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Some properties of the supersoluble formation and the supersoluble residual of a group

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Abstract

Purpose: In this paper, We determine the finite group G = HK such that K is a supersoluble subgroup of G, and H is not a supersoluble subgroup of G.

Methods: Let p, q, r be primes such that p < q < r, and p, q are not a divisor of r - 1, and p is not a divisor of q - 1. Let X be a group of order p, and let F = GF(q) and L = GF(r) such that the filed F contains a primitive pth root of unity. Let V be a simple FX-module, and let $Y = V \rtimes X$ and W also be a faithful simple LY-module. Let $G = W \rtimes Y$, $H = W \rtimes X$, and $K = W \rtimes V$.

Results: Then, we determine that *K* is a supersoluble subgroup of *G*, and *H* is not a supersoluble subgroup of *G*.

Conclusions: We characterize the supersoluble residual of group G.

Keywords: Supersoluble, Formation, \mathfrak{X} -residual, Supersoluble residual, FX-module

Introduction

This paper continues a thread of research in finite soluble groups initiated by Ballester-Bolinhes et al. [1]. It is shown in [2] that a finite group G, which is the product of two normal supersoluble subgroups, is supersoluble if and only if G' is nilpotent. Asaad and Shaalan (Theorem 3.8 in [3]) proved the following generalization of Baer's result:

Assume that a finite group G is the product of the supersoluble subgroups H and K. Assume further that G' is nilpotent. If H commutes with every subgroup of K and K commutes with every subgroup of H, then G is supersoluble.

They also prove an analogous result by considering K nilpotent instead of G' (Theorem 3.2). Later, Carocca [4] presented extensions of the preceding result considering p-supersolubility instead of supersolubility. Following Carocca [4], we say that the subgroups H and K of a group G are mutually permutable if H commutes with every subgroup of K and K commutes with every subgroup of H. If G = HK and H and K are mutually permutable, we say that G is the mutually permutable product of the subgroups H and K.

It is known that the class \mathfrak{U} of all finite supersoluble groups is a formation. This means that if a finite group *G* is supersoluble and *N* is a normal subgroup of *G*, then *G*/*N* is supersoluble, and if *M* and *N* are two normal subgroups of a finite group *G*, then *G*/($M \cap N$) is supersoluble, provided that *G*/*M* and *G*/*N* are supersoluble. Consequently, every finite group *G* has a smallest normal subgroup with a supersoluble quotient. This subgroup is called the supersoluble residual of *G*, and it is denoted by $G^{\mathfrak{U}}$. It is clear that $G^{\mathfrak{U}}$ is epimorphism-invariant, and so, it is a characteristic subgroup of *G* (see Lemma 2.4, Chapter II in [5]).

This paper focuses on the study of supersoluble subgroups and the supersoluble residual of the group G = [W] [V] X as a semidirect product and considers the subgroups $H = W \rtimes X$ and $K = W \rtimes V$ of G such that Xis the cyclic group of order p, and V is an irreducible and faithful X-module over GF(q), and $Y = V \rtimes X$ is the corresponding semidirect product, and W is an irreducible and faithful Y-module over GF(r) such that p, q and r are primes. We determine that G is the mutually permutable product of the subgroups H and K. Moreover, H is not a supersoluble subgroup of G. On the other hand, $K \in \mathfrak{U}$ and $H^{\mathfrak{U}} < W$. However, $G^{\mathfrak{U}} = W$.

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Methods

Preliminaries

Whenever possible, we follow the notation and terminology of [5,6]. All groups considered are finite.

Definition 2.1. [4]. Let G be a group and H and K be subgroups of G. We say that H and K are mutually permutable if H commutes with every subgroup of K and K commutes with every subgroup of H.

Definition 2.2. [5]. A class of groups is a collection \mathfrak{X} of groups with the property that if $G \in \mathfrak{X}$ and if $H \cong G$, then $H \in \mathfrak{X}$. We will often use the term \mathfrak{X} -group to describe a group belonging to \mathfrak{X} .

Class \mathfrak{U} denotes the class of finite supersoluble groups.

Definition 2.3. [5]. If \mathfrak{X} and \mathfrak{I} are classes of groups, we define their class product \mathfrak{XI} as follows:

 $\mathfrak{XI} = (G : G \text{ has a normal subgroup } N \in \mathfrak{X} \text{ with } G/N \in \mathfrak{I}).$

If $\mathfrak{X} = \emptyset$ or $\mathfrak{I} = \emptyset$, we have the obvious interpretation $\mathfrak{X}\mathfrak{I} = \emptyset$. For powers of a class, we set $\mathfrak{X}^0 = (1)$, and for $n \in \mathbb{N}$, make the inductive definition $\mathfrak{X}^n = (\mathfrak{X}^{n-1})\mathfrak{X}$.

Definition 2.4. [5].

(a) A class map c is called a closure operation if, for all classes X and J, the following three conditions are satisfied:

Co1: $\mathfrak{X} \subseteq c\mathfrak{X}$ *(we say* c *is expanding);*

- Co2: $c\mathfrak{X} = c(c\mathfrak{X})$ (we say c is idempotent); Co3: If $\mathfrak{X} \subseteq \mathfrak{I}$, then $c\mathfrak{X} \subseteq c\mathfrak{I}$ (we say is
- monotonic).
- (b) A class X is said to be c-closed if X = cX. (If c is a closure operation, it is clear from Co2 that cℑ is c-closed for any class ℑ.) We adopt the convention that the empty class Ø is c-closed for every closure operation c.
- (c) The product AB of two class maps is defined by composition; thus,

(AB) $\mathfrak{X} = A(B\mathfrak{X})$

for all classes \mathfrak{X} .

Definition 2.5. [5]. For a class of groups, we define: $Q\mathfrak{X}=(G: \exists H \in \mathfrak{X} \text{ and an epimorphism from H onto } G);$

 $R_{0}\mathfrak{X}=(G:\exists N_{i} \trianglelefteq G(i=1,\ldots,r) \text{ with } G/N_{i} \in \mathfrak{X} \text{ and } \bigcap_{i=1}^{\prime} N_{i}=1);$ $E_{\phi}\mathfrak{X}=(G:\exists N \trianglelefteq G \text{ with } N \le \Phi(G) \text{ and } G/N \in \mathfrak{X}).$ **Definition 2.6.** [5]. A formation is a class of groups that is closed under both Q and R_0 .

Corollary 2.7. Let \mathfrak{X} be a class of groups, then \mathfrak{X} is a formation if and only if the following two conditions are satisfied for the class \mathfrak{X} :

- (1) If $G \in \mathfrak{X}$ and $N \trianglelefteq G$, then $G/N \in \mathfrak{X}$.
- (2) If N₁ and N₂ are normal subgroups of group G such that G/N₁ ∈ X and G/N₂ ∈ X and N₁ ∩ N₂ = 1, then G ∈ X.

Proof. Straightforward.

Definition 2.8. [5]. An E_{ϕ} -closed classs is called saturated.

Corollary 2.9. Let \mathfrak{X} be a formation. Then, \mathfrak{X} is saturated if and only if for all finite groups $G, G/\Phi(G) \in \mathfrak{X}$ implies $G \in \mathfrak{X}$.

Some properties of the supersoluble formation

Proof. Straightforward.

We study in this section some properties of the supersoluble formation \mathfrak{U} . The next result includes the definition of the \mathfrak{X} -residual $G^{\mathfrak{X}}$ of a group G; it always exists if the class $\mathfrak{X}(\neq \emptyset)$ is R_0 -closed, and it is epimorphism-invariant when \mathfrak{X} is a formation.

Corollary 3.1. The class \mathfrak{U} is a saturated formation.

Proof. By Huppert's Theorem [7], it is straightforward. \Box

Lemma 3.2. (Lemma 2.4, Chapter II in [5]). Let \mathfrak{X} be an R_0 -closed class and G a finite group. Then the set L = $\{N \leq G : G/N \in \mathfrak{X}\}$, partially ordered by inclusion, has a unique minimal element, denoted by $G^{\mathfrak{X}}$ and called the \mathfrak{X} -residual of G. It is a characteristic subgroup, and if \mathfrak{X} is a formation and $\varepsilon : G \to \varepsilon(G)$ is an epimorphism, then $\varepsilon(G)^{\mathfrak{X}} = \varepsilon(G^{\mathfrak{X}})$.

Corollary 3.3. Let G be a finite group. Then,

- (1) If $H \leq G$ and $G/H \in \mathfrak{U}$, then $G^{\mathfrak{U}} \leq H$;
- (2) If $A \leq G$ and $H \leq G$, then $(\frac{HA}{A})^{\mathfrak{U}} = \frac{H^{\mathfrak{U}}A}{A}$;
- (3) If $H \leq G$, then $H^{\mathfrak{U}} \leq G^{\mathfrak{U}}$.

Proof. Straightforward.

Lemma 3.4. Let G be a finite group and H be a subgroup of G such that $(\frac{G}{A})^{\mathfrak{U}} = (\frac{HA}{A})^{\mathfrak{U}}$ where A is a normal subgroup of G. Then, $G^{\mathfrak{U}}A = H^{\mathfrak{U}}A$. Moreover, if $A \leq G^{\mathfrak{U}}$, then $G^{\mathfrak{U}} = H^{\mathfrak{U}}A$.

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Proof. By Corollary 3.3, $(\frac{HA}{A})^{\mathfrak{U}} = \frac{H^{\mathfrak{U}}A}{A}$. On the other hand, $(\frac{G}{A})^{\mathfrak{U}} = (\frac{GA}{A})^{\mathfrak{U}} = \frac{G^{\mathfrak{U}}A}{A}$. So, $\frac{G^{\mathfrak{U}}A}{A} = \frac{H^{\mathfrak{U}}A}{A}$, and then, $G^{\mathfrak{U}}A = H^{\mathfrak{U}}A$. If $A \leq G^{\mathfrak{U}}$, then $G^{\mathfrak{U}} = G^{\mathfrak{U}}A$. Therefore, $G^{\mathfrak{U}} = H^{\mathfrak{U}}A$.

Proposition 3.5. Let G be a finite group, A be a minimal normal subgroup of G, and H be a subgroup of G. If $(\frac{G}{A})^{\mathfrak{U}} = (\frac{HA}{A})^{\mathfrak{U}}$, then either $A \leq G^{\mathfrak{U}}$ or $H^{\mathfrak{U}} = G^{\mathfrak{U}}$.

Proof. By Lemma 3.4, $G^{\mathfrak{U}}A = H^{\mathfrak{U}}A$. So $H^{\mathfrak{U}}(A \cap G^{\mathfrak{U}}) = H^{\mathfrak{U}}A \cap G^{\mathfrak{U}} = G^{\mathfrak{U}}A \cap G^{\mathfrak{U}} = G^{\mathfrak{U}}$; therefore, $H^{\mathfrak{U}}(A \cap G^{\mathfrak{U}}) = G^{\mathfrak{U}}$. On the other hand, $1 \leq A \cap G^{\mathfrak{U}} \leq A$ and $A \cap G^{\mathfrak{U}} \leq G$. So, either $A \cap G^{\mathfrak{U}} = A$ or $A \cap G^{\mathfrak{U}} = 1$, and the proof is completed.

The supersoluble residual of a group

All modules are right modules unless the contrary is stated.

Definition 4.1. A module is said to be simple (irreducible) if

- (1) it is non-zero, and
- (2) the only proper submodule that it possesses is the zero submodule.

An *R*-module *M* is called *R*-semisimple if *M* is a direct product of finitely many simple *R*-submodules.

Definition 4.2. [8]. If G is a group and R is any ring with an identity element, the group ring RG is defined to be the set of all formal sums $\sum_{x \in G} r_x x$ where $r_x \in R$ and $r_x = 0$ with

finitely many exceptions, together with the rules of addition and multiplication

$$(\sum_{x} r_{x}x) + (\sum_{x} r'_{x}x) = \sum_{x} (r_{x} + r'_{x})x;$$

and

$$(\sum_{x} r_x x)(\sum_{x} r'_x x) = \sum_{x} (\sum_{yz=x} r_y r'_z) x.$$

It is very simple to verify with these rules that RG is a ring with identity element $1_R 1_G$, which is simply written as 1.

Remark 4.3. If F is a field, then FG, in addition to being a ring, has a natural F-module structure given by

$$f(\sum_{x} f_x x) = \sum_{x} (ff_x)x, \qquad (f \in F)$$

Thus, FG is a vector space over F and $Dim_F(FG) = |G|$ *.*

Definition 4.4. The product of all the abelian minimal normal subgroups of a group G is called the abelian component of the socle and is denoted by Soc(G).

Theorem 4.5. (*Theorem 10.3, Chapter B in [5]*). Let G be a finite group and K an arbitrary field. Then, the following conditions are equivalent:

- (a) G has a faithful simple module over K;
- (b) $Soc_{\mathfrak{U}}(G)$ has a subgroup N such that
 - (1) $Core_G(N)=1$, and
 - (2) $Soc_{\mathfrak{U}}(G)/N$ is cyclic and is a p'-group if char(K) = p > 0.

Corollary 4.6. Let p,q be primes and X be a group of order p. Let F be the Galois field F = FG(q). Then, X has a faithful simple module over F.

Lemma 4.7. Let F be a field and X be a finite group. If V is a irreducible FX-module, then V is a vector space over F of finite dimension.

Proposition 4.8. (Lemma 9.2, Chapter B in [5]). Let G be an abelian group of order n, let K be a field, and let V be a simple KG-module. If either

- the polynomial xⁿ − 1 splits into a product of linear factors in K[x](in particular, if K contains a primitive nth root of unity), or
- (2) V is absolutely irreducible,

then $Dim_K(V) = 1$.

Corollary 4.9. Let p,q be primes and X be a group of order p, let F=GF(q) and F contain a primitive pth root of unity. If V is a simple FX-module, then $Dim_F(V)=1$. Moreover, |V| = q.

Lemma 4.10. Let *K* be a field of prime characteristic *p* and *G* be a finite group. Let *W* be a *KG*-module. Then, *W* is an elementary abelian *p*-group.

Proof. For every $w \in W$, $pw = \underbrace{w + ... + w}_{p \text{ times}} = w(1_F 1_G) + ... + w(1_F 1_G) = w(\underbrace{1_F 1_G + ... 1_F 1_G}_{p \text{ times}}) = w((1_F + ... + 1_F) 1_G) = 0$. So, the abelian group (W, +) is a

 $\dots + 1_F(1_G) = 0$. So, the abelian group (W, +) is a p-elementary abelian group.

Results and discussion

Theorem 5.1. Let p,q,r be distinct primes such that p < q < r. Let X be a group of order p, and let F = GF(q) and www.SID.ir

 $\overline{K} = GF(r)$ such that the field F contains a primitive pth root of unity. Let V be a simple FX-module over F, and let $Y = V \rtimes_{\varphi} X$ such that for all $x \in X$ and for all $v \in V$, $v\varphi_x = v_x(=v(1_Fx))$ where $\varphi_x \in Aut(V)$ and W also be a simple KY-module over \overline{K} . If $G = W \rtimes_{\psi} Y$ = such that for all $y \in Y$, $\psi(y) = \psi_y$, and for all $w \in W$, $w\psi_y = wy$ and $H = W \rtimes X$ and $K = W \rtimes V$. Then, G is the product of the mutually permutable subgroups H and K.

Proof. It is easy to verify that φ and ψ are well defined because, by Lemma 4.7, *V* is a vector space over *F* of finite dimension, and *W* is a vector space over \overline{K} of finite dimension. By Lemma 4.10, $|W| = r^{\alpha}$ such that α is a nonnegative integer. On the other hand, by Corollary 4.9, |V| = q. Thus, $|G| = pqr^{\alpha}$ and $|H| = pr^{\alpha}$ and $|K| = qr^{\alpha}$. Therefore, |G:K| = p and $K \leq G$. Therefore, *K* commutes with every subgroup of *H*. Let *x* be an arbitrary element of *X* and *w* be an arbitrary element of *W*. Let $V = \langle v \rangle$; then,

$$(0, (v, o)) + (w, (0, x)) = (w\psi_{-(v,0)}, (v, 0) + (0, x))$$
$$= (w\psi_{-(v,0)}, (v, x)).$$

Now, let $t \in \mathbb{Z}$, $w' \in W$ and $x' \in X$. Then,

$$(w', (0, x')) + (0, (tv, 0)) = (w' + 0\psi_{-(0, x')}, (0, x') + (tv, 0))$$
$$= (w', ((tv)\varphi_{-x'}, x')).$$

Let x' = x and $w' = w\psi_{-(\nu,0)}$. There is a $t \in \mathbb{Z}$ such that $\nu\varphi_x = t\nu$, $(t\nu)\varphi_x^{-1} = \nu$; this means that $(t\nu)\varphi_{-x} = \nu$. Therefore,

$$\begin{aligned} &(0, (v, o)) + (w, (0, x)) = (w\psi_{-(v,0)}, (v, x)) \\ &= (w', ((tv)\varphi_{-x}, x)) = (w', (0, x)) + (0, (tv, 0)) \\ &= (w\psi_{-(v,0)}, (0, x)) + (0, (tv, 0)) \\ &= (w\psi_{-(v,0)}, (0, x)) + t(0, (v, 0)). \end{aligned}$$

Let $h \in H$ and $v_1 \in V$, then $v_1 = mv$ where $m \in \mathbb{Z}_q$, so $v_1 + h = mv + h$. Consequently, $v_1 + h = (m-1)v + v + h$; therefore, $v_1 + h = (m - 1)v + h' + tv$ where $t \in \mathbb{Z}$ and $h' \in H$. There is a $s \in \mathbb{Z}$ such that tv = s(mv), so $v_1 + h = (m-1)v + h' + s(mv)$. Therefore, $v_1 + h = h_1 + s'v_1$ where $h_1 \in H$ and $s' \in \mathbb{Z}$. Now, let $K_1 \leq K$ and $|K_1| = qr^{\beta}$ where $0 \leq \beta \leq \alpha$. We prove that *H* commutes with *K*₁. Let $W' = \{(w, (0, 0)) | w \in W\}$ and $V' = \{(0, (v, 0)) | v \in V\}$ and $X' = \{(0, (0, x)) | x \in X\}$. We know that $W' \leq G$. Let $T \in Syl_r(K_1)$, then $n_r(K_1) = 1$. This means that $T \leq K_1$. Let $S \in Syl_q(K_1)$, then $S \in Syl_q(K)$. Therefore, $S = V'^k$ where $k \in K$. On the other hand, K = W' + V'. So, $k = w_1 + v_1$ such that $v_1 \in V'$ and $w_1 \in W'$; therefore, $V'^{k} = -v_1 - w_1 + V' + w_1 + v_1$. We know that $w_1 + v_1 = v_1 + w'_1$ where $w'_1 \in W'$, so $(V')^k = -(w_1 + w'_1)^k$ v_1) + V' + (w_1 + v_1) = -(v_1 + w'_1) + V' + (v_1 + w'_1) =

 $-w'_1 - v_1 + V' + v_1 + w'_1 = w'_1 + V' + w_1 = (V')^{w'_1}$. Therefore, $S = (V')^{w'_1}$. $S \cap T = 1$, so $K_1 = S + T = T + S$. Let $h \in H, t \in T$, and $s \in S$, then T is a r-subgroup of G and $T \leq \mathcal{N}_G(H)$. Therefore, (s + t) + h = s + (t + h) = $s + (h' + t) = -w'_1 + (w'_1 + s - w'_1) + w'_1 + (h' + t),$ where $h' \in H$. Let $h_1 = w'_1 + h'_1$ where $h_1 \in H$, then $(s + t) + h = -w'_1 + (w'_1 + s - w'_1) + h_1 + t = -w'_1 + w'_1 + t_1 + t_2 = -w'_1 + w'_1 + w'_1 + t_2 = -w'_1 + w'_1 + w'_1 + w'_1 + w'_2 = -w'_1 + w'_2 = -w'_2 =$ $h'_1 + m(w'_1 + s - w'_1) + t$ where $\dot{h'_1} \in H$ and $m \in \mathbb{Z}_q$. On the other hand, $m(w'_{1} + s - w'_{1}) = w'_{1} + ms - w'_{1}$ and $ms - w'_1 = w'' + ms$ where $w'' \in W'$. Therefore, $(s+t) + h = -w'_1 + h'_1 + w'_1 + w'' + ms + t \in H + K_1.$ This implies that $K_1 + H \subseteq H + K_1$. Consequently, $K_1 + H = H + K_1$; this means that *H* commutes with K_1 . Let $L \leq K$ and $|L| = r^m$ such that $0 \leq m \leq \alpha$, then L is a r-subgroup of *G* and $L \leq W' \leq H \leq \mathcal{N}_G(H)$. Therefore, L+H = H+L. Now, let $L \le K$ and |L| = q, then $L = (V')^k$ where $k \in K$. We know that K = W' + V' (= V' + W'), so let $k = x + w_1$ such that $x \in V'$ and $w_1 \in W'$. Therefore, $(V')^k = (V')^{x+w_1} = (V')^{w_1}$. Let $t \in L$ and $h \in H$, then $l + h = -w_1 + (w_1 + l - w_1) + w_1 + h =$ $-w_1 + (w_1 + l - w_1) + h_1$, where $h_1 \in H$. Therefore, $l + h = -w_1 + h' + m(w_1 + l - w_1)$, where $h' \in H$ and $m \in \mathbb{Z}_q$. So, $l + h = -w_1 + h' + w - 1 + ml - w_1$. This yields $l + h = -w_1 + h' + w_1 + w'_1 + ml$ where $w'_1 \in W'$. Therefore, $l + h = -w_1 + h' + w_1 + w'_1 + ml \in$ H + L; this means that H commutes with L. Consequently, *H* commutes with every subgroup of *K*. Let $(w, (v, x)) \in$ G, then $(w, (v, 0)) + (0, (0, x)) = (w + 0\psi_{-(v,0)}, (v, 0) +$ $(0,x) = (w, (v,x)), \text{ where } (w, (v,0)) \in K \text{ and } (0, (0,x)) \in K$ H. This implies G = H + K, and the proof is completed.

Theorem 5.2. Let the conditions of Theorem 5.1 be valid and p, q to be not a divisor of r - 1 and p to be not a divisor of q - 1. If the simple KY-module W will be faithful over K, then H is not a supersoluble subgroup of G.

Proof. Let H be supersoluble. We also let $|W| = r^{\alpha}$ where α is a non-negative integer. If |W| = 1, then Aut(W) = 1; this means that $Y = ker\psi = 1$ (because W is a faithful simple KY-module over K), a contradiction. Let |W| = r, then $Aut(W) \cong \mathbb{Z}_{r-1}$. Therefore, $\frac{Y}{keryl} \hookrightarrow \mathbb{Z}_{r-1}$. This implies that $Y \hookrightarrow \mathbb{Z}_{r-1}$, a contradiction. Thus, $|W| = r^{\alpha}$ where $\alpha \ge 2$. If X is a maximal subgroup of H, then by Huppert's Theorem [7], |H : X|is a prime, a contradiction. Therefore, X is not a maximal subgroup of H. Let M be a maximal subgroup H such that *M* contains *X*. Let $|H : M| = p_1$ where p_1 is a prime, and let |M| = pk where $k \in \mathbb{Z}$, then $|H| : M| = p_1$, so $p_1|r^{\alpha}$. This implies that $p_1 = r$. So, $|M| = pr^{\alpha-1}$. Let $|Core_H(M)| = r^m p^n$ where $0 \le n \le 1$ and $0 \le n$ $m \leq \alpha - 1$. On the other hand, $\frac{H}{Core_H(M)} \hookrightarrow S_{|H:M|}$ where $S_{|H:M|}$ is the symmetric group on |H| : M| letters.

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Therefore, $r^{\alpha-m}p^{1-n}|r!$. If $\alpha-m \ge 2$, then $r^2|r^{\alpha-m}p^{1-n}|r!$, a contradiction. So, $\alpha - m = 1$; this means that $|Core_H(M)| = r^{\alpha-1}p^n$. If n = 1, then $M = Core_H(M)$. This yields $M \leq H$. If n = 0, then $|Core_H(M)| = r^{\alpha - 1}$. So, $\left|\frac{H}{Core_{H}(M)}\right| = pr$. On the other hand, $\left|\frac{M}{Core_{H}(M)}\right| = p$. Therefore, $\frac{M}{Core_{H}(M)} \in Syl_{p}(\frac{H}{Core_{H}(M)})$ and $n_{p}(\frac{H}{Core_{H}(M)})|r$, then $n_{p}(\frac{H}{Core_{H}(M)}) = 1$; this implies that $\frac{M}{Core_{H}(M)} \leq \frac{M}{Core_{H}(M)}$ $\frac{H}{Core_{H}(M)}$. So, $M \leq H$. Therefore, M is supersoluble. If $\alpha = 2$, then |M| = pr. We know that $X \in Syl_p(M)$ and $n_p(M) = 1$, then $X \leq H$, a contradiction. So, $\alpha \geq 3$. X is not a maximal subgroup of M. Therefore, M has a maximal subgroup M_1 such that $X \leq M_1$. Similarly, we prove that $s|M_1| = pq^{\alpha-2}$ and $M_1 \leq M$. Let $M_0(=$ *M*), ..., $M_{\alpha-2}$ be subgroups of *G* such that $X \leq M_i$ and $M_i \leq M_{i-1}$ and $|M_i| = pr^{\alpha-i-1}$, $(i = 1, ..., \alpha - 2)$. So, $|M_{\alpha-2}| = pr$. Therefore, $n_p(M_{\alpha-2}) = 1$ and $X \leq M_{\alpha-2}$, then $X \in syl_p(M_{\alpha-2})$. This means that $XchM_{\alpha-2} \trianglelefteq M_{\alpha-3}$, so $X \leq M_{\alpha-3}$. Inductively, we have $XchM \leq H$. So, $X \leq$ H, a contradiction. Consequently, we imply that H is not supersoluble.

Theorem 5.3. Let p,q be primes such that p < q. Let G be a finite group and W,X be subgroups of G such that G = WX and $|W| = q^{\alpha} (\alpha \in \mathbb{N})$ and |X| = p. Also, let W be an abelian subgroup of G. If [W, X] < W, then $G^{\mathfrak{U}} < W$.

Proof. Let T = [W, X], so $T = [W, X] \leq \langle W, X \rangle = G$. Let $w \in W$ and $x \in T$; therefore, [wT, xT] = T. Thus, $[\frac{W}{T}, \frac{XT}{T}] = 1$, then $\frac{W}{T} \leq C_{\frac{G}{T}}(\frac{XT}{T})$. If $|X \cap T| = p$, then $X \cap T = X$; this means that $X \leq T$, a contradiction. Consequently, $|X \cap T| = 1$. This yields $|\frac{XT}{T}| = p$, then $\frac{XT}{T}$ is abelian. So, $\frac{XT}{T} \leq C_{\frac{G}{T}}(\frac{XT}{T})$. On the other hand, $n_q(G) = 1$. We have $\frac{G}{T} = \frac{W}{T}\frac{XT}{T} \leq C_{\frac{G}{T}}(\frac{XT}{T})$; this yields $\frac{G}{T} = C_{\frac{G}{T}}(\frac{XT}{T})$, so $\frac{XT}{T} \leq \frac{G}{T}$. So, $\frac{G}{T} \in \mathfrak{U}$; therefore, $G^{\mathfrak{U}} \leq T < W$, and the proof is completed.

Proposition 5.4. Let p be a prime, K = GF(p), H be a finite group, and W be an irreducible KH-module. Then, $G = W \rtimes_{\varphi} H$ is a group such that for all $h \in H$, $\varphi(h) = \varphi_h$ and for all $w \in W$ $w\varphi_h = wh(=w(1_K h))$, and W also is a minimal normal subgroup of G.

Proof. It is easy to verify that the φ is well defined; this means that for every $h \in H$, $\varphi_h \in Aut(W)$. Thus, G is a group, and $W' = \{(w, 0) | w \in W\}$ is a normal subgroup of G. Let $T \trianglelefteq G$ and $T \le W'$ and also $W_1 = \{w \in W | (w, 0) \in T\}$; this implies that $W_1 \le W$. $G_1 = T + H_1 \le$ G where $H_1 = \{(0, h) | h \in H\}$. Let $w \in W_1$ and $h \in H$. So, $(0, -h) + (w, 0) = (w\varphi_h, -h) \in G_1$, and this yields $(w\varphi_h, 0) \in T$, so $w\varphi_h \in W_1$. Let a be an arbitrary element of K. $w(\underline{1_Kh + ... + 1_Kh}) = w(1_Kh) + ... + (1_Kh) = w\varphi_h + \underline{w_h + w_h}$ $\begin{array}{ll} \dots + w\varphi_h \in W_1. \text{ So, } w(ah) \in W_1. \text{ Now, let } w_1 \in W \text{ and } \lambda = \\ \sum\limits_{h \in H} a_h h \in KH, \text{ then } w\lambda = \\ \sum\limits_{x \in H} w\lambda_x \text{ such that } \lambda_x = \\ \sum\limits_{h \in H} b_h^x h \\ \text{where } b_h^x = \begin{cases} a_x h = x \\ 0h \neq x \end{cases} \text{ So, } w\lambda_x = w(a_x x); \text{ therefore,} \\ 0h \neq x \end{cases} \text{ for } w\lambda_x = \\ \sum\limits_{x \in H} w(a_x x) = w(\sum\limits_{x \in h} a_x x) = w\lambda. \text{ This means} \\ \text{that for every } w \in W \text{ and } \lambda = \\ \sum\limits_{x \in H} a_x x, w\lambda = \\ \sum\limits_{x \in H} w(a_x x). \\ \text{Thus, for all } x \in H \text{ and for all } w \in W_1, w(a_x x) \in W_1. \text{ So,} \\ w\lambda \in W_1; \text{ this means that } W_1 \text{ is a KH-module. So, either} \\ W_1 = 0 \text{ or } W_1 = W \text{ because } W \text{ is an irreducible KH-module and } W_1 \leq W. \text{ Therefore, either } T = 0 \text{ or } T = W'; \\ \text{this implies that } W' \text{ is a minimal normal subgroup of } \\ \text{G.} \\ \end{array}$

Theorem 5.5. By hypothesis of Theorem 5.1, $G^{\mathfrak{U}} = W'$ such that $W' = \{(w, (0, 0)) | w \in W\}$.

Proof. We know that |Y| = pq, then if M is a maximal subgroup of Y, then either |M| = p or |M| = q. By Huppert's Theorem [7], Y is supersoluble. On the other hand, $\frac{G}{W'} \cong Y$, so $\frac{G}{W'}$ is supersoluble, and then, $G^{\mathfrak{U}} \leq W'$. We know that $G^{\mathfrak{U}} \neq 1$ (because by Theorem 5.2, H is not a supersoluble subgroup of G), by Proposition 5.4, W' is a minimal subgroup of G, then $G^{\mathfrak{U}} = w'$.

Proposition 5.6. [5]. Let V be a simple KG-module, let $N \leq G$, and let W be a simple submodule of V_N . Then, the subset $W_g = \{wg | w \in W\}$ of V is a simple submodule of V_N , and $V = \bigoplus_{g \in G} W_g$. In particular, V_N is a semisimple KN-module.

Proposition 5.7. (*Proposition 3.2 in [9]*). Let M be an *R*-module. Then, the following statements are equivalent:

(a) *M* has a family $\{S_i\}_{i \in I}$ of simple submodules such that $M = \bigoplus S_i(d.s)$;

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- (b) M has a family of simple submodules whose sum is M itself;
- (a) every submodule of M is a direct summand of M.

Theorem 5.8. Let the hypothesis of Theorem 5.1 be valid. Then, $K \in \mathfrak{U}$.

Proof. We know that $|K| = r^{\alpha}q$ where $\alpha \in \mathbb{N}$. If $\alpha = 1$, then by Huppert's Theorem [7], $K \in \mathfrak{U}$. Let $\alpha \ge 2$ and W_1 be a simple $\overline{K}V$ -module of W_V where W_V is a semisimple $\overline{K}V$ -module. By Proposition 4.8, $Dim_K(W_1) = 1$. Therefore, $|W_1| = r$. By Clifford's Theorem [5,10], $W = \bigoplus_{y \in Y} W_1 y$ such that for all $y \in Y$, $W_1 y$ is a simple $\overline{K}V$ -module of W_V . By Proposition 5.7, $W = \bigoplus_{i \in I} W_1 y_i(d.s)$

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where $\{y_i | i \in I\} \subseteq Y$. So $|W| = |\bigoplus_{i \in I} W_1 y_i| = r^{|I|}$. If |W| = r^{α} , then $|I| = \alpha$. Therefore, W_V has a KV-module W' such that $|W'| = r^{\alpha - 1}$. Now, let *M* be a maximal subgroup of *K* such that $|M| = q^{\gamma} r^{\beta}$ where $0 \leq \beta \leq \alpha$ and $0 \leq \gamma \leq 1$. If $\gamma = 0$, then $|M| = r^{\beta}$. We know that $n_r(K) = 1$ and M is a *r*-subgroup of *K*, then $M < W_I = \{(w, (0, 0)) | w \in W\}$. Therefore, $M = W_I$. Consequently, |K: M| = q. If $\gamma = 1$, then $|M| = r^{\beta}q$. Let $\beta = 0$, then |M| = q, so $M = V_I^k$ where $V_I = \{(0, (v, 0)) | v \in V\}$ and $k \in K$. Therefore, V_I is a maximal subgroup of K. Let $W'_1 = \{(w, (0, 0)) | w \in$ W_1 }, then $G_1 = W'_1 + V_I$ is a subgroup of K. Consequently, $V_I \leq G_1$, a contradiction. Therefore, $\beta \geq 1$. Let $W'' \in Syl_r(M)$ and $V_1 \in Syl_g(M)$, then $M = W'' + V_1$. This implies that $M = W'' + (V_I)^k$ where $k \in K$. We know that $n_r(K) = 1$, then $W'' \leq W_I$; on the other hand, $M^{-k} = (W'')^{-k} + V_I$. Let $S = \{w | (w, (0, 0)) \in$ $(W'')^{-k}$. Let $w \in S$ and $v \in V$. (0, (-v, 0)) + (w, (0, 0)) = $(w\psi_{(\nu,0)}, (-\nu, 0) + (0, 0)) = (w\psi_{(\nu,0)}, (-\nu, 0)) \in M^{-k}.$ Therefore, there are $w_1 \in S$ and $v_1 \in V$ such that $(w\psi_{(\nu,0)}, (-\nu, 0)) = (w_1, (0, 0)) + (0, (\nu_1, 0))$. On the other hand, $(w_1, (0, 0)) + (0, (v, 0)) = (w_1, (0, 0) + (v_1, 0)) =$ $(w_1, (v_1, 0))$. So, $w\psi_{(v,0)} = w_1 \in S$. Consequently, there exists $i \in I$ such that $(W_1y_i)' = \{(w, (0, 0)) | w \in W_1y_i\} \leq$ $(W'')^{-k}$. Since, if for every $i \in I$, $\{(w, (0, 0) | w \in W_1 y_i\} \leq$ $(W'')^{-k}$ then $W_I \leq (W'')^{-k}$. Therefore, $(w'')^{-k} = W_I$. This yields $\beta = \alpha$, and this means that $M^{-k} = K$, a contradiction. Since $(W_1y_i)' \leq (W')^{-k}$, this implies that $(W_1y_i)' \cap (W'')^{-k} = 1$. So, $|(W_1y_i)' + (W'')^{-k}| = r^{\beta+1}$. Let $G' = ((W_1y_i)' + (W'')^{-k}) + V_I$; the G'is a subgroup of K. We know that $|G'| = r^{\beta+1}q, M^{-k} \leq G'$ and M^{-k} is a maximal subgroup of K. Consequently, G' = K and $\beta + 1 = \alpha$. So, $|M^{-k}| = r^{\alpha - 1}q$ and |K : M| = r. By Huppert's Theorem [7], K is supersoluble, and the proof is completed.

Conclusions

All our previous results show that the subgroup K of the finite group G = HK is a supersoluble subgroup of G, and the subgroup H is not a supersoluble subgroup of G. Let p, q, r be primes such that p < q < r, and p, q are not a divisor of r - 1, and p is not a divisor of q - 1. Let X be a group of order p, and let F = GF(q) and L = GF(r) such that the filed F contains a primitive pth root of unity. Let V be a simple FX-module, and let $Y = V \rtimes X$ and W also be a faithful simple LY-module. Let $G = W \rtimes Y, H = W \rtimes X$, and $K = W \rtimes V$. Then, we determine that K is a supersoluble subgroup of G, and H is not a supersoluble subgroup of G, and we also characterize the supersoluble residual of group G.

Competing interests

The author declares that he has no competing interests.

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References

- Ballester-Bolinhes, A, Pedraza-Aguilera, MC, Pe'rez-Ramps, MD: On finite products of totally permutable group. Bull. Austral. Math. Soc. 53, 441–445 (1996)
- Baer, R: Classes of finite groups and their properties. Illinois J. Math. 1, 115–187 (1957)
- Asaad, M, Shaalan, A: On the supersolvability of finite groups. Arch. Math. 53, 318–326 (1989)
- 4. Carocca, A: p-supersolvability of factorized finite groups. Hokkaido Math. J. **21**, 395–403 (1992)
- 5. Doerk, K, Hawkes, T: Finite soluble groups. De Gruyter, New York (1992)
- Ghalandarzadeh, SH, Malakoti Rad, P, Shirinkam, S: Multiplication modules and Cohens theorem. Mathematical Sci: QJ. 2(3), 251–260 (2008)
- Huppert, B: Normalteiler und maximale Untergruppen endlicher Gruppen. Math. Z. 60, 409–434 (1954)
- Robinson, DJS: A course in the theory of groups. 2nd edn. Springer, New York (1996)
- Sharpe, DW, Vamos, P: Injective module. Cambridge University Press, Cambridge (1972)
- Mohamadzadeh, B. Yousofzadeh, A: A note on weak amenability of semigroup algebras. Mathematical Sci.: Q. J. 3(3), 241–246 (2009)

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