

ORIGINAL RESEARCH

Open Access

# Adapted linear approximation for singular integrals

Mostefa Nadir<sup>1,2,3</sup>

## Abstract

**Purpose:** The purpose of this work is to present an approximation for singular integrals of Cauchy type kernel on a smooth-oriented contour.

**Methods:** For the method, we use a small modification of the spline functions in order to eliminate the singularity. Noting that this approximation is due to the idea of Sanikidze's *Approximate solution of singular integral equations in the case of closed contours of integrations*.

**Results:** This approximation represents a good approach for any singular integral given on the curve in the sense of Cauchy principal value.

**Conclusions:** This approximation is destined to solve numerically the singular integral equations with Cauchy type kernel on a smooth-oriented contour.

**Keywords:** Singular integral, Interpolation; Hölder space, Hölder condition, Approximation theory, Spline functions

**2000 MSC:** 45D05; 45E05; 45L05; 45L10; 65R20

## Introduction

The method of numerical solution of singular integral equations of the first and second kind is convenient for application of electronic computers is necessary [1]. For equations of the first kind, many methods are developed in aerodynamics such as the method of discrete vortices. We find analogous equations in the theory of cracks.

The main consideration of the present work is the construction of a new approximation of a singular integral in order to use it for a numerical solution of singular integral equations.

$$a(t_0)\varphi(t_0) + \frac{b(t_0)}{\pi i} \int_{\Gamma} \frac{\varphi(t)}{t - t_0} dt + \int_{\Gamma} k(t, t_0)\varphi(t) dt = f(t_0), \quad (1)$$

where, under  $\Gamma$ , we designate a smooth-oriented contour;  $t$  and  $t_0$  are points on  $\Gamma$ ; and  $a(t)$ ,  $b(t)$ ,  $k(t, t_0)$  and  $f(t)$  are

given functions on  $\Gamma$ . Our schemes describe the quadrature method for the approximation of singular integral operator:

$$F(t_0) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(t)}{t - t_0} dt, \quad t, t_0 \in \Gamma, \quad (2)$$

with Cauchy kernel by a sequence of numerical integration operators. For the existence of the principal value of this integral for a given density  $\varphi(t)$ , we will need more than mere continuity. In other words, the density  $\varphi(t)$  has to satisfy the Hölder condition  $H(\mu)$  [2]. So, we note that singular integral operators of the first kind with Cauchy kernel have index zero. In particular, injective singular operators of the first kind are bijective and have bounded inverse.

## Methods

Let  $t = t(s) = x(s) + iy(s)$ , where  $s \in [a, b]$  is the parametric complex equation of the curve  $\Gamma$  with the respect to some parameter  $s$ . Consider that  $N$  is an arbitrary natural number; generally, we take it large enough and divide the interval  $[a, b]$  into  $N$  equal subintervals of  $[a, b]$ :

$$[a, b] = \{a = s_0 < s_1 < \dots < s_N = b\},$$

Correspondence: mostefanadir@yahoo.fr

<sup>1</sup>Laboratory of Pure and Applied Mathematics, University of Msila, Msila, 28000, Algeria

<sup>2</sup>Laboratory of Signals Analysis and Systems, University of Msila, Msila, 28000, Algeria

Full list of author information is available at the end of the article

be called  $I_1$  to  $I_N$ , so that, we have  $I_{\sigma+1} = [s_\sigma, s_{\sigma+1}]$ .

$$s_\sigma = a + \sigma \frac{l}{N}, \quad l = b - a, \quad \sigma = 0, 1, 2, \dots, N.$$

Further, fixing a natural number  $M > 1$  and divide each of segments  $[s_\sigma, s_{\sigma+1}]$  by the equidistant points,

$$s_{\sigma k} = s_\sigma + k \frac{h}{M}, \quad h = \frac{l}{N}, \quad k = 0, 1, \dots, M.$$

In other words, we have, for the subinterval  $[s_\sigma, s_{\sigma+1}]$ , the following subdivision [1-7]:

$$[s_\sigma, s_{\sigma+1}] = \{s_\sigma = s_{\sigma 0} < s_{\sigma 1} < \dots < s_{\sigma M} = s_{\sigma+1}\}$$

Denoted by

$$t_\sigma = t(s_\sigma), \quad t_{\sigma k} = t(s_{\sigma k}); \quad \sigma = 0, 1, 2, \dots, N; \\ k = 0, 1, \dots, M.$$

It means that the interval  $[a, b]$  is divided into  $N_1 = MN$  equal subintervals.

For an arbitrary number  $\sigma = 0, 1, 2, \dots, N - 1$ , we define the spline function  $S_1(\varphi; t, \sigma)$  depends of  $\varphi, t$  and  $\sigma$ . This latter represents the linear approximation of the function density  $\varphi(t)$  on the subinterval  $[t_\sigma, t_{\sigma+1}]$  of the curve  $\Gamma$ . Noting that the interval  $[t_\sigma, t_{\sigma+1}]$  is divided into subintervals  $[t_{\sigma k}, t_{\sigma(k+1)}]$  of length  $(t_{\sigma(k+1)} - t_{\sigma k})$  and interpolates the density function  $\varphi(t)$  with respect to the values  $\varphi(t_{\sigma k})$  and  $\varphi(t_{\sigma(k+1)})$  at the points  $t_{\sigma k}$  and  $t_{\sigma(k+1)}$ , respectively, with a linear polynomial, the following formula gives, for  $t_{\sigma k} \leq t \leq t_{\sigma(k+1)}$ :

$$S_1(\varphi; t, \sigma) = \frac{(t_{\sigma(k+1)} - t)}{(t_{\sigma(k+1)} - t_{\sigma k})} \varphi(t_{\sigma k}) \\ + \frac{(t - t_\sigma)}{(t_{\sigma(k+1)} - t_{\sigma k})} \varphi(t_{\sigma(k+1)}). \quad (3)$$

This spline function exists and is uniquely called a linear interpolating polynomial. Defined for an arbitrary numbers  $\sigma$  and  $\nu$  such that  $0 \leq \sigma, \nu \leq N - 1$ , the function  $\beta_{\sigma\nu}(\varphi; t, t_0)$  depends on  $\varphi, t$  and  $t_0$  by this equation:

$$\beta_{\sigma\nu}(\varphi; t, t_0) = \begin{cases} U(\varphi; t, \sigma) - V(\varphi; t_0, \sigma, \nu), & t \neq t_0 \\ 0, & t = t_0 \end{cases} \quad (4)$$

where the points  $t$  and  $t_0$  belong, respectively, to the arcs  $t_\sigma \widehat{t}_{\sigma+1}$  and  $t_\nu \widehat{t}_{\nu+1}$  with  $t_\alpha \widehat{t}_{\alpha+1}$  designating the smallest arc with ends  $t_\alpha$  and  $t_{\alpha+1}$ . The density  $\varphi$  represents still a given function on the curve  $\Gamma$  and of the class  $H(\mu)$  [1,5,7]. The function  $U(\varphi; t, \sigma)$  represents a modified linear interpolation of the function density  $\varphi(t)$  on the subinterval  $[t_\sigma, t_{\sigma+1}]$  of the curve  $\Gamma$ .

Indeed, for  $t_{\sigma k} \leq t \leq t_{\sigma(k+1)}$ , we put

$$U(\varphi; t, \sigma) = \frac{t_{\sigma(k+1)} - t}{t_{\sigma(k+1)} - t_{\sigma k}} \varphi(t_{\sigma k}) \frac{t - t_0}{t_{\sigma k} - t_0} \\ + \frac{t - t_{\sigma k}}{t_{\sigma(k+1)} - t_{\sigma k}} \varphi(t_{\sigma(k+1)}) \frac{t - t_0}{t_{\sigma(k+1)} - t_0},$$

and the function  $V(\varphi; t, \sigma, \nu)$  is given by

$$V(\varphi; t_0, \sigma, \nu) = \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t_{\sigma(k+1)} - t)}{(t_{\sigma k} - t_0)(t_{\sigma(k+1)} - t_{\sigma k})} \\ + \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t - t_{\sigma k})}{(t_{\sigma(k+1)} - t_0)(t_{\sigma(k+1)} - t_{\sigma k})},$$

where the function  $\varphi$  represents a given function on the curve  $\Gamma$  of the class  $H(\mu)$ . Noting that  $\psi_{\sigma\nu}(\varphi; t, t_0)$  is the quadratic approximation of the density  $\varphi(t)$  at the point  $t \in [t_\sigma, t_{\sigma+1}]$ ,  $t_0 \in [t_\nu, t_{\nu+1}]$  and  $0 \leq \sigma, \nu \leq N - 1$  by

$$\psi_{\sigma\nu}(\varphi; t, t_0) = \varphi(t_0) + \beta_{\sigma\nu}(\varphi; t, t_0), \quad (5)$$

and replacing this latter in the singular integral (2)

$$F(t_0) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(t)}{t - t_0} dt,$$

we obtain the following approximation:

$$S(\varphi, t_0) = \frac{1}{\pi i} \int_{\Gamma} \frac{\psi_{\sigma\nu}(\varphi; t, t_0)}{t - t_0} dt \\ = \varphi(t_0) + \frac{1}{\pi i} \int_{\Gamma} \frac{\beta_{\sigma\nu}(\varphi; t, t_0)}{t - t_0} dt. \quad (6)$$

## Results and discussion

**Theorem 1.** Let  $\Gamma$  be a smooth contour oriented, and let  $\varphi$  be a function density defined on  $\Gamma$  satisfies the Hölder condition  $H(\mu)$ , then the following estimation

$$|F(t_0) - S(\varphi; t_0)| \leq \max\left(\frac{C_1 \ln(MN)}{(MN)^\mu}, \frac{C_2}{N^\mu}\right) N, \quad M > 1$$

holds, where the constant  $C_1, C_2$  depends only of the contour  $\Gamma$  and the Holder's constant.

*Proof.* For any points  $t \in [t_\sigma, t_{\sigma+1}]$  and  $t_0 \in [t_\nu, t_{\nu+1}]$  with the conditions  $t_{\sigma k} \leq t \leq t_{\sigma(k+1)}$  and  $t_{\nu k} \leq t_0 \leq t_{\nu(k+1)}$ , we have the equation below:

$$\varphi(t) - \psi_{\sigma\nu}(\varphi; t, t_0) = \varphi(t) - \varphi(t_0) \\ - \left\{ \frac{t_{\sigma(k+1)} - t}{t_{\sigma(k+1)} - t_{\sigma k}} \varphi(t_{\sigma k}) \frac{t - t_0}{t_{\sigma k} - t_0} \right. \\ + \frac{t - t_{\sigma k}}{t_{\sigma(k+1)} - t_{\sigma k}} \varphi(t_{\sigma(k+1)}) \frac{t - t_0}{t_{\sigma(k+1)} - t_0} \\ - \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t_{\sigma(k+1)} - t)}{(t_{\sigma k} - t_0)(t_{\sigma(k+1)} - t_{\sigma k})} \\ \left. - \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t - t_{\sigma k})}{(t_{\sigma(k+1)} - t_0)(t_{\sigma(k+1)} - t_{\sigma k})} \right\}. \quad (7)$$

□

**Table 1 Exact principal value of singular integral, approximate calculation of the integral, and error for  $N_1 = 30$  (Example 1)**

$N_1 = 30$	Exact solution	Approximate solution	Error
	3.3415e-001 -2.3328e-002i	3.3402e-001 -2.2816e-002i	5.2769e-004
	3.3666e-001 -4.6999e-002i	3.3642e-001 -4.6499e-002i	5.5707e-004
	3.4106e-001 -7.1367e-002i	3.4068e-001 -7.0895e-002i	6.0630e-004
	3.4770e-001 -9.6807e-002i	3.4717e-001 -9.6386e-002i	6.8121e-004
	3.5714e-001 -1.2372e-001i	3.5642e-001 -1.2339e-001i	7.9148e-004

Taking into account expression (7), we get the following equation:

$$\frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(t) - \psi_{\sigma\nu}(\varphi; t, t_0)}{t - t_0} dt = \frac{1}{\pi i} \sum_{\sigma=0}^{N-1} \int_{t_{\sigma} t_{\sigma+1}} \frac{\varphi(t) - \psi_{\sigma\nu}(\varphi; t, t_0)}{t - t_0} dt; \tag{8}$$

hence,

$$F(t_0) - S(\varphi; t_0) = \frac{1}{\pi i} \sum_{\sigma=0}^{N-1} \sum_{k=0}^{M-1} \int_{t_{\sigma k} t_{\sigma(k+1)}} \frac{\varphi(t) - \varphi(t_0)}{t - t_0} - \left\{ \frac{t_{\sigma(k+1)} - t}{t_{\sigma(k+1)} - t_{\sigma k}} \varphi(t_{\sigma k}) \frac{t - t_0}{t_{\sigma k} - t_0} + \frac{t - t_{\sigma k}}{t_{\sigma(k+1)} - t_{\sigma k}} \varphi(t_{\sigma(k+1)}) \frac{t - t_0}{t_{\sigma(k+1)} - t_0} - \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t_{\sigma(k+1)} - t)}{(t_{\sigma k} - t_0)(t_{\sigma(k+1)} - t_{\sigma k})} - \frac{S_1(\varphi; t_0, \nu)(t - t_0)(t - t_{\sigma k})}{(t_{\sigma(k+1)} - t_0)(t_{\sigma(k+1)} - t_{\sigma k})} \right\} \frac{1}{t - t_0} dt,$$

seeing that the equalities  $t_{\sigma k} - t_0 = 0$  and  $t_{\sigma(k+1)} - t_0$  are possible only when  $\sigma = \nu - 1, \nu + 1$  and  $\nu$ . For the two first cases, the integral (8) exists when  $t_{\sigma k}$  tends to  $t_0$  or  $t_{\sigma(k+1)}$  tends to  $t_0$ ; on the other case, if  $\sigma = \nu$ , we can easily see that the function  $\beta_{\sigma\sigma}(\varphi; t, t_0)$  contains  $(t_{\sigma k} - t_0)$  and  $(t_{\sigma(k+1)} - t_0)$  as factors for the points  $t, t_0 \in [t_{\sigma}, t_{\sigma+1}]$

with the conditions  $t_{\sigma k} \leq t, t_0 \leq t_{\sigma(k+1)}$ , so we have the equation below:

$$\beta_{\sigma\sigma}(\varphi; t, t_0) = U(\varphi; t, \sigma) - V(\varphi; t_0, \sigma, \sigma).$$

Therefore,

$$\beta_{\sigma\sigma}(\varphi; t, t_0) = \frac{(t_{\sigma(k+1)} - t)(t - t_0)}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma k} - t_0)} (\varphi(t_{\sigma k}) - S_1(\varphi; t_0, \sigma)) + \frac{(t - t_{\sigma k})(t - t_0)}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma(k+1)} - t_0)} (\varphi(t_{\sigma(k+1)}) - S_1(\varphi; t_0, \sigma)). \tag{9}$$

It is easy to see that

$$\beta_{\sigma\sigma}(\varphi; t, t_0) = (t - t_0)Q(\varphi; t, t_0),$$

where

$$Q(\varphi; t, t_0) = \frac{(t_{\sigma(k+1)} - t)}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma k} - t_0)} (\varphi(t_{\sigma k}) - S_1(\varphi; t_0, \sigma)) + \frac{(t - t_{\sigma k})}{(t_{\sigma(k+1)} - t_{\sigma k})(t_{\sigma(k+1)} - t_0)} (\varphi(t_{\sigma(k+1)}) - S_1(\varphi; t_0, \sigma)).$$

**Table 2 Exact principal value of singular integral, approximate calculation of integral, and error for  $N_1 = 20$  (Example 2)**

$N_1 = 20$	Exact solution	Approximate solution	Error
	-2.7101e-001 -1.4676e-001i	-2.6184e-001 -1.4383e-001i	9.6317e-003
	-1.6706e-001 -2.0230e-001i	-1.6597e-001 -1.9663e-001i	5.7670e-003
	-9.0558e-002 -2.0774e-001i	-9.2594e-002 -2.0463e-001i	3.7191e-003
	-3.9496e-002 -2.0259e-001i	-4.2143e-002 -2.0147e-001i	2.8746e-003
	3.9496e-002 -2.0259e-001i	3.6897e-002 -2.0468e-001i	3.3353e-003

Note: Many examples confirm the efficiency of this approximation.

Passing now to the estimation of the expression (8), for  $t \in [t_\sigma, t_{\sigma+1}]$  and  $t_0 \in [t_\nu, t_{\nu+1}]$  with the conditions  $\sigma \neq \nu - 1, \nu + 1$  and  $\nu$ , we obtain the equation below:

$$\left| \frac{1}{\pi i} \sum_{\sigma=0}^{N-1} \sum_{k=0}^{M-1} \int_{t_{\sigma k} t_{\sigma(k+1)}} \frac{\varphi(t) - \varphi(t_0)}{t - t_0} - \left\{ \frac{t_{\sigma(k+1)} - t}{t_{\sigma(k+1)} - t_{\sigma k}} \varphi(t_{\sigma k}) \frac{t - t_0}{t_{\sigma k} - t_0} + \frac{t - t_{\sigma k}}{t_{\sigma(k+1)} - t_{\sigma k}} \varphi(t_{\sigma(k+1)}) \frac{t - t_0}{t_{\sigma(k+1)} - t_0} - \frac{t - t_0}{t_{\sigma(k+1)} - t_{\sigma k}} \left( \frac{t_{\sigma(k+1)} - t}{t_{\sigma k} - t_0} + \frac{t - t_{\sigma k}}{t_{\sigma(k+1)} - t_0} \right) S_1(\varphi; t_0, \nu) \right\} \frac{1}{t - t_0} dt \right| = O\left(\frac{\ln MN}{M^\mu N^\mu}\right).$$

Indeed, it is clear that

$$\max_{t_0 \in [t_\nu, t_{\nu+1}]} \left| \frac{1}{\pi i} \sum_{\sigma=0}^{N-1} \sum_{k=0}^{M-1} \int_{t_{\sigma k}}^{t_{\sigma(k+1)}} \frac{\varphi(t) - \varphi(t_0)}{t - t_0} \right| = O\left(\frac{\ln MN}{M^\mu N^\mu}\right);$$

also, we estimate the expression

$$\left| \frac{1}{\pi i} \sum_{\sigma=0}^{N-1} \sum_{k=0}^{M-1} \int_{t_{\sigma k} t_{\sigma(k+1)}} - \left\{ \frac{t_{\sigma(k+1)} - t}{t_{\sigma(k+1)} - t_{\sigma k}} \varphi(t_{\sigma k}) \frac{t - t_0}{t_{\sigma k} - t_0} + \frac{t - t_{\sigma k}}{t_{\sigma(k+1)} - t_{\sigma k}} \varphi(t_{\sigma(k+1)}) \frac{t - t_0}{t_{\sigma(k+1)} - t_0} - \frac{t - t_0}{t_{\sigma(k+1)} - t_{\sigma k}} \left( \frac{t_{\sigma(k+1)} - t}{t_{\sigma k} - t_0} + \frac{t - t_{\sigma k}}{t_{\sigma(k+1)} - t_0} \right) S_1(\varphi; t_0, \nu) \right\} \frac{1}{t - t_0} dt \right|$$

$$\approx \left| \frac{1}{\pi i} \sum_{\sigma=0}^{N-1} \sum_{k=0}^{M-1} \int_{t_{\sigma k} t_{\sigma(k+1)}} \frac{\varphi(t_{\nu k}) - \varphi(t_{\sigma k})}{t_{\nu k} t_{\sigma k}} + \frac{\varphi(t_{\nu(k+1)}) - \varphi(t_{\sigma(k+1)})}{t_{\nu(k+1)} - t_{\sigma(k+1)}} dt \right| = O\left(\frac{\ln MN}{M^\mu N^\mu}\right).$$

Naturally, the estimation given above is obtained with the help of the density  $\varphi$  taken as an element of the Hölder space  $H(\mu)$  [2]. Further, for the cases where  $\sigma = \nu - 1, \nu + 1$  and  $\nu$ , we obtain using again the condition  $\varphi \in H(\mu)$  and the condition of smoothness of  $\Gamma$ . With this, we obtain the following estimation:

$$\left| \frac{1}{\pi i} \int_{t_\nu, t_{\nu+1}} \frac{\varphi(t) - \varphi(t_0)}{t - t_0} dt \right| \leq A \sum_{\kappa=0}^{M-1} \int_{s_{\nu \kappa}}^{s_{\nu(\kappa+1)}} |s - s_0|^{\mu-1} ds = O(N^{-\mu})$$

## Numerical experiments

Using our approximation, we apply the algorithms to singular integrals, and we present results concerning the accuracy of the calculations. In this numerical experiments, Table 1 represents the exact principal value of the singular integral, and Table 2 corresponds to the approximate calculation produced by our approximation at point value interpolation.

**Example 1.** Consider the singular integral,

$$I = F(t_0) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(t)}{t - t_0} dt,$$

where the curve  $\Gamma$  designates the unit circle, and the function density  $\varphi$  is given by the following expression:

$$\varphi(t) = \frac{1}{t - 2}.$$

**Example 2.** Consider the singular integral,

$$I = F(t_0) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(t)}{t - t_0} dt,$$

where the curve  $\Gamma$  designates the unit circle, and the function density  $\varphi$  is given by the following expression:

$$\varphi(t) = \frac{t}{t^2 - 4}.$$

## Conclusion

We remark for this approximation based on the trapezoidal technical convergence to the exact value of the singular integrals given by the Cauchy principal value. Also, these numerical calculations show that the accuracy improves with increased number of subdivisions.

## Competing interests

The author declares that he has no competing interests.

## Author's information

MN is a PhD degree holder and a full professor in the University of Msila.

## Acknowledgements

The author would like to thank the Editor-in-Chief Prof. K. Maleknejad, the Managing Editor M. Khodabin, and the referees.

## Author details

<sup>1</sup>Laboratory of Pure and Applied Mathematics, University of Msila, Msila, 28000, Algeria. <sup>2</sup>Laboratory of Signals Analysis and Systems, University of Msila, Msila, 28000, Algeria. <sup>3</sup>Department of Mathematics, Faculty of Mathematics and Informatics, University of Msila, Msila, 28000, Algeria.

Received: 7 May 2012 Accepted: 7 July 2012

Published: 18 September 2012

## References

- Nadir, M: Problèmes aux limites qui se reduisent aux equations intégrales de Fredholm. In *Seminaire de l'Institut de Mathématiques et Informatique*, Annaba (1985)
- Muskhlishvili, NI: *Singular Integral Equations*. P. Noordhoff, Moscow (1953)
- Nadir, M: Opérateurs intégraux et bases d'ondelettes. *Far East J. Sci.* **6**(6), 977-995 (1998)

4. Sanikidze, J: On the approximate calculation of singular line integral. In *Seminar of Institute of Applied Mathematics*, Tbilissi (1970)
5. Sanikidze, J: Approximate solution of singular integral equations in the case of closed contours of integration. In *Seminar of Institute of Applied Mathematics*, Tbilissi (1971)
6. Antidze, J: On the approximate solution of singular integral equations. In *Seminar of Institute of Applied Mathematics*, Tbilissi (1975)
7. Nadir, M: On the approximation of singular integrals of Cauchy types. In *Proceeding on Dynamic System and Applications*, Turkey (2004)

doi:10.1186/2251-7456-6-36

**Cite this article as:** Nadir: Adapted linear approximation for singular integrals. *Mathematical Sciences* 2012 **6**:36.

Archive of SID

**Submit your manuscript to a SpringerOpen<sup>®</sup> journal and benefit from:**

- ▶ Convenient online submission
- ▶ Rigorous peer review
- ▶ Immediate publication on acceptance
- ▶ Open access: articles freely available online
- ▶ High visibility within the field
- ▶ Retaining the copyright to your article

---

Submit your next manuscript at ▶ [springeropen.com](http://springeropen.com)

---