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Covering dimension and normality in *L*-topological spaces

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Abstract

Purpose: In this paper, we extend the concept of covering dimension of general topological spaces to *L*-topological spaces using α-*Q*-covers and quasi-coincidence relation.

Methods: Dimension theory is a branch of topology devoted to the definition and study of the notion of dimension in certain classes of topological spaces. The dimension of a general topological space *X* can be defined in three different ways: the small inductive dimension *indX*, the large inductive dimension *IndX*, and the covering dimension *dimX*. The covering dimension *dim* behaves somewhat better than the other two dimensions, i.e., that for the dimension *dim*, a large number of theorems of the classical theory can be extended to general topological spaces. Also, there is a substantial theory of covering dimension for normal spaces.

Results: A characterization of covering dimension in the weakly induced *L*-topological spaces is obtained. Moreover, a characterization of covering dimension for fuzzy normal spaces is also obtained.

Conclusions: Finally, This paper provides some brief sketches regarding the topics covering dimension in *L*-topological spaces and covering dimension for fuzzy normal spaces. The neighborhood structure used for the investigations is the quasi-coincident neighborhood structure.

Keywords: Fuzzy topology, Covering dimension, Fuzzy normal spaces

AMS subject classification: 54A40, 03E72, 54D20

Introduction

Adnadjevic [1,2] introduced the concept of generalized fuzzy spaces (GF spaces) and defined two dimension functions, *F-ind* and *F-Ind*. Later, Cuchillo and Tarres [3] extended them into fuzzy topological spaces in the case of zero dimensionality. Ajmal and Kohli [4] have studied the concept of covering dimension in fuzzy topological spaces. however, all these studies have been done in the [0,1] fuzzy topological spaces. In this paper, an attempt is made to extend the notions of covering dimension (*dim*) to *L*-topological spaces (*L-ts*) using quasi-coincidence relation.

Let L be a complete lattice. Its universal bounds are denoted by $\perp$ and $\top$. Thus, $\perp \leq \alpha \leq \top$ for all $\alpha \in L$. We set $\bigvee \phi = \perp$ and $\bigwedge \phi = \top$. A unary operation $'$ on L is a quasi-complementation. It is an involution (i.e., $\alpha'' = \alpha$ for all $\alpha \in L$) that inverts the ordering (i.e., $\alpha \leq \beta$ implies $\beta' \leq \alpha'$). In (L'), DeMorgan's laws hold $(\bigvee A)' = \bigwedge\{\alpha'; \alpha \in A\}$ and $(\bigwedge A)' = \bigvee\{\alpha'; \alpha \in A\}$ for every $A \subset L$. Moreover, in particular, $\perp' = \top$ and $\top' = \perp$.

A molecule or co-prime element in a lattice L is a joined irreducible element in L, and the set of all nonzero co-prime elements of L is denoted by $M(L)$. Also, we denote $A_{(\alpha)} = \{x \in X : A(x) \not\leq \alpha\}$ and $A_{[\alpha]} = \{x \in X : A(x) \leq \alpha\}$.

A complete lattice L is completely distributive if it satisfies either of the logically equivalent CD1 or CD2 as given below:

CD1. $\bigwedge_{i\in I}(\bigvee_{j\in J_i} a_{i,j}) = \bigvee_{\phi\in\Pi_{i\in I}J_i}(\bigwedge_{i\in I} a_{i,\phi(i)})$
CD2. $\bigvee_{i\in I}(\bigwedge_{j\in J_i} a_{ij}) = \bigwedge_{\phi\in\Pi_{i\in I}J_i}(\bigvee_{i\in I} a_{i,\phi(i)})$

for all $\{\{a_{i,j} : j \in J_i\} : i \in I\} \subset P(L) \setminus \{\phi\}, I \neq \phi$

If (L') is a complete lattice, then for a set X, L^X is the complete lattice of all maps from X into L called L-sets or

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L-subsets of X. Under point wise ordering, $a \le b$ in L^X if and only if $a(x) \le b(x)$ in L for all $x \in X$. The constant member of L^X with value α is denoted by α itself.

Clearly, L^X has a quasi-complementation $'$ defined point wisely $\alpha'(x) = \alpha(x)'$ for all $\alpha \in L$ and $x \in X$. Thus, DeMorgan's laws are inherited by $(L^X,')$.

Let $(L,')$ be a complete lattice and X be any nonempty set. A subfamily $\tau \subset L^X$ which is closed under the formation of sups and finite infs (both formed in L^X) is called an L-topology on X, and its members are called open L-sets. The pair (X, τ) is called an L-topological space (L-ts). Quasi-complements of open L-sets are called closed L-sets. This means that we are considering the category of L-**Top**.

For each $(X, T) \in$ **Top**, we can define an L-fuzzy space $(X, \omega_L(T))$ with $\omega_L(T) =\ll \mathbf{C}((X,T),(L,\mathbf{SUP}(L))) \gg$, that is, $\omega_L(T)$ precisely consists of all continuous mappings from (X, T) to $(L, \mathbf{SUP}(L))$, where $\mathbf{SUP}(L)$is the upper topology generated by the sub-basic sets. Again, given $(Y, S) \in$ **L-Top**, we can define a topological space $(Y, \iota_L(S))$ with $\iota_L(S) = \bigvee\{V^{\leftarrow}(SUP(L)); V \in S\}$.

We know that the set of all nonzero co-prime elements in a completely distributive lattice is $\bigvee$ -generating. Moreover, for a continuous lattice L and a topological space (X, T), $T = \iota_L\omega_L(T)$ is not true in general. By the proposition of Kubiak (see Proposition 3.5 in [5]), we know that one sufficient condition for $T = \iota_L\omega_L(T)$ is that L is completely distributive.

In 1988, Wang extended the Lowen functor ω for completely distributive lattices as follows: for a topological space (X, T), $(X, \omega(T))$ is called the induced space of (X, T), where $\omega(T) = \{A \in L^X : \forall a \in M(L), A_{(\alpha')} \in T\}$. In 1992, Kubiak also extended the Lowen functor ω_L for a complete lattice L. In fact, when L is completely distributive, $\omega_L = \omega$.

An L-topological space (X, τ) is called a weakly induced space if $\forall\alpha \in M(L), \forall A \in \tau$; it is true that $A_{(\alpha')} \in [\tau]$, where $[\tau]$ is the set of all crisp open sets in τ.

Based on these facts, in this paper, we use a complete, completely distributive lattice L in L^X. For a standardized basic fixed-basis terminology, we follow Hoehle and Rodabaugh [6]. We take q to denote the quasi-coincidence relation. Also, $L - Pnt(X)$ denotes the collection of all L-fuzzy points in the L-*ts* (X, τ).

Methods

In this work, the neighborhood structure used for the investigations is the quasi-coincident neighborhood structure(Q-nbd). Also, there are several definitions of fuzzy compactness on fuzzy topological space introduced by many authors. These notions are defined using various tools such as fuzzy cover, Q-cover, etc. Among which, only N-compactness and related tools are considered in this work for the investigations.

Covering dimension

Definition 1. Let (X, τ) be an L-*ts*. A fuzzy point x_α is quasi-coincident with $A \in L^X$ (and write $x_\alpha \prec A$) if $x_\alpha \not\le A'$. Also, A quasi-coincides with B at x (AqB at x) if $A(x) \not\le B'(x)$. We say that A is quasi-coincident with B and write AqB if AqB at x for some $x \in X$. Further, $A\neg\, qB$ means A does not quasi-coincide with B. We say that $U \in \tau$ is a quasi-coincident *nbd* of x_α (Q-*nbd*) if $x_\alpha qU$. The family of all Q-*nbd*s of x_α is denoted by $Q_\tau(x_\alpha)$ or $Q(x_\alpha)$ [7].

Definition 2. Let (X, τ) be an L-*ts*, $A \in L^X$. $\Phi \subset L^X$ is called a Q-cover of A if for every $x \in Supp(A)$, there exists $U \in \Phi$ such that $x_{A(x)} \prec U$. Φ is a Q-cover of (X, τ) if Φ is a Q-cover of $\top$. If $\alpha \in M(L)$, then $C \in \tau$ is an α-Q-*nbd* of A if $C \in Q(x_\alpha)$ for every $x_\alpha \le A$. Φ is called an α-Q-cover of A if for every $x_\alpha \le A$, there exists $U \in \Phi$ such that $x_\alpha \prec U$. Φ is called an open α-Q-cover of A if $\Phi \subset \delta$ and Φ is an α-Q-cover of A. $\Phi_0 \subset L^X$ is called a sub α-Q-cover of A if $\Phi_0 \subset \Phi$ and Φ_0 is also an α-Q-cover of A [7].

Definition 3. Let $\boldsymbol{U} = \{U_\lambda : \lambda \in \Lambda\}$, not all zero, be a family of L-subsets of an L-ts X. The order of a fuzzy point x_α in $\boldsymbol{U}$ is the number of elements of $\boldsymbol{U}$ which are quasi-coincident with x_α. We denote it by Ord(x_α,$\boldsymbol{U}$). The order of a collection **U** is defined as the largest integer n such that for every x_α with $\alpha \in M(L)$, x_α quasi-coincides with $(n+1)$ members of $\boldsymbol{U}$; that is, Ord$(x_\alpha, \boldsymbol{U}) = n + 1$ for all $\alpha \in M(L)$.

Definition 4. Let (X, τ) be an L-*ts*, $A \in L^X$. Then, α-dim$A$ is the least integer n such that every finite open α-Q-cover of A has an open α-Q-cover refinement of order not exceeding n. Also, dim $A = n$ if α-dim $A = n$ for every $\alpha \in M(L)$. dim$(X, \tau) = n$ if dim $\top = n$. Where a collection $\boldsymbol{A}$ refines a collection $\boldsymbol{B}$($\boldsymbol{A} < \boldsymbol{B}$) if for every $A \in \boldsymbol{A}$, there exists $B \in \boldsymbol{B}$ such that $A \le B$.

Remarks 1. dim $X = -1$ if and only if X is void and dim$X = n$ if it is true that dim$X \le n$ and dim$X \le n - 1$ is not true. Also, dim $X = \infty$ if it is not true for any integer n that dim$X \le n$.

Theorem 1. Let (X, τ) be an L-*ts*. The following are then equivalent:

(i) dim $X \le n$
(ii) For every $\alpha \in M(L)$, every finite α-Q-cover $\{U_1, U_2, \ldots . U_k\}$ of $\top$ by open L-subsets, there is an open α-Q-cover $\{V_1, V_2, \ldots . V_k\}$ of order not exceeding n such that $V_i < U_i$ for $i = 1, 2, 3, \ldots k$.
(iii) If $\{U_1, U_2, \ldots . U_{n+2}\}$ is an open α-Q-cover of $\top$, then there exists an open α-Q- cover

$\{V_1, V_2, \ldots . V_{n+2}\}$ of $\top$ such that $V_i < U_i$ and $\underset{1\le i\le n+2}{Inf} V_i < \alpha$, where $\alpha \in M(L)$.

Proof.

(i) $\Rightarrow$ (ii)

Let $\dim X \le n$, $\alpha \in M(L)$ and $\boldsymbol{U} = \{U_1, U_2, \ldots . U_k\}$ be a finite open α-Q-cover of $\top$. Now, if $\boldsymbol{U}$ has a refinement $\boldsymbol{W}$ with order not exceeding n and if $W \in \boldsymbol{W}$, there exists some i such that $W_i < U_i$ and suppose that each W is associated with a unique U_i containing it and take $V_i = Sup\{W : W < U_i\}$. Clearly, each U_i is open and $W_i < U_i$ for some i. Now, since order of $\boldsymbol{W}$ is not exceeding n, it follows that each $x_\alpha \in M(L^X)$ quasi-coincides with at most $n+1$ members of $\boldsymbol{W}$, and each $W \in \boldsymbol{W}$ is associated with a unique U_i. Hence, x_α quasi-coincides with at most $n+1$ members of $\{V_i\}$. Hence, $\{V_i\}$ is an α-Q-cover of $\top$ with order not exceeding n.

(i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) are obvious.

(iii) $\Rightarrow$ (ii)

Let $\boldsymbol{U} = \{U_1, U_2, \ldots . U_k\}$ be a finite open an α-Q-cover of $\top$. Assume that $k > n+1$. Define the collection $\{G_i : 1 \le i \le n+2\}$ as follows: $G_i = U_i$ if $i \le n+1$ and $G_{n+2} = \underset{n+2\le i\le k}{Sup} U_i$. Now, clearly, $\{G_i : 1 \le i \le n+2\}$ is an open α-Q-cover of $\top$, and by hypothesis of (iii), there is an open α-Q-cover $\{H_1, H_2, \ldots H_{n+2}\}$ such that $H_i < G_i$ and $\underset{1\le i\le n+2}{Inf} H_i < \alpha$.

Now, take $W_i = U_i$. If $i \le n+1$ and $W_i = U_i \wedge H_{n+2}$ if $i > n+1$, then clearly, the collection $\boldsymbol{W} = \{W_1, W_2, \ldots . W_k\}$ is an open α-Q-cover of $\top$ with the property that $W_i < U_i$ and $\underset{1\le i\le n+2}{Inf} W_i < \alpha$. Now, if there exists a subset B of $\{1, 2, 3, \ldots . k\}$ with $n+2$ elements such that $\underset{i\in B}{Inf} W_i > \alpha$, we will renumber the family $\boldsymbol{W}$ to give a family $\boldsymbol{P} = \{P_1, P_2, \ldots . P_k\}$ such that $\underset{1\le i\le n+2}{Inf} P_i > \alpha$. Now, proceeding in a manner similar to the construction above, we obtain an an α-Q-cover $\boldsymbol{W}' = \{W'_1, W'_2, \ldots . W'_k\}$ by open fuzzy sets with $W_i < P_i$ and $\underset{1\le i\le n+2}{Inf} W_i < \alpha$.

Now, again, if $\boldsymbol{C}$ is a subset of $\{1, 2, \ldots k\}$ with $n+2$ elements such that $\underset{i\in P}{Inf} P_i > \alpha$, then $\underset{Wi\in W}{Inf} W'_i < \alpha$. By repeating this process for a finite number of times, we will end up with an open α-Q-cover $\{V_1, V_2, \ldots . V_k\}$ of $\top$ with order not exceeding n and $V_i < U_i$.

This completes the proof. □

Theorem 2. In a weakly induced *L-ts*, the following are equivalent:

(i) $\dim(X, \tau) \le n$.
(ii) There exists an $\alpha \in M(L)$ such that $\alpha-\dim(X, \tau) \le n$.
(iii) $\dim(X, [\tau]) \le n$.

Proof.

(i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii)

Let $\boldsymbol{U} = \{U_1, U_2, \ldots . U_k\} \subset [\tau]$ be a finite open cover of X, then $\{\chi_U : U \in \boldsymbol{U}\}$ is an open α-Q-cover of $\top$. Since $\alpha-\dim(X, \tau) \le n$, it follows that $\{\chi_U : U \in \boldsymbol{U}\}$ has an open refinement $\boldsymbol{V}$ of order not exceeding n. Now, consider $\boldsymbol{W} = \{V_{(\alpha')} : V \in \boldsymbol{V}\}$, where $V_{(\alpha')} = \{x \in X : V(x) \not\le \alpha'\}$. By the weakly induced property, $\boldsymbol{W}$ is an open cover of X. Now, we will prove that $\boldsymbol{W}$ has an order not exceeding n.

For, if possible, let order of $\boldsymbol{W}$ be greater than n. Therefore, there exists $x \in X$ which belongs to at least $n+2$ members of $\boldsymbol{W}$, that is

1. $x \in \{x \in X : V(x) \not\le \alpha'\}$ for at least $n+2$ members of $\boldsymbol{V}$,
2. $V(x) \not\le \alpha'$ for at least $n+2$ members of $\boldsymbol{V}$ or $x_\alpha \prec V$ for at least $n+2$ members of $\boldsymbol{V}$. This is a contradiction to that order of $\boldsymbol{V}$ is not exceeding n.

(iii) $\Rightarrow$ (i)

Let $\boldsymbol{U} \subset [\tau]$ be an open α-Q-cover of $\top$ where $\alpha \in M(L)$. Since (X, τ) is weakly induced, it follows that $\{U_{(\alpha')} : U \in \boldsymbol{U}\}$ is an open cover of X, and it has an open refinement of order not exceeding n say $\boldsymbol{V}$. For every $V \in \boldsymbol{V}$, let U_V be such that $V < U_{V_{(\alpha')}}$. Consider $\boldsymbol{W} = \{\chi_V \wedge U_V : V \in \boldsymbol{V}, V < U_{V_{(\alpha')}}\}$. This is an open refinement of $\boldsymbol{U}$ with order not exceeding n. For, if possible, let order of $\boldsymbol{W}$ be greater than n, then there exists $x_\alpha \in M(L^X)$ which quasi-coincides with at least $n+2$ members of $\boldsymbol{W}$, that is

1. $x_\alpha \not\le (\chi_V \wedge U_V)'$ for at least $n+2$ members of $\boldsymbol{W}$,
2. $x_\alpha \not\le \chi'_V \vee U'_V$ for at least $n+2$ members of $\boldsymbol{W}$,
3. $x_\alpha \prec \chi_V$ or $x_\alpha \prec U_V$ for at least $n+2$ members of $\boldsymbol{V}$.

In both cases, x∈V for at most $n+2$ members of $\boldsymbol{V}$, and this is a contradiction.

This completes the proof. □

Definition 5. A refinement $\{b_t : t \in \mathbf{T}\}$ of $\{a_s : s \in \mathbf{S}\}$ is said to be precise if $T = S$ and $b_s \le a_s$ for each s $\in S$.

Theorem 3. In a weakly induced *L-ts*, the following are equivalent:

(i) dim $(X, \tau) \leq$ n.
(ii) For every $\alpha \in M(L)$, every finite α-Q-cover of $\top$ by open L- sets has a precise open refinement of order not exceeding n.
(iii) There exists an $\alpha \in M(L)$ such that every finite α-Q-cover of $\top$ by open L- sets has a precise open refinement of order not exceeding n.
(iv) If $\{U_1, U_2, \ldots..U_{n+2}\}$ is an open α-Q-cover of $\top$, then there exists an open α-Q- cover $\{V_1, V_2, \ldots..V_{n+2}\}$ of $\top$ such that $V_i < U_i$, where $\alpha \in M(L)$.
(v) There exists an $\alpha \in M(L)$ such that α-dim$(X,\tau) \leq n$.
(vi) dim $(X, [\tau]) \leq n$.

Proof. Equivalence of (i), (v), and (vi) follows from Theorem 2. All other implications except (iii) $\Rightarrow$ (i) follows from Theorem 1.

(iii) $\Rightarrow$ (i)

By Theorem 2, it is enough to prove that dim$(X, [\tau]) \leq n$. Let $\boldsymbol{U} \subset [\tau]$ be a finite open cover of X. Then $\{\chi_U : U \in \boldsymbol{U}\}$ is a finite open α-Q-cover of $\top$, and it has a precise open refinement of order not exceeding n. Let it be $\boldsymbol{V} = \{V_1, V_2, \ldots V_k\}$. Let $\boldsymbol{W} = \{V_{i(\alpha')} : i = 1, 2, 3, \ldots\ldots\}$. By weakly induced property, $\boldsymbol{W}$ is an open cover of X. Also, it is easy to show that order of $\boldsymbol{W}$ is not exceeding n, and hence, dim $(X, [\tau]) \leq n$.
This completes the proof. □

Normal spaces

Definition 6. (X, τ) is called normal if for every closed L-subset P and every open L-subset U in (X, τ) such that $P \leq U$, there exists an open L-subset V in (X, τ) such that $P \leq V \leq clV \leq U$ [7].

Definition 7. Let (X, τ) be an L-ts. An α-Q-cover $\{U_\lambda : \lambda \in \wedge\}$ of X is said to be shrinkable if there exists an open α-Q-cover $\{V_\lambda : \lambda \in \wedge\}$ of X such that cl $V_\lambda \leq U_\lambda$ for each $\lambda \in \wedge$

Definition 8. Let $\mathbf{A} = \{A_t : t \in T\} \subseteq L^X$, $D \in L^X$, $\alpha \in M(L)$. If $\forall x_\alpha \leq D$, $\exists P \in \eta(x_\alpha)$ and a finite subset T_0 of T such that $\forall t \in T - T_0$, $A_t \leq P$, then $\mathbf{A}$ is called α-locally finite in D. If there exists $\gamma \in \beta^*(\alpha)$ such that $\mathbf{A}$ is γ-locally finite in D, then $\mathbf{A}$ is called α^--locally finite in D [7].

Theorem 4. The following are equivalent in an L-ts (X, τ):

(i) X is normal.
(ii) For every $\alpha \in M(L)$, every point finite α-Q-cover of $\top$ by open L- sets is shrinkable.
(iii) For every $\alpha \in M(L)$, every open α-Q-cover of $\top$ has a locally finite refinement by closed L- sets.

Proof.

(i) $\Rightarrow$ (ii).

Let $\alpha \in M(L)$ and $\{U_\lambda : \lambda \in \Lambda\}$ be a point finite α-Q-cover of a normal L-ts X. Also, let Λ be well ordered. We will construct a shrinking of $\{U_\lambda : \lambda \in \Lambda\}$ by induction. Let $\mu \in \Lambda$ and for each $\lambda \leq \mu$ suppose that there is an open L-set V_λ such that $clV_\lambda \leq U_\lambda$ and for each $\nu \leq \mu$, $\{\bigvee_{\lambda \leq \nu} V_\lambda\} \vee \{\bigvee_{\lambda > \nu} U_\lambda\}$ is an α-Q-cover of X.

Let $x_\alpha \in M(L^X)$. Now, since $\{U_\lambda : \lambda \in \Lambda\}$ is point finite and Λ is well ordered, there exists a largest element $\xi \in \Lambda$ such that $x_\alpha q U_\xi$. Now if $\xi \geq \mu$ then $x_\alpha q \bigvee_{\lambda \geq \mu} U_\lambda$ and if $\xi < \mu$ then $x_\alpha q \bigvee_{\lambda < \mu} V_\lambda$. Thus, $\{\bigvee_{\lambda \geq \mu} U_\lambda \bigvee \bigvee_{\lambda < \mu} V_\lambda\}$ is an α-Q-cover of X. Therefore, we have $\{\bigvee_{\lambda \geq \mu} U_\lambda \bigvee \bigvee_{\lambda < \mu} V_\lambda\}' \leq U_\mu$, and since X is normal, there exists an open L-set V_μ such that $\{\bigvee_{\lambda > \mu} U_\lambda \bigvee \bigvee_{\lambda \leq \mu} V_\lambda\}' \leq V_\mu \leq clV_\mu \leq U_\mu$.

Thus, we have $clV_\mu \leq U_\mu$ and $\{\bigvee_{\lambda \leq \mu} U_\lambda \bigvee \bigvee_{\lambda < \mu} V_\lambda\}$ is an α-Q-cover of X. Thus, the construction of a shrinking of $\{U_\lambda : \lambda \in \Lambda\}$ is complete by induction.

(ii) $\Rightarrow$ (iii) is clear.

(iii) $\Rightarrow$ (i)

Let X be a space in which for any $\alpha \in M(L)$, every point finite α-Q-cover has a locally finite refinement by closed L-sets. Let A, B be such that $A \neg q B$. Now clearly $\{A', B'\}$ is an open α-Q-cover of X. Thus, by assumption, $\{A', B'\}$ has a locally finite refinement by closed L-sets say $\mathbf{F}$. Let E be the union of members of $\mathbf{F}$ which are not quasi-coincident with A and F be the union of members of $\mathbf{F}$ which are not quasi-coincident with B. Clearly, E and F are closed and take $U = E'$ and $V = F'$. Now, clearly, U is not quasi-coincident with V and $A \leq U$ and $B \leq V$. Then, Theorem 9.2.11 of [7] X is normal.
This completes the proof. □

Definition 9. $\{A_\lambda : \lambda \in \wedge\}, \{B_\lambda : \lambda \in \wedge\}$ are said to be similar if for each finite subset μ of $\wedge$, the sets $\wedge_{\lambda \in \mu} A_\lambda$ and $\wedge_{\lambda \in \wedge} B_\lambda$ are either both zero or both non zero.

Proposition 1. Let $\{U_\lambda : \lambda \in \wedge\}$ be a locally finite collection of open L-sets of a normal space X and $\{F_\lambda : \lambda \in \wedge\}$ be a family of closed L-sets such that $F_\lambda < U_\lambda$ for $\lambda \in \wedge\}$. Then there exists a family $\{G_\lambda : \lambda \in \wedge\}$ of open L-sets such that $F_\lambda < G_\lambda < ClG_\lambda < U_\lambda$ and the families $\{F_\lambda : \lambda \in \wedge\}$ and $\{clG_\lambda : \lambda \in \wedge\}$ are similar.

Proof. Let $\bigwedge$ be well ordered with a least element. Now, by induction, we will construct a family $\{G_\lambda : \lambda \in \bigwedge\}$ of open L-sets such that $F_\lambda < G_\lambda < clG_\lambda < U_\lambda$ and for each ν of $\bigwedge$, the family $\{K_\lambda^\nu : \lambda \in \bigwedge\}$ given by $K_\lambda^\nu = clG_\lambda$ if $\lambda \leq \nu$ and $K_\lambda^\nu = F_\lambda$ if $\lambda > \nu$ is similar to $\{F_\lambda : \lambda \in \bigwedge\}$. Let $\mu \in \bigwedge$ and G_λ has been defined for $\lambda < \mu$ such for each $\nu < \mu$, the family $\{K_\lambda^\nu : \lambda \in \bigwedge$ is similar to $\{F_\lambda : \lambda \in \bigwedge\}$.

Let $\{L_\lambda : \lambda \in \bigwedge\}$ be the family defined as $L_\lambda = clG_\lambda$ if $\lambda \leq \mu$ and $L_\lambda = F_\lambda$ if $\lambda > \mu$. Now $\{L_\lambda : \lambda \in \bigwedge\}$ is similar to $\{F_\lambda : \lambda \in \bigwedge\}$. For, suppose that $\lambda_1, \lambda_2, \ldots .\lambda_\nu \in \bigwedge$ and $\lambda_1 < \lambda_2 < \ldots < \lambda_j < \mu < \lambda_{j+1} < \ldots . < \lambda_\nu$. Then $\bigwedge_i \{L_{\lambda_i} : 1 \leq i \leq \nu\} = \bigwedge_i \{K_{\lambda_i}^{\lambda_j} : 1 \leq j \leq \nu\}$ so that $\wedge L_{\lambda_i} = 0$ if and only if $\wedge F_{\lambda_i} = 0$. Since $L_\lambda < U_\lambda$ for each λ, the family $\{L_\lambda : \lambda \in \bigwedge\}$ is locally finite. Thus, if Γ is the set of finite subsets of $\bigwedge$ and for each $\gamma \in \Gamma$, $\{E_\gamma : \gamma \in \Gamma\}$ is a locally finite family of closed L-sets. Hence, $E = \vee\{E_\nu : E_\nu \wedge F_\mu = 0\}$ is a closed set which is disjoint from F_μ. Therefore, there exists an open L-set G_μ such that $F_\mu < G_\mu < clG_\mu < U_\mu$ and $clG_\mu \wedge E = 0$.

Now, the open L-sets are defined for $\lambda \leq \mu$, and it remains to show that the collection $\{K_\lambda^\nu : \lambda \in \bigwedge\}$ is similar to $\{F_\lambda : \lambda \in \wedge\}$. For that, it is sufficient to show that the collections $\{K_\lambda^\nu : \lambda \in \bigwedge\}$ and $\{L_\lambda : \lambda \in \wedge\}$ are similar. Suppose that $\lambda_1, \lambda_2, \ldots .\lambda_\nu \in \bigwedge$ and that $\underset{1 \leq i \leq \nu}{\wedge} L_{\lambda_i} = 0$. It must be shown that $\underset{1 \leq i \leq \nu}{\wedge} K_{\lambda_i}^\mu = 0$. Suppose that $\lambda_1 < \lambda_2 < \ldots < \lambda_j \leq \mu < \lambda_{j+1} < \ldots . < \lambda_\nu$. If $\lambda_j \neq \mu$, there is nothing to prove. If $\lambda_j = \mu$ then $L_{\lambda_1} \wedge \ldots . \wedge L_{\lambda_{j-1}} \wedge F_\mu \wedge \ldots . \wedge L_{\lambda_{j+1}} \wedge \ldots . \wedge L_{\lambda_\nu} = 0$, and hence, by construction $L_{\lambda_1} \wedge \ldots . \wedge L_{\lambda_{j-1}} \wedge clG_\mu \wedge \ldots . \wedge L_{\lambda_{j+1}} \wedge \ldots . \wedge L_{\lambda_\nu} = 0$. Thus, we have $\underset{1 \leq i \leq \nu}{\wedge} K_{\lambda_i}^\mu = 0$ as required.

This completes the proof. □

Theorem 5. Let (X, τ)be an $L - ts$. Then (X, τ) is normal if and only if for every two L-closed subsets P and Q in (X, τ) such that P does not quasi-coincide with Q, there exists open subsets U and V in (X, τ) such that $P \leq U, Q \leq V$ and U does not quasi-coincide with V [7].

Result. If $dimX = 0$, then X is a fuzzy normal space

Proof. Let $P, Q \in \tau'$ be such that $P \neg qQ$. Now, $\{P', Q'\}$ is an open α-Q-cover of X. For every $x_\alpha \in M(L^X)$, if $x_\alpha \leq P'$ then $x_\alpha qP'$ and if $x_\alpha \not\leq P'$, then $x_\alpha qQ'$. Since $dimX = 0$, by Theorem 1, there exists a refinement $\{U, V\}$ of $\{P', Q'\}$ with order zero such that $U \leq P'$ and $V \leq Q'$. Therefore, we have $P \leq V$ and $Q \leq U$ with $U \neg qV$. Hence, X is normal.

This completes the proof. □

Theorem 6. For every closed subspace A of an L-ts (X, τ), $dimA \leq dimX$

Proof. Suppose $dimX \leq n$. Let $\{U_1, U_2, \ldots U_k\}$ be an open α-Q-cover of A. Now clearly $U_i = A \wedge V_i$ for some $V_i \in \tau$. Now, $\{V_1, V_2, \ldots V_k, A'\}$ is a finite open α-Q-cover of X. Since $dimX \leq n$, it has an open refinement $\mathbf{W}$ of order not exceeding n. Take $\mathbf{V} = \{W \wedge A : W \in \mathbf{W}\}$. This is an open refinement of $\{U_1, U_2, \ldots U_k\}$.

This completes the proof. □

Definition 10. Let (X, τ) be an $L - ts$. (X, τ) is called T_1 if for every two distinguished molecules e and d in (X, τ) such that $e \not\leq d$, there exists $U \in Q_\delta(e)$ such that $d \not\ll U$ [7].

Theorem 7. Let (X, τ) be an $L - ts$. Then (X, τ) is T_1 if and only if every molecule in (X, τ) is a closed subset [7].

Definition 11. Let (X, τ) be an $L - ts$, $A, B \in L^X$. A and B are called separated if $clA \cap B = A \cap clB = \perp$. A is called connected, if there does not exist separated $C, D \in L^X - \{\perp\}$ such that $A = C \cup D$.(X, τ) is connected if $\top$ is connected [7].

Definition 12. (X, τ) is totally disconnected if it contains no connected subspace that consists more than one molecule.

Theorem 8. If (X, τ) is a T_1 space with $dimX = 0$, then X has a basis consisting of open and closed L-sets.

Proof. Let (X, τ) be a T_1 space with $dimX = 0$. Let $U \in \tau$ and $x_\alpha \in M(L^X)$ such that $x_\alpha qU$. Now x_α is a closed set, and hence, $\{U, (x_\alpha)'\}$ is an open α-Q-cover of $\top$. By Theorem 1, there exists an open α-Q-cover $\{V, W\}$ such that $V < U, W < (x_\alpha)'$ and $U \wedge W < \alpha$. Thus, V is an open and closed L-set such that $x_\alpha \in V \subset U$.

This completes the proof. □

Theorem 9. If (X, τ) is a normal space, then the following are equivalent:

(i) $\dim X \leq n$
(ii) For every $\alpha \in M(L)$, every finite α-Q-cover $\{U_1, U_2, \ldots ..U_k\}$ of $\top$ by open L-subsets , there is an open α-Q-cover $\{V_1, V_2, \ldots ..V_k\}$ of order not exceeding n such that $clV_i < U_i$ for $i = 1,2,3,\ldots k$.
(iii) For every $\alpha \in M(L)$, every finite α-Q-cover $\{U_1, U_2, \ldots ..U_k\}$ of $\top$, there exists a closed α-Q-cover $\{F_1, F_2, \ldots ..F_k\}$ of order not exceeding n such that $F_i < U_i$ for $i = 1,2,3,\ldots k$.

(iv) For every $\alpha \in M(L)$, every finite α-Q-cover of $\top$ by open L-subsets has a finite refinement by closed sets of order not exceeding n.

(v) If $U_1, U_2, \ldots ..U_k\}$ is an open α-Q-cover of X, there exists an α-Q-cover $\{F_1, F_2, \ldots ..F_k\}$ by closed L-subsets such that $F_i < U_i$ and $\underset{1 \le i \le n+2}{InfF_i} < \alpha$.

Proof.

(i) $\Rightarrow$ (ii)

Suppose that $\dim X \le n$, $\alpha \in M(L)$ and $\{U_1, U_2, \ldots ..U_k\}$ be an α-Q-cover of $\top$ by open L-sets. Then by Theorem 4, there exists an open open α-Q-cover $\{W_1, W_2, \ldots ..W_k\}$ of order not exceeding n such that $W_i < U_i$. Given that X is normal, therefore, by Theorem 4, there exists an open α-Q-cover $\{V_1, V_2, \ldots ..V_k\}$ such that $clV_i < W_i$ for each i. Then $\{V_1, V_2, \ldots ..V_k\}$ is an α-Q-cover by open L-sets with the required properties.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is clear.

(iv) $\Rightarrow$ (v)

Let $\mathbf{U} = \{U_1, U_2, \ldots ..U_{n+2}\}$ be an α-Q-cover of $\top$. Then, by hypothesis, $\mathbf{U}$ has a finite closed α-Q-cover refinement $\mathbf{E}$ of order not exceeding n. If $E \in \mathbf{E}$, then $E \le U_i$ for some i. Associate the set E_i with sets U_i containing it and let $\mathbf{F}_i = \bigvee\{E_i : E_i < U_i\}$. Clearly, F_i is closed, and $F_i < U_i$ and $\{F_1, F_2, \ldots ..F_{n+2}\}$ is an α-Q-cover of $\top$ such that $\bigwedge F_{(i)} < \alpha$.

(v) $\Rightarrow$ (i)

Let $\{U_1, U_2, \ldots ..U_{n+2}\}$ be an α-Q-cover of $\top$ by open L-sets. By hypothesis, there exists an α-Q-cover $\{F_1, F_2, \ldots ..F_{n+2}\}$ by closed L-sets such that $F_i < U_i$ and $\wedge F_i < \alpha$. Now, by Proposition 1, there exist open L-sets $\{V_1, V_2, \ldots ...V_{n+2}\}$ such that $F_i < V_i < U_i$ for each i and $\{V_i\}$ are similar to $\{F_i\}$. Thus, $\{V_1, V_2, \ldots \ldots V_{n+2}\}$ is an open α-Q-cover of $\top$ with $V_i < U_i$ and $\wedge V_i < \alpha$. Then, by Theorem 1, $dimX \le n$.

This completes the proof. □

Results and discussion

In this paper, the notions of covering dimension *dim* is extended to L-topological spaces using the order of an α-Q-cover in terms of quasi-coincident neighborhood. A characterization of covering dimension in the weakly induced L-topological spaces is also obtained. Moreover, a characterization of covering dimension for fuzzy normal spaces is also obtained.

Conclusions

This paper provides some brief sketches regarding the topics covering dimension in L-topological spaces and covering dimension for fuzzy normal spaces. In this paper, the neighborhood structure used for the investigations is the quasi-coincident neighborhood structure (Q-nbd). There are also other types of neighborhood structures, for example, the remote neighborhood (R-nbd), in fuzzy topology. All the investigations, which have been done in this paper, can be carried out using these neighborhood structures and related tools. Also, there are several definitions of fuzzy compactness on fuzzy topological space introduced by many authors. These notions are defined using various tools such as fuzzy cover, Q-cover, α-Q-cover, etc. Among which, only N-compactness and related tools are considered in this work for the investigations. It is also possible to extend these discussions in terms of other notions of compactness, and it can obtain various notions in different ways.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors, TB and SJJ, carried out the proof. Both authors read and approved the final manuscript.

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