{\kappa-1}{p}} \|f\|_{L^{p,\kappa}(w)})^p \\ &\leq C\|\vec{b}\|^p w(Q)^\kappa \|f\|^p_{L^{p,\kappa}(w)}. \end{aligned} \tag{3.4}$$

The third inequality is obtained by Lemma A.

For $II_2(x)$, note that $\lambda_i = (b_i)_{Q,w} = \frac{1}{w(Q)} \int_Q b_i(x) w(x)dx$, and $i = 1, 2$. By Hölder inequality and inequality, $|x_0 - y| \leq C|x - y|$, we get

$$\begin{aligned}
&\int_Q |II_2(x)|^p w(x)dx \\
&\quad \le \int_Q w(x)(\int_{R^n\setminus 2Q} \frac{|(b_1(y)-(b_1)_{Q,w})(b_2(y)-(b_2)_{Q,w})|}{|x-y|^n}|f(y)|dy)^p dx \\
&\quad \le Cw(Q)(\int_{R^n\setminus 2Q} \frac{|(b_1(y)-(b_1)_{Q,w})(b_2(y)-(b_2)_{Q,w})|}{|x_0-y|^n}|f(y)|dy)^p \\
&\quad \le Cw(Q)(\sum_{i=1}^{\infty}\frac{1}{|2^iQ|}\int_{2^{i+1}Q\setminus 2^iQ}|(b_1(y)-(b_1)_{Q,w}) \\
&\qquad \times (b_2(y)-(b_2)_{Q,w})||f(y)|dy)^p \\
&\quad \le Cw(Q)\{\sum_{i=1}^{\infty}\frac{1}{|2^iQ|}(\int_{2^{i+1}Q}|f(y)|^p w(y)dy)^{\frac{1}{p}} \\
&\qquad \times (\int_{2^{i+1}Q}|(b_1(y)-(b_1)_{Q,w})|^{2p'}w(y)^{-\frac{p'}{p}}dy)^{\frac{1}{2p'}} \\
&\qquad \times (\int_{2^{i+1}Q}|(b_2(y)-(b_2)_{Q,w})|^{2p'}w(y)^{-\frac{p'}{p}}dy)^{\frac{1}{2p'}}\}^p.
\end{aligned}$$

Indeed, by Lemma B, we know BMO(R^n) is equivalent to BMO(w), $w \in A_\infty$. Let $W = w^{-\frac{p'}{p}} \in A_{p'} \subset A_\infty$, $b_i \in$ BMO(R^n), $i = 1, 2$, for any cube Q, and by using Lemma B and C, we show that

$$(\frac{1}{W(2^{i+1}Q)}\int_{2^{i+1}Q}|b_i(y)-(b_i)_{Q,w}|^{2p'}W(y)dy)^{\frac{1}{2p'}} \le Ci\|b_i\|_*.$$

Thus, since $w \in A_p$, it yields

$$\begin{aligned}
\int_Q |II_2(x)|^p w(x)dx &\le Cw(Q)(\sum_{i=1}^{\infty}\frac{i^2}{|2^iQ|}\|b_1\|_*\|b_2\|_* W(2^{i+1}Q)^{\frac{1}{p'}} \\
&\quad \times (\int_{2^{i+1}Q}|f(y)|^p w(y)dy)^{\frac{1}{p}})^p \\
&\le Cw(Q)\|\vec{b}\|^p\|f\|^p_{L^{p,\kappa}(w)}(\sum_{i=1}^{\infty}\frac{i^2}{w(2^{i+1}Q)^{\frac{1-\kappa}{p}}})^p \\
&\le Cw(Q)^\kappa\|\vec{b}\|^p\|f\|^p_{L^{p,\kappa}(w)}. \qquad (3.5)
\end{aligned}$$

The last inequality is obtained by Lemma A and D'Alembert judge method of positive series.

For $II_3(x)$, by the inequality of $|x_0 - y| \le C|x - y|$ since $w \in A_p \subset A_\infty$ by Lemma B, we have

$$\begin{aligned}
&\int_Q |II_3(x)|^p w(x)dx \\
&\quad \le C\int_Q |(b_1(x)-\lambda_1) \\
&\qquad \times \int_{R^n\setminus 2Q}\frac{|b_2(y)-\lambda_2|}{|x-y|^n}|f(y)|dy|^p w(x)dx \\
&\quad \le C\int_Q |(b_1(x)-\lambda_1)|^p w(x)dx \\
&\qquad \times (\int_{R^n\setminus 2Q}\frac{|b_2(y)-\lambda_2|}{|x_0-y|^n}|f(y)|dy)^p \\
&\quad \le Cw(Q)\|b_1\|_*^p(\int_{R^n\setminus 2Q}\frac{|b_2(y)-\lambda_2|}{|x_0-y|^n}|f(y)|dy)^p.
\end{aligned}$$

By the Hölder inequality Lemma B and C, we yield

$$\begin{aligned}
&\int_{R^n\setminus 2Q}\frac{|b_2(y)-\lambda_2|}{|x_0-y|^n}|f(y)|dy \\
&\quad \le C\sum_{i=1}^{\infty}\frac{1}{|2^iQ|}\int_{2^{i+1}Q}|b_2(y)-\lambda_2||f(y)|dy \\
&\quad \le C\sum_{i=1}^{\infty}\frac{1}{|2^iQ|}(\int_{2^{i+1}Q}|f(y)|^p w(y)dy)^{\frac{1}{p}} \\
&\qquad \times (\int_{2^{i+1}Q}|b_2(y)-\lambda_2|^{p'}w(y)^{-\frac{p'}{p}}dy)^{\frac{1}{p'}} \\
&\quad \le C\|b_2\|_*\|f\|_{L^{p,\kappa}(w)} \\
&\qquad \times \sum_{i=1}^{\infty} i\frac{w(2^{i+1}Q)^{\frac{\kappa}{p}}}{|2^iQ|}(\int_{2^{i+1}Q}w(y)^{-\frac{p'}{p}}dy)^{\frac{1}{p'}} \\
&\quad \le C\|b_2\|_*\|f\|_{L^{p,\kappa}(w)}\sum_{i=1}^{\infty}\frac{i}{w(2^{i+1}Q)^{\frac{1-\kappa}{p}}}
\end{aligned}$$

and indeed for $0 < \kappa < 1$, by using Lemma A, we have

$$\begin{aligned}
\sum_{i=1}^{\infty}\frac{i}{w(2^{i+1}Q)^{\frac{1-\kappa}{p}}} &\le \sum_{i=1}^{\infty}\frac{i}{D^{(i+1)\frac{1-\kappa}{p}}}w(Q)^{\frac{\kappa-1}{p}} \\
&\le Cw(Q)^{\frac{\kappa-1}{p}}.
\end{aligned}$$

Thus, we conclude that

$$\int_Q |II_3(x)|^p w(x)dx \le Cw(Q)^\kappa\|\vec{b}\|^p\|f\|^p_{L^{p,\kappa}(w)}. \qquad (3.6)$$

In the same method, we shall get the result of $II_4(x)$ as

$$\int_Q |II_4(x)|^p w(x)dx \le Cw(Q)^\kappa\|\vec{b}\|^p\|f\|^p_{L^{p,\kappa}(w)} \qquad (3.7)$$

in which together with 3.1–3.7, the proof of theorem is finished.

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2 To deal with this result, we split f as above by $f = f_1 + f_2$, which yields

$$\begin{aligned}
&\lambda w(\{x \in Q : |T_{\vec{b}}f(x)| > \lambda\}) \\
&\quad \le \lambda w(\{x \in Q : |T_{\vec{b}}f_1(x)| > \lambda/2\}) \\
&\qquad + \lambda w(\{x \in Q : |T_{\vec{b}}f_2(x)| > \lambda/2\}) \\
&\quad = III + IV,
\end{aligned}$$

for any cube Q and $\lambda > 0$. For the term III, since we use Lemma F, it follows that

$$\begin{aligned}
III &\le C\int_{R^n}\Phi(|f_1|)(x)w(x)dx \\
&\le Cw(Q)^\kappa\|f\|_{L^{\Phi,\kappa}(w)}.
\end{aligned}$$

For the last term IV, without loss of generality, we still assume $m = 2$. By homogeneity it is enough to assume

$\lambda/2 = \|b_1\|_* = \|b_2\|_* = 1$ and hence, we only need to prove that

$$w(\{x \in Q : |T_{\vec{b}} f_2(x)| > 1\}) \leq Cw(Q)^\kappa \|f\|_{L^{\Phi,\kappa}(w)},$$

for any cube Q and $0 < \kappa < 1$. In fact, by Lemma E, we get

$$\begin{aligned} &w(\{x \in Q : |T_{\vec{b}} f_2(x)| > 1\}) \\ &\leq \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{x \in Q : |T_{\vec{b}} f_2(x)| > t\}) \\ &\leq C\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{x \in Q : M_\Phi f_2(x) > t\}) \\ &= C\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{x \in Q : M(\Phi|f_2|)(x) > t\}). \qquad (3.8) \end{aligned}$$

where $\Phi(t) = t\log^m(e+t)$. We use the Fefferman-Stein maximal inequality

$$\int_{x:Mf(x)>t} \phi(t)dx \leq \frac{C}{t} \int_{R^n} |f(x)| M\phi(x)dx,$$

for any functions f and $\phi \geq 0$. This yields

$$\begin{aligned} &w(\{x \in Q : M(\Phi|f_2|)(x) > t\}) \\ &\leq \frac{1}{t} \int_{\{x \in R^n : M(\Phi|f_2|)(x) > t\}} \chi_Q(x) w(x)dx \\ &\leq \frac{C}{t} \int_{R^n} \Phi(|f_2|)(x) M(w\chi_Q)(x)dx \\ &= \frac{C}{t} \left(\int_{3Q} + \int_{R^n \setminus 3Q}\right) \Phi(|f_2|)(x) M(w\chi_Q)(x)dx \\ &= \frac{C}{t}(IV_1 + IV_2). \qquad (3.9) \end{aligned}$$

For IV_1, since $w \in A_1$, it follows that

$$\begin{aligned} IV_1 &\leq C \int_{3Q} \Phi(|f|)(x) w(x)dx \\ &\leq Cw(3Q)^\kappa \|\Phi(|f|)\|_{L^{1,\kappa}(w)} \\ &\leq Cw(Q)^\kappa \|f\|_{L^{\Phi,\kappa}}. \qquad (3.10) \end{aligned}$$

To estimate the term IV_2, we first consider the form

$$\frac{1}{|R|} \int_{Q \cap R} w(y)dy,$$

for any $x \in R^n \setminus 3Q$, $x \in R$, and $R \cap Q \neq \emptyset$. By simple geometric observation, we have

$$\begin{aligned} \frac{1}{|R|} \int_{Q \cap R} w(y)dy &\leq C \frac{1}{|x - x_0|^n} \int_Q w(y)dy \\ &= \frac{C}{|x - x_0|^n} w(Q). \end{aligned}$$

Therefore, we obtain

$$M(w\chi_Q)(x) \leq \frac{C}{|x - x_0|^n} w(Q).$$

Since $w \in A_1$ satisfies the doubling condition and Lemma A, we can estimate the term IV_2 as follows

$$\begin{aligned} IV_2 &\leq C \int_{R^n \setminus 3Q} \frac{\Phi(|f|)(x)}{|x - x_0|^n} w(Q)dx \\ &\leq Cw(Q) \sum_{i=1}^\infty \frac{1}{|3^i Q|} \int_{3^{i+1}Q} \Phi(|f|)(x)dx \\ &\leq Cw(Q) \|\Phi(|f|)\|_{L^{1,\kappa}(w)} \sum_{i=1}^\infty \frac{1}{w(3^i Q)^{1-\kappa}} \\ &\leq Cw(Q)^\kappa \|f\|_{L^{\Phi,\kappa}}. \qquad (3.11) \end{aligned}$$

The last inequality is similar to 3.4. Note that $t\Phi(\frac{1}{t}) > 1$, from 3.8–3.11, we conclude

$$w(\{x \in Q : |T_{\vec{b}} f_2(x)| > 1\}) \leq Cw(Q)^\kappa \|f\|_{L^{\Phi,\kappa}(w)}.$$

Thus, the proof of Theorem *1.2* is completed.

Conclusions

Let $1 < p < \infty$, $0 < \kappa < 1$, if $b_j \in \mathrm{BMO}(R^n)$, $1 \leq j \leq m$, $w \in A_p$, we have

$$\|T_{\vec{b}} f\|_{L^{p,\kappa}(w)} \leq C\|\vec{b}\| \|f\|_{L^{p,\kappa}(w)}$$

and

$$\lambda w(x \in Q : |T_{\vec{b}} f(x)| > \lambda) \leq C\|\vec{b}\| \|f\|_{L^{\Phi,\kappa}(w)},$$

which extends the results of the commutator in weighted Morrey space. In particular, for $m = 1$, this result is also new in classical commutator.

Competing interests
The author declares that he has no competing interest.

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