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On the existence and uniqueness of solutions of a class of initial value problems of fractional order

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Abstract
In this paper, sufficient conditions for the existence and uniqueness of solution are studied for a class of initial value problem of fractional order, involving the Caputo-type derivative of a hypergeometric fractional operator applying fixed point theory. Examples are also provided to illustrate the results.

Keywords: Caputo-type fractional derivative, Initial value problem, Fixed point theorems

Introduction

Fractional calculus is nearly as old as the classical calculus [1]. The fractional differential equations are used for the better modeling of the physical processes in various fields of science and engineering disciplines such as fluid mechanics, viscoelasticity, mathematical biology, bioengineering, control theory, signal processing, circuit analysis, seismology, etc. The interest of the study of fractional-order models among researchers and scientists is due to the fact that fractional-order models are more accurate than integer-order models, i.e. fractional-order models can provide more degrees of freedom than the integer-order system. For more details on the theory of fractional calculus, refer to the monographs [2-7].

In many applications, the results governed by the fractional differential equations cannot be solved explicitly. In such situations, we often resort to geometric or numerical analysis of the fractional differential equations for information about the solution instead of solving them. However, before such analysis, it is necessary to know whether solutions actually exist along with their domain. Furthermore, it is well known that specifying an initial value is enough to uniquely determine a solution. Therefore, the question of the existence of solutions for fractional differential equations is in its infancy as very few results are available in the literature. In the last several decades, many researchers have studied different types of nonlinear fractional differential equations; we mention [8-21] and survey paper [22] and the references therein.

The classical fractional calculus is based on several definitions for the operators of integration and differentiations of arbitrary order [23]. Among the various definitions of fractional derivatives, the Riemann-Liouville and Caputo's fractional derivatives are widely used in the literature. However, the Riemann-Liouville fractional derivative leads to a conflict of interest between the well-established mathematical theory, such as the initial value problem of fractional order and non-zero problem related to the derivative of a constant. The main advantage on Caputo's fractional derivative is that it allows consideration of easily interpretable initial conditions [4].

While studying the scale-invariant solutions of a time fractional diffusion-wave equation, Gorenflo et al. [24] introduced the Caputo-type modification of the Erdéyi-Kober fractional derivative which was further studied by Luchko and Trujillo [25]. Recently, Rao et al. [26] introduced and studied the Caputo-type modification of the Saigo fractional operators [27-29].

Motivated by the work [14,30], we investigate the existence of solution of nonlinear fractional initial value problems of the form

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$$^{C}D_{0+}^{\alpha,\beta,\gamma}u(t) = f\left(t, u(t), {}^{C}D_{0+}^{\sigma,\delta,\gamma}u(t)\right), \quad t \in (0,1], \tag{1}$$

$$u^{(k)}(0) = c_k, \tag{2}$$

where $\alpha \in (m-1, m)$, $\sigma \in (n-1, n)$, $m, n \in \mathbb{N}$, $\alpha > \sigma$; β, γ, δ are real numbers and $c_k \geq 0$; $k = 0, 1, 2, \cdots, m-1$.

The equations like the simple harmonic fractional oscillator [31,32] and the Bagley-Torvik equation [33] are the particular case of the initial value problems (1) and (2).

This paper comprises of three sections. In the 'Preliminaries' section, we provide some basic definitions and properties of the fractional operators. We also state Banach's contraction mapping principle, Schauder's fixed point theorem and the nonlinear alternative of Leray and Schauder. The results related to our main findings have been discussed in the 'Related results' section. In the 'Main results' section, we have established some sufficient conditions for the existence and uniqueness for the initial value problems (1) and (2).

Preliminaries

For the convenience of the reader, we present here some notations and lemmas that will be used in the proof of our main results.

Definition 2.1 ([27-29])**.** The left-sided Saigo fractional integral operator of order $\alpha > 0$ involving the Gauss hypergeometric function for a real valued continuous function $f(t)$ defined on $\mathbb{R}_+ = (0, \infty)$ is

$$\mathcal{I}_{0+}^{\alpha,\beta,\gamma}f(t) = \frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} \times {}_2F_1\left(\alpha+\beta, -\gamma; \alpha; 1-\frac{\tau}{t}\right)f(\tau)\,d\tau, \tag{3}$$

where β and γ are real numbers.

The semi group properties of the operator (3) will play an important role in obtaining our findings.

$$\mathcal{I}_{0+}^{\alpha,\beta,\gamma}\ \mathcal{I}_{0+}^{\sigma,\delta,\alpha+\gamma}f(t) = \mathcal{I}_{0+}^{\alpha+\sigma,\beta+\delta,\gamma}f(t) \tag{4}$$

$$\mathcal{I}_{0+}^{\alpha,\beta,\gamma}\ \mathcal{I}_{0+}^{\sigma,\delta,\gamma-\beta-\sigma-\delta}f(t) = \mathcal{I}_{0+}^{\alpha+\sigma,\beta+\delta,\gamma-\sigma-\delta}f(t) \tag{5}$$

$$\mathcal{I}_{0+}^{\alpha,\beta,\gamma}\ \mathcal{I}_{0+}^{\sigma,\delta,\nu}f(t) = \mathcal{I}_{0+}^{\sigma,\delta,\nu}\mathcal{I}_{0+}^{\alpha,\beta,\gamma}f(t) \tag{6}$$

Erdélyi-Kober and Riemann-Liouville fractional integral operators $\mathcal{E}_{0+}^{\alpha,\gamma}$ and $\mathcal{I}_{0+}^{\alpha}$, respectively, are obtained by using the following relation:

$$\mathcal{I}_{0+}^{\alpha,0,\gamma}f(t) = \mathcal{E}_{0+}^{\alpha,\gamma}f(t) \ \text{ and } \ \mathcal{I}_{0+}^{\alpha,-\alpha,\gamma}f(t) = \mathcal{I}_{0+}^{\alpha}f(t).$$

Definition 2.2 ([27])**.** The left-sided generalized fractional operator of order α is defined as

$$^{L}\mathcal{D}_{0+}^{\alpha,\beta,\gamma}f(t) = \mathcal{D}^n\,\mathcal{I}_{0+}^{n-\alpha,-\beta-n,\alpha+\gamma-n}f(t), \tag{7}$$

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, β and γ are real numbers.

Lemma 2.3 ([34])**.** *If* $\Re(\gamma) > 0$, $\Re(\sigma) > 0$, $\Re(\gamma+\sigma-\alpha-\beta) > 0$, *then*

$$\int_0^1 x^{\gamma-1}(1-x)^{\sigma-1}{}_2F_1(\alpha,\beta;\gamma;x)\,dx = \frac{\Gamma(\gamma)\Gamma(\sigma)\Gamma(\gamma+\sigma-\alpha-\beta)}{\Gamma(\gamma+\sigma-\alpha)\Gamma(\gamma+\sigma-\beta)}. \tag{8}$$

Using the above formula (8), the following lemma can be easily established:

Lemma 2.4. *Let* $\alpha > 0$, β *and* γ *be real. Then, for* $\mu > \max\{0, -(\gamma-\beta)\} - 1$,

$$\mathcal{I}_{0+}^{\alpha,\beta,\gamma}\left(t^{\mu}\right) = \frac{\Gamma(\mu+1)\Gamma(\mu+\gamma-\beta+1)}{\Gamma(\mu-\beta+1)\Gamma(\alpha+\mu+\gamma+1)}t^{\mu-\beta}, \tag{9}$$

and for $\mu > \max\{0, -(\alpha+\beta+\gamma)\} - 1$,

$$^{L}\mathcal{D}_{0+}^{\alpha,\beta,\gamma}\left(t^{\mu}\right) = \frac{\Gamma(\mu+1)\Gamma(\mu+\alpha+\beta+\gamma+1)}{\Gamma(\mu+\beta+1)\Gamma(\mu+\gamma+1)}t^{\mu+\beta}. \tag{10}$$

Rao et al. [26] defined the Caputo-type modification of the Saigo fractional differential operator of (3) in the following way:

Definition 2.5. The Caputo-type modification of the Saigo fractional operator of order α is defined as

$$^{C}\mathcal{D}_{0+}^{\alpha,\beta,\gamma}f(t) = \mathcal{I}_{0+}^{n-\alpha,-\beta-n,\alpha+\gamma-n}\mathcal{D}^n f(t), \tag{11}$$

where $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, β and γ are real numbers.

When $\beta = -\alpha$, the Caputo-type modification of the Saigo fractional operator reduces to the classical Caputo's fractional operator of order α:

$$^{C}\mathcal{D}_{0+}^{\alpha,-\alpha,\gamma}f(t) = {}^{C}\mathcal{D}_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)d\tau, \quad t > 0,\ n = [\alpha]+1. \tag{12}$$

We shall make use of the following space functions introduced by Dimovski [35].

Definition 2.6. The space of function $\mathbf{C}_{\lambda}^{m}$, $\lambda \in \mathbb{R}$, $m \in \mathbb{N}$ contains all the function $f(t)$, $t > 0$ such that $f(x) =$

$t^{\rho} f_1(t)$ with $\rho > \lambda$ and $f \in \mathbf{C}^m[0,\infty)$. Clearly, $\mathbf{C}_\lambda^m \subseteq \mathbf{C}_\lambda$, $m \in \mathbb{N}$ with $\mathbf{C}_\lambda^0 \subseteq \mathbf{C}_\lambda$.

Lemma 2.7 ([26]). *Let β and γ be reals, and let $\lambda \geq \max\{0, \beta, -(\alpha+\beta+\gamma)\} - 1$. Then, the Caputo-type fractional derivative ${}^C\mathcal{D}_{0+}^{\alpha,\beta,\gamma}$ is a left-inverse operator to the fractional integral $\mathcal{I}_{0+}^{\alpha,\beta,\gamma}$ for the functions on the space $\mathbf{C}_{\lambda+n}^n$*

$$ {}^C\mathcal{D}_{0+}^{\alpha,\beta,\gamma}\ \mathcal{I}_{0+}^{\alpha,\beta,\gamma} f(t) = f(t), \qquad for \quad f(t) \in \mathbf{C}_{\lambda+n}^n. \tag{13}$$

Lemma 2.8 ([26]). *Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, β and γ are reals, and let $\lambda \geq \max\{0, -(\alpha+\beta+\gamma)\} - 1$. Then, for $f(t) \in \mathbf{C}_{\lambda+n}^n$ then the following relation for composition holds true*

$$\mathcal{I}_{0+}^{\alpha,\beta,\gamma}\ {}^C\mathcal{D}_{0+}^{\alpha,\beta,\gamma} f(t) = f(t) - \sum_{i=0}^{n-1} \eta_i x^i, \tag{14}$$

where

$$\eta_i = \lim_{x\to 0} \frac{1}{i!} \mathcal{D}^i f(t).$$

Lemma 2.9. *Let α, β, γ, σ and δ be real such that $m-1 < \sigma < \alpha < m$, $m \in \mathbb{N}$, $\alpha > \beta$ and $\sigma > \delta$. Then, for $f(t) \in \mathbf{C}_{\lambda+m}^m$,*

$$ {}^C\mathcal{D}_{0+}^{\alpha,\beta,\gamma} f(t) = {}^C\mathcal{D}_{0+}^{(\alpha-m+k),(\beta+m-k),(\gamma+m-k)} f^{(m-k)}(t), \tag{15}$$

and

$$ {}^C\mathcal{D}_{0+}^{\alpha,\beta,\gamma} f(t) = {}^C\mathcal{D}_{0+}^{\alpha-\sigma,\beta-\delta,\gamma+\sigma}\ {}^C\mathcal{D}_{0+}^{\sigma,\delta,\gamma} f(t) \tag{16}$$

where $k \in \{1, \cdots, m-1\}$.

Proof. Let $f(t) \in \mathbf{C}_{\lambda+m}^m$. Since $\alpha - m + k \in (k-1, k)$ for $k \in \{1, \cdots, m-1\}$, by using Definition 2.5, we get

$$\begin{aligned} {}^C\mathcal{D}_{0+}^{\alpha,\beta,\gamma} f(t) &= \mathcal{I}_{0+}^{m-\alpha,-\beta-m,\alpha+\gamma-m} f^{(m)}(t) \\ &= \mathcal{I}_{0+}^{k-(\alpha-m+k),-(\beta+m-k)-k,(\alpha-m+k)+(\gamma+m-k)-n} \\ &\quad \times \left(f^{(m-k)}(t)\right)^{(k)} \\ &= {}^C\mathcal{D}_{0+}^{\alpha-m+k,\beta+m-k,\gamma+m-k} f^{(m-k)}(t). \end{aligned}$$

Again, from (4) to (6) and (11), we have

$$\begin{aligned} {}^C\mathcal{D}_{0+}^{\alpha,\beta,\gamma} f(t) &= \mathcal{I}_{0+}^{(m-\sigma)+(\sigma-\alpha),(-\delta-m)+(\delta-\beta),\alpha+\gamma-m} f^{(m)}(t) \\ &= \mathcal{I}_{0+}^{\sigma-\alpha,\delta-\beta,\alpha+\gamma-m}\ \mathcal{I}_{0+}^{m-\sigma,-\delta-m,\sigma+\gamma-m} f^{(m)}(t) \\ &= \mathcal{I}_{0+}^{m-(\alpha-\sigma),-(\beta-\delta)-m,\alpha+\gamma-m} \mathcal{D}^m \left({}^C\mathcal{D}_{0+}^{\sigma,\delta,\gamma} f(t)\right) \\ &= {}^C\mathcal{D}_{0+}^{\alpha-\sigma,\beta-\delta,\gamma+\sigma}\ {}^C\mathcal{D}_{0+}^{\sigma,\delta,\gamma} f(t). \end{aligned}$$

□

To prove sufficient conditions for the uniqueness solution of the initial value problems (1) and (2), we will use Banach contraction mapping principle:

Theorem 2.10 ([36]). *Let (X, d) be a complete metric space, and let $F : X \to X$ be a contraction with Lipschitzian constant L. Then, F has a unique fixed point $u \in X$. Furthermore, for any $x \in X$, we have*

$$\lim_{n\to\infty} F^n(x) = u \tag{17}$$

with

$$d(F^n(x), u) \leqslant \frac{L^n}{1-L} d(x, F(x)). \tag{18}$$

We also state Schauder's fixed point theorem and the Leray-Schauder-type nonlinear alternative which will be used to prove the existence result of (1) and (2).

Theorem 2.11 ([36]). *Let E be a closed, convex subset of normed linear space X. Then, every compact continuous map $\mathcal{T} : E \to E$ has at least one fixed point.*

By $\bar{U}$ and ∂U, we denote the closure of the set U and the boundary of U, respectively.

Theorem 2.12 ([37]). *Let X be a normed linear space, $\mathfrak{C} \subset X$ be a convex set and U be open in $\mathfrak{C}$ with $0 \in U$. Let $T : \bar{U} \to \mathfrak{C}$ be continuous and compact mapping. Then, either*

i The mapping T has a fixed point in $\bar{U}$, or
ii There exists $v \in \partial U$ and $\lambda \in (0,1)$ with $v = \lambda Tv$.

Related results

In this section, we mention some results, which are used in the later part of our discussion.

Lemma 3.1 ([38,39]). *Let $c > b > 0$ and $t < 1$, $t \neq 0$. Then,*

$${}_2F_1(a,b;c;t) < \begin{cases} J & a < -1, \\ \min\{H, J_1\} & a \in (-1,0), \end{cases} \tag{19}$$

where

$$\begin{aligned} J &:= \left(1 - \frac{b}{c}\right) + \frac{b}{c}(1-t)^{-a}, \\ H &:= \left(1 - \frac{bt}{c}\right)^{-a}, \\ J_1 &:= \left(\frac{b}{c}\right)(1-t)^{c-a-b} + \left(1 - \frac{b}{c}\right)(1-t)^{-b}. \end{aligned}$$

If $a \leq c-1$, then $\min\{H, J_1\} = H$.

The hypergeometric term in Saigo operator's integrand is strictly positive [38,39]

$$ {}_2F_1(\alpha+\beta, -\gamma; \alpha; t) > 0, \quad t \in (0,1], \tag{20}$$

where $\alpha > \beta$.

Let $X = \mathbf{C}[0,1]$ be a Banach space of all continuous function endowed with the sup norm

$$\|\nu\|_* = \sup_{t\in[0,1]} \|\nu(t)\|. \tag{21}$$

Let B be the non-empty closed subspace of X defined as

$$B = \{\nu \in X : \|\nu\|_* \le M, M > 0\}. \tag{22}$$

First, we prove that the solution of initial value problems (1) and (2) is equivalent to the solution of an integral equation.

Lemma 3.2. *Let α, β, γ, σ and δ be reals such that $m-1 < \sigma < \alpha < m$, $m \in \mathbb{N}$, $\gamma > 0$, $\alpha > \beta$, $\sigma > \delta$, and let f be a continuous function defined in $[0,1]$. Then, $u \in \mathbf{C}^m_{\lambda+m}[0,1]$, $\lambda \ge \max\{\delta-(\alpha+\beta+\gamma), -(\sigma+\delta+\gamma), 0\}-1$ is a solution of the initial value problem* (1) *and* (2) *if and only if*

$$u(t) = \sum_{k=0}^{m-1} \frac{t^k}{k!}c_k + \frac{t^{-\sigma-\delta}}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \times {}_2F_1\left(\sigma+\delta, -\gamma; \sigma; 1-\frac{s}{t}\right)\nu(s)\,ds, \tag{23}$$

where $\nu \in \mathbf{C}_{\lambda+m}[0,1]$ is a solution of

$$\nu(t) = \frac{t^{\sigma+\delta-\alpha-\beta}}{\Gamma(\alpha-\sigma)} \int_0^t (t-s)^{\alpha-\sigma-1} \times {}_2F_1\left(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-\frac{s}{t}\right) \times f(s, u(s), \nu(s))\,ds. \tag{24}$$

Proof. Consider $u \in \mathbf{C}^m_{\lambda+m}[0,1]$. By (16), the initial value problem (1) and (2) can be expressed as

$${}^C\mathcal{D}^{\alpha-\sigma,\beta-\delta,\gamma+\sigma}_{0+}\ {}^C\mathcal{D}^{\sigma,\delta,\gamma}_{0+} u(t) = f\left(t, u(t), {}^C\mathcal{D}^{\sigma,\delta,\gamma}_{0+} u(t)\right).$$

Replacing ${}^C\mathcal{D}^{\sigma,\delta,\gamma}_{0+} u(t)$ by the function $\nu(t)$, it reduces to

$${}^C\mathcal{D}^{\alpha-\sigma,\beta-\delta,\gamma+\sigma}_{0+} \nu(t) = f(t, u(t), \nu(t)).$$

Applying Lemma 2.8 and using the initial condition (2), we obtain

$$\nu(t) = \frac{t^{\sigma+\delta-\alpha-\beta}}{\Gamma(\alpha-\sigma)} \int_0^t (t-s)^{\alpha-\sigma-1} \times {}_2F_1\left(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-\frac{s}{t}\right) \times f(s, u(s), \nu(s))\,ds.$$

Secondly, applying Lemma 2.8 and the initial condition (2) on $\nu(t) = {}^C\mathcal{D}^{\sigma,\delta,\gamma}_{0+} u(t)$, we obtain (23).

Conversely, let $\nu \in \mathbf{C}_{\lambda+m}[0,1]$ be the solution of (24). Since f is continuous function, then by Lemmas 2.7 and 2.8, it reduces to

$${}^C\mathcal{D}^{\alpha-\sigma,\beta-\delta,\gamma+\sigma}_{0+} \nu(t) = f(t, u(t), \nu(t)) \quad \text{for } t \in (0,1].$$

Again, since $u \in \mathbf{C}^m_{\lambda+m}[0,1]$, using Lemmas 2.7 and 2.8 and the initial condition (2) on (23), we get

$$\nu(t) = {}^C\mathcal{D}^{\sigma,\delta,\gamma}_{0+} u(t).$$

Hence, we arrive with the desired result (1). □

Lemma 3.3. *Let α, β, γ, σ and δ be reals such that $\gamma > 0$ and $n-1 < \sigma < n \le m-1 < \alpha < m$, $m, n \in \mathbb{N}$, $\gamma > 0$, $\alpha > \beta$, $\sigma > \delta$, and let f be a continuous function on $[0,1]$. Then, $u \in \mathbf{C}^m_{\lambda+m}[0,1]$, $\lambda \ge \max\{\delta, -(\alpha+\beta+n), 0\}-1$ is a solution of the initial value problem* (1) *and* (2) *if and only if*

$$u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!}c_k + \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1}\nu(s)\,ds, \tag{25}$$

where $\nu \in \mathbf{C}^n_{\lambda+m}[0,1]$ is a solution of

$$\nu(t) = \sum_{k=0}^{m-n-1} \frac{t^k}{k!}c_{n+k} + \frac{t^{-\alpha-\beta}}{\Gamma(\alpha-n)} \int_0^t (t-s)^{\alpha-n-1} \times {}_2F_1\left(\alpha+\beta, -\gamma-n; \alpha-n; 1-\frac{s}{t}\right) \times f(s, u(s), \chi(s))\,ds \tag{26}$$

and

$$\chi(t) = \frac{t^{\sigma+\delta}}{\Gamma(n-\sigma)} \int_0^t (t-s)^{n-\sigma-1} \times {}_2F_1\left(-\sigma-\delta, n-\gamma-\sigma; n-\sigma; 1-\frac{s}{t}\right)\nu(s)\,ds. \tag{27}$$

Proof. Let $u \in \mathbf{C}^m_{\lambda+m}[0,1]$ be the solution of initial value problems (1) and (2), then applying Lemma 2.9 and substituting $\nu(t) = \mathcal{D}^n u(t)$, we have

$${}^C\mathcal{D}^{\alpha-n,\beta+n,\gamma+n}_{0+} \nu(t) = f(t, u(t), \chi(t)),$$

where $\chi(t)$ is defined in (27).

Applying Lemma 2.8 and the initial condition (2), we can write

$$\nu(t) = \sum_{k=0}^{m-n-1} \frac{t^k}{k!}c_{n+k} + \frac{t^{-\alpha-\beta}}{\Gamma(\alpha-n)} \int_0^t (t-s)^{\alpha-n-1} \times {}_2F_1\left(\alpha+\beta, -\gamma-n; \alpha-n; 1-\frac{s}{t}\right) \times f(s, u(s), \chi(s))\,ds.$$

Again for $\nu(t) = \mathcal{D}^n u(t)$, we obtain

$$u(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!}c_k + \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1}\nu(s)\,ds.$$

Conversely, let $\nu \in \mathbf{C}^n_{\lambda+m}[0,1]$ be the solution of (26). Then, we have

$$\begin{aligned} u^{(n)}(t) =& \nu(t) \\ =& \sum_{k=0}^{m-n-1} \frac{t^k}{k!} c_{n+k} + \frac{t^{-\alpha-\beta}}{\Gamma(\alpha-n)} \int_0^t (t-s)^{\alpha-n-1} \\ & \times {}_2F_1\left(\alpha+\beta, -\gamma-n; \alpha-n; 1-\frac{s}{t}\right) \\ & \times f(s, u(s), \chi(s))\, ds \\ =& \sum_{k=0}^{m-n-1} \frac{t^k}{k!} c_{n+k} + \mathcal{I}_{0+}^{\alpha-n,\beta+n,\gamma+n} f\left(t, u(t), {}^C\mathcal{D}_{0+}^{\sigma,\delta,\gamma} u(t)\right). \end{aligned}$$

Since f is continuous and $m-n-1 < \alpha-n \leq m-n$, then we have

$$\begin{aligned} {}^C\mathcal{D}_{0+}^{\alpha-n,\beta+n,\gamma+n} u^{(n)}(t) = & \sum_{k=0}^{m-n-1} \frac{c_{n+k}}{k!} {}^C\mathcal{D}_{0+}^{\alpha-n,\beta+n,\gamma+n} t^k \\ & + f\left(t, u(t), {}^C\mathcal{D}_{0+}^{\sigma,\delta,\gamma} u(t)\right). \end{aligned}$$

In view of Lemma 2.9, we finally get

$${}^C D_{0+}^{\alpha,\beta,\gamma} u(t) = f\left(t, u(t), {}^C D_{0+}^{\sigma,\delta,\gamma} u(t)\right).$$

Obviously, using (25), it can be easily shown that $\nu^{(m-n)} = u^{(m)} \in \mathbf{C}_{\lambda+m}[0,1]$. This proves that $u(t)$ is the solution of the initial value problems (1) and (2). □

Main results

In this section, the main objective is to find the sufficient conditions for the existence and uniqueness of solution of the initial value problem (1) and (2). Here, two cases are investigated: $m-1 < \sigma < \alpha < m$ and $n-1 < \sigma < n \leq m-1 < \alpha < m$.

When $m-1 < \sigma < \alpha < m$

Throughout this section, we suppose that β, γ and δ are real numbers such that $\gamma > 0$, $\alpha > \beta$, $\sigma > \delta$, $\gamma > \delta - 1$ and $\beta > \delta - 1$.

To facilitate our discussion, let the following assumptions be satisfied:

(H1) $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exist three non-negative functions $L_1(t)$, $L_2(t)$ and $L_3(t)$ in $\mathbf{C}[0,1]$ such that

- (i) $g(0,0,0) = 0$ and $g(t,0,0) \equiv L_1(t) \neq 0$ uniformly continuous on compact subinterval $(0,1]$.
- (ii) $\left\|g(t,x,y) - g(t,\bar{x},\bar{y})\right\| \leq L_2(t)\|x-\bar{x}\| + L_3(t)\left\|y-\bar{y}\right\|$.

(H2) $g(t,x,y) = t^{\delta-\beta} f(t,x,y)$.

(H3) $0 < p < \infty$, $0 < q < \infty$, $r < 1$ such that $M = \frac{p+q}{1-r} > 0$.

H4 $\mho := \frac{\Gamma(\beta-\delta+1)\Gamma(\gamma+\sigma+1)}{\Gamma(\alpha+\beta+\gamma-\delta+1)}$.

For the notational convenience, we denote the following:

$$\begin{aligned} p :=& \sup_{t\in[0,1]} \frac{1}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \\ & \times {}_2F_1(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-x)\, L_1(tx)\, dx, \\ q :=& \sup_{t\in[0,1]} \sum_{k=0}^{m-1} \frac{c_k t^k}{\Gamma(k+1)\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta+k} \\ & \times {}_2F_1(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-x)\, L_2(tx)\, dx, \end{aligned}$$

and

$$\begin{aligned} r :=& \sup_{t\in[0,1]} \frac{1}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \\ & \times {}_2F_1(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-x) \\ & \times \left\{ \frac{t^{-\delta} x^{-\delta} L_2(tx)\, \Gamma(\gamma-\delta+1)}{\Gamma(1-\delta)\,\Gamma(\gamma+\sigma+1)} + L_3(tx) \right\} dx. \end{aligned}$$

Let $\nu \in \bar{B}$. Consider the mapping Υ by

$$\begin{aligned} \Upsilon\nu(t) :=& \frac{t^{\sigma+\delta-\alpha-\beta}}{\Gamma(\alpha-\sigma)} \int_0^t (t-s)^{\alpha-\sigma-1} \\ & \times {}_2F_1\left(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-\frac{s}{t}\right) \\ & \times f(s, u(s), \nu(s))\, ds, \\ :=& \frac{t^{\delta-\beta}}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} \\ & \times {}_2F_1(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-x) \\ & \times f(tx, u(tx), \nu(tx))\, dx \\ :=& \frac{1}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \\ & \times {}_2F_1(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-x) \\ & \times g(tx, u(tx), \nu(tx))\, dx, \end{aligned} \tag{28}$$

where $u(t)$ is from (23).

Let $\lambda \geq \max\{\delta-(\alpha+\beta+\gamma), -(\sigma+\delta+\gamma), 0\} - 1$. Then, in view of Lemma 3.2 and by assumption (H1), the initial value problem (1) and (2) are equivalent to that of the operator Υ which has a fixed point in B. We shall now state and prove the uniqueness solution of the initial value problem (1) and (2).

Theorem 4.1. *Let the assumptions (H1), (H2) and (H3) hold. Then, the initial value problem* (1) *and* (2) *has a unique solution on* $[0,1]$.

Proof. Here, we shall use the Banach contraction principle to prove that Υ has a fixed point.

Let $v \in B$, then

$$|u| = \left| \sum_{k=0}^{m-1} \frac{t^k}{k!} c_k + \frac{t^{-\sigma-\delta}}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \times {}_2F_1\left(\sigma+\delta, -\gamma; \sigma; 1-\frac{s}{t}\right) v(s)\, ds \right|$$
$$\leq \left| \sum_{k=0}^{m-1} \frac{t^k}{k!} c_k \right| + \frac{t^{-\sigma-\delta}}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \times {}_2F_1\left(\sigma+\delta, -\gamma; \sigma; 1-\frac{s}{t}\right) |v(s)|\, ds.$$

Since, $c_k \geq 0$; $k = 1, 2, \cdots, m-1$, and using the formula (8), we have

$$\|u\| \leq \sum_{k=0}^{m-1} \frac{t^k}{k!} c_k + \frac{\Gamma(\gamma-\delta+1)}{\Gamma(1-\delta)\Gamma(\gamma+\sigma+1)} \|v\|\, t^{-\delta}.$$

Consider the operator Υ defined in (28), then

$$\|\Upsilon v\|_* = \sup_{t\in[0,1]} \frac{1}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \times {}_2F_1(\alpha+\beta-\sigma-\delta, \sigma-\gamma; \alpha-\sigma; 1-x) \times |g(tx, u(tx), v(tx))|\, dx$$
$$\leq \sup_{t\in[0,1]} \frac{1}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \times {}_2F_1(\alpha+\beta-\sigma-\delta, \sigma-\gamma; \alpha-\sigma; 1-x) \times \left\{ \left| g(tx, u(tx), v(tx)) - g(tx, 0, 0) \right| + \left| g(tx, 0, 0) \right| \right\} dx$$
$$\leq \sup_{t\in[0,1]} \frac{1}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \times {}_2F_1(\alpha+\beta-\sigma-\delta, \sigma-\gamma; \alpha-\sigma; 1-x) \times \{L_1(tx) + L_2(tx)\|u\| + L_3(tx)\|v\|\}\, dx$$
$$\leq p + q + rM \leq M. \tag{29}$$

This proves that $\Upsilon B \subset B$.

Next, we prove that Υ is a contraction map. Let $v_1, v_2 \in B$. Then, for $t \in [0,1]$, we have

$$|u_1 - u_2| = \left| \frac{t^{-\sigma-\delta}}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \times {}_2F_1\left(\sigma+\delta, -\gamma; \sigma; 1-\frac{s}{t}\right)(v_1 - v_2)\, ds \right|$$
$$\leq \frac{t^{-\sigma-\delta}}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \times {}_2F_1\left(\sigma+\delta, -\gamma; \sigma; 1-\frac{s}{t}\right) \|v_1 - v_2\|\, ds.$$

Using (8), we have

$$\|u_1 - u_2\| \leq \frac{\Gamma(\gamma-\delta+1)}{\Gamma(1-\delta)\Gamma(\gamma+\sigma+1)} t^{-\delta} \|v_1 - v_2\|. \tag{30}$$

Hence,

$$\|\Upsilon v_1(t) - \Upsilon v_2(t)\| \leq \frac{1}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \times {}_2F_1(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-x) \times \left\| g(tx, u_1(tx), v_1(tx)) - g(tx, u_2(tx), v_2(tx)) \right\| dx$$
$$\leq \frac{1}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \times {}_2F_1(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-x) \times \{L_2(tx)\|u_1 - u_2\| + L_3(tx)\|v_1 - v_2\|\}\, dx$$
$$\leq \frac{1}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \times {}_2F_1(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-x) \times \left\{ L_2(tx) \frac{(tx)^{-\delta}\Gamma(\gamma-\delta+1)}{\Gamma(1-\delta)\Gamma(\gamma+\sigma+1)} + L_3(tx) \right\} \|v_1 - v_2\|\, dx$$
$$\leq r\|v_1 - v_2\|_*.$$

Thus,

$$\|\Upsilon v_1(t) - \Upsilon v_2(t)\|_* \leq r\|v_1 - v_2\|_*.$$

According to assumption (H3), $r < 1$ implies that the operator Υ is a contraction in B. As a consequence of Theorem 2.10, the operator Υ has a unique fixed point. Thus, this fixed point is a solution of the initial value problem (1) and (2). □

Next, we state and prove the existence of the solutions for the initial value problem (1) and (2) by using Schauder's fixed point theorem 2.11.

Theorem 4.2. *Let the assumptions (H1) to (H4) hold. Then, the initial value problems* (1) *and* (2) *has at least one solution in the space B.*

Proof. Consider the operator Υ defined in (28) and $E = \{v \in B : \|v\|_* \leq \kappa\}$, where

$$\|\Upsilon v\|_* \leq p + q + rM := \kappa.$$

In view of the proof of Theorem 4.1, the operator Υ maps E into itself. To prove that the operator Υ is compact and continuous, we shall divide the proof into several steps.

Step 1: Υ is continuous.
Let $\{v_n\}$ be a sequence in E such that $v_n \to v$ as

$n \to \infty$. Then, from Lemma 3.2 and (30) for $t \in [0,1]$, we obtain

$$\|u_n - u\| \le \frac{\Gamma(\gamma-\delta+1)}{\Gamma(1-\delta)\,\Gamma(\gamma+\sigma+1)} t^{-\delta}\,\|v_n - v\|.$$

Then,

$$\begin{aligned}
&\|\Upsilon v_n(t) - \Upsilon v(t)\| \\
&\le \frac{1}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \\
&\quad \times {}_2F_1(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-x) \\
&\quad \times \|g(tx, u_n(tx), v_n(tx)) \\
&\quad - g(tx, u(tx), v(tx))\|\, dx \\
&\le \frac{1}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \\
&\quad \times {}_2F_1(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-x) \\
&\quad \times \left\{ L_2(tx) \frac{(tx)^{-\delta}\Gamma(\gamma-\delta+1)}{\Gamma(1-\delta)\Gamma(\gamma+\sigma+1)} \right. \\
&\quad \left. + L_3(tx) \right\} \|v_n - v\|\, dx \\
&\le r\,\|v_n - v\|_*.
\end{aligned}$$

This implies that

$$\|\Upsilon v_n(t) - \Upsilon v(t)\|_* \le r\,\|v_n - v\|_*.$$

Since $v_n \to v$ as $n \to \infty$, hence $\|\Upsilon v_n(t) - \Upsilon v(t)\|_* \to 0$ as $n \to \infty$.

Step 2: The operator Υ is bounded in E into itself. The proof is similar to the proof of Theorem 4.1.

Step 3: The operator Υ is equicontinuous on E. Let $v \in E$ and t_1, $t_2 \in [0,1]$ such that $t_1 < t_2$. Then,

$$\begin{aligned}
&\|\Upsilon v(t_2) - \Upsilon v(t_1)\| \\
&\le \frac{1}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \\
&\quad \times {}_2F_1(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-x) \\
&\quad \times \left\{ \|g(t_2x, u(t_2x), v(t_2x)) \right. \\
&\quad - g(t_2x, 0, 0)\| + \|g(t_2x, 0, 0) - g(t_1x, 0, 0)\| \\
&\quad \left. + \|g(t_1x, 0, 0) - g(t_1x, u(t_1x), v(t_1x))\| \right\} dx \\
&\le \frac{1}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \\
&\quad \times {}_2F_1(\alpha+\beta-\sigma-\delta, -\sigma-\gamma; \alpha-\sigma; 1-x) \\
&\quad \times \{\|L_1(t_2x) - L_1(t_1x)\| + (L_2(t_1x) \\
&\quad + L_2(t_2x))\,\|u\| \\
&\quad + (L_3(t_1x) + L_3(t_2x))\,\|v\|\}\, dx.
\end{aligned}$$

But by (H1), $L_1(t)$ is uniformly continuous in [0,1]. So, for the given $\varepsilon > 0$, we find $\rho > 0$ such that $\| t_2 - t_1 \| < \rho$, then $\|L_1(t_2) - L_1(t_1)\| < \varepsilon = \frac{\rho}{\mho}$. Hence,

$$\|\Upsilon v(t_2) - \Upsilon v(t_1)\|_* \le \rho + 2q + 2r\kappa,$$

which is independent of v.

Thus, the operator Υ is relatively compact. As a consequence of the Arzelà-Ascoli theorem, the operator Υ is compact and continuous. By Theorem 2.11, we conclude that the operator Υ has at least one solution of the initial value problem (1) and (2). This completes the proof. □

Theorem 4.3. *Let the assumptions (H1) to (H4) hold. Then, the initial value problem* (1) *and* (2) *have a solution.*

Proof. Let $U = \{v \in B : \|v\|_* < R\}$ with $R = \frac{p+q}{1-r} > 0$.

Consider the operator Υ defined in (28). Then, by (H1) and the Arzelà-Ascoli theorem, it can be easily shown that the operator $\Upsilon : \bar{U} \to \bar{U}$ is compact and continuous.

Next, we show that U is a priori bounds. If possible, assume that there is a solution $v \in \partial U$ such that

$$v = \lambda\,\Upsilon v \quad \text{with } \lambda \in (0,1). \tag{31}$$

By the assumption that v is a solution for $\lambda \in (0,1)$, one can obtain

$$\begin{aligned}
\|v\|_* &= \sup_{t\in[0,1]} \left| \frac{\lambda}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \right. \\
&\quad \times {}_2F_1(\alpha+\beta-\sigma-\delta, \sigma-\gamma; \alpha-\sigma; 1-x) \\
&\quad \times g\,(tx, u(tx), v(tx)) |\, dx \\
&< \sup_{t\in[0,1]} \frac{\lambda}{\Gamma(\alpha-\sigma)} \int_0^1 (1-x)^{\alpha-\sigma-1} x^{\beta-\delta} \\
&\quad \times {}_2F_1(\alpha+\beta-\sigma-\delta, \sigma-\gamma; \alpha-\sigma; 1-x) | \\
&\quad \times g\,(tx, u(tx), v(tx)) |\, dx \\
&\le p + q + r\,\|v\|_*.
\end{aligned}$$

Therefore $v \notin \partial U$. By Theorem 2.12, Υ has a fixed point in $\bar{U}$, which is a solution of initial value problems (1) and (2). This completes the proof. □

Suppose that according to the Theorem 4.3, v_0 is the fixed point. So by Lemma 3.2, for $t \in [0,1]$ one can obtain

$$\begin{aligned}
u_0(t) &= \sum_{k=0}^{m-1} \frac{t^k}{k!} c_k + \frac{t^{-\sigma-\delta}}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \\
&\quad \times {}_2F_1\left(\sigma+\delta, -\gamma; \sigma; 1-\frac{s}{t}\right) v_0(s)\, ds.
\end{aligned} \tag{32}$$

Example 4.4. Consider the fractional differential equation

$$\begin{aligned}
{}^C\mathcal{D}_{0+}^{1.7,1,1.5} u(t) &= \frac{\left(|u(t)| + \left|{}^C\mathcal{D}_{0+}^{1.5,1.3,1.5} u(t)\right|\right)}{120\sqrt{\pi}\left(1 + |u(t)| + \left|{}^C\mathcal{D}_{0+}^{1.5,1.3,1.5} u(t)\right|\right)} \\
&\quad + 14t^6, \qquad t \in [0,1],
\end{aligned} \tag{33}$$

with

$$u(0) = 2, \qquad u'(0) = 5. \tag{34}$$

Set

$$g(t,u,v) := \frac{t^{0.3}\,(u+v)}{120\sqrt{\pi}\,(1+u+v)} + 14t^{6.3}.$$

Let $t \in [0,1]$ and $u, v, \bar{u}, \bar{v} \in [0,\infty)$, then

$$\begin{aligned}\left|g(t,u,v) - g(t,\bar{u},\bar{v})\right| &= \frac{t^{0.3}}{120\sqrt{\pi}}\left|\frac{u+v}{1+u+v} - \frac{\bar{u}+\bar{v}}{1+\bar{u}+\bar{v}}\right| \\ &\le \frac{t^{0.3}}{120\sqrt{\pi}}\left\{\frac{|u-\bar{u}|+|v-\bar{v}|}{(1+u+v)(1+\bar{u}+\bar{v})}\right\} \\ &\le \frac{t^{0.3}}{120\sqrt{\pi}}\left\{|u-\bar{u}|+|v-\bar{v}|\right\}.\end{aligned}$$

Thus,

$$\left\|g(t,u,v) - g(t,\bar{u},\bar{v})\right\| \le L_2(t)\,\|u-\bar{u}\| + L_3(t)\,\|v-\bar{v}\|,$$

where

$$L_2(t) = \frac{t^{0.3}}{120\sqrt{\pi}} \quad \text{and} \quad L_3(t) = \frac{t^{0.3}}{120\sqrt{\pi}}.$$

Also, $g(t,0,0) := L_1(t) = 14t^{6.3}$. Hence, the initial value problem (33) and (34) satisfy (H1).

One can easily verify that $p \simeq 9.9561$, $q \simeq 0.0356$ and $r \simeq 0.0065 < 1$.

Take $M > 10.0571$. Hence, by Theorem 4.1 and Theorem 4.2 the initial value problem (33) and (34) has at least one solution defined on $[0,1]$.

When $n-1 < \sigma < n \le m-1 < \alpha < m$

Throughout this section, we suppose that β, γ and δ are real numbers such that $\gamma > 0$, $\alpha > \beta$, $\sigma > \delta$, $\gamma + \delta > -1$ and $n + \beta > -1$.

We take the following assumptions to be satisfied:

(H5) $g(t,u,v) := t^{-n-\beta} f(t,u,v)$.
(H6) $0 < p^* < \infty$, $0 < q^* < \infty$, $r^* < 1$ such that $M = \frac{p^*+q^*}{1-r^*} > 0$.
(H7) $\Xi := \frac{\Gamma(n+\beta+1)\Gamma(n+\gamma+1)}{\Gamma(\alpha+\beta+\gamma+n+1)}$, $n, m \in \mathbb{N}$.

We denote

$$\begin{aligned} p^* &= \sup_{t\in[0,1]} |\Phi(t)| + \sup_{t\in[0,1]} \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1}\, x^{n+\beta} \\ &\quad \times {}_2F_1(\alpha+\beta, -\gamma-n; \alpha-n; 1-x)\, L_1(xt)\,dx, \\ q^* &= \sup_{t\in[0,1]} \frac{1}{\Gamma(\alpha-n)} \sum_{k=0}^{n-1} \frac{t^k}{k!} |c_k| \int_0^1 (1-x)^{\alpha-n-1}\, x^{n+k+\beta} \\ &\quad \times {}_2F_1(\alpha+\beta, -\gamma-n; \alpha-n; 1-x)\, L_2(xt)\,dx \end{aligned}$$

and

$$\begin{aligned} r^* &= \sup_{t\in[0,1]} \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta} \\ &\quad \times {}_2F_1(\alpha+\beta, -\gamma-n; \alpha-n; 1-x) \\ &\quad \times \left(\frac{(xt)^n}{\Gamma(n+1)} L_2(xt) + \frac{\Gamma(\sigma+\delta+\gamma+1)(xt)^{n+\delta}}{\Gamma(n+\delta+1)\,\Gamma(\gamma+1)} L_3(xt) \right) dx. \end{aligned}$$

Let $v \in \bar{B}$. Define the mapping $\mathbf{T}$ by

$$\begin{aligned} \mathbf{T}v(t) &:= \sum_{k=0}^{m-n-1} \frac{t^k}{k!} c_{n+k} + \frac{t^{-\alpha-\beta}}{\Gamma(\alpha-n)} \int_0^t (t-x)^{\alpha-n-1} \\ &\quad \times {}_2F_1\left(\alpha+\beta, -\gamma-n; \alpha-n; 1-\frac{x}{t}\right) \\ &\quad \times f(x, u(x), \chi(x))\,dx \\ &:= \Phi(t) + \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta} \\ &\quad \times {}_2F_1(\alpha+\beta, -\gamma-n; \alpha-n; 1-x) \\ &\quad \times g(tx, u(tx), \chi(tx))\,dx, \end{aligned} \tag{35}$$

where $u(t)$ and $\chi(t)$ are from (25) and (27), respectively, and $\Phi(t) := \sum_{k=0}^{m-n-1} \frac{t^k}{k!} c_{n+k}$.

Theorem 4.5. *Let the assumptions (H1), (H5) and (H6) hold. Then, the initial value problem* (1) *and* (2) *has a unique solution on* $[0,1]$.

Proof. By Lemma 3.3, the initial value problem (1) and (2) are transformed to the integral equation (26). Consider the operator $\mathbf{T}$ defined in (35). Here, we shall make use of the Banach contraction principle to prove that $\mathbf{T}$ has a fixed point. First, we shall prove $\mathbf{T}B \subset B$. Let $v \in B$, then

$$\begin{aligned} \|u(t)\| &\le \sum_{k=0}^{n-1} \frac{t^k}{k!} |c_k| + \frac{1}{\Gamma(n)} \int_0^t (t-s)^{n-1} \|v(s)\|\,ds \\ &\le \sum_{k=0}^{n-1} \frac{t^k}{k!} |c_k| + \frac{\|v\|\,t^n}{\Gamma(n+1)}. \end{aligned} \tag{36}$$

Since $\sigma + \delta > -1$, then we have

$$\begin{aligned} \|\chi(t)\| &\le \frac{t^{n+\delta}}{\Gamma(n-\sigma)} \int_0^1 (1-s)^{n-\sigma-1} \\ &\quad \times {}_2F_1(-\sigma-\delta, n-\gamma-\sigma; n-\sigma; 1-s)\,\|v(s)\|\,ds \\ &\le \frac{\Gamma(\sigma+\delta+\gamma+1)\,\|v\|\,t^{n+\delta}}{\Gamma(n+\delta+1)\,\Gamma(\gamma+1)}. \end{aligned} \tag{37}$$

Hence, in view of (H1), (36) and (37), we have

$$\begin{aligned}
&\|\mathbf{T}\nu\|_* \\
&\le \sup_{t\in[0,1]} |\Phi(t)| + \sup_{t\in[0,1]} \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta} \\
&\quad \times {}_2F_1(\alpha+\beta,-\gamma-n;\alpha-n;1-x) \left\| g(tx,u(tx),\chi(tx)) \right\| dx \\
&\le \sup_{t\in[0,1]} |\Phi(t)| + \sup_{t\in[0,1]} \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta} \\
&\quad \times {}_2F_1(\alpha+\beta,-\gamma-n;\alpha-n;1-x) \\
&\quad \times \left\{ \left\| g(tx,0,0) \right\| + \left\| g(tx,u(tx),\chi(tx)) - g(tx,0,0) \right\| \right\} dx \\
&\le \sup_{t\in[0,1]} |\Phi(t)| + \sup_{t\in[0,1]} \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta} \\
&\quad \times {}_2F_1(\alpha+\beta,-\gamma-n;\alpha-n;1-x) \\
&\quad \times \left\{ L_1(xt) + \sum_{k=0}^{n-1} \frac{(xt)^k}{k!} |c_k| L_2(xt) \right. \\
&\quad \left. + \left(\frac{(xt)^n}{\Gamma(n+1)} L_2(xt) + \frac{\Gamma(\sigma+\delta+\gamma+1)(xt)^{n+\delta}}{\Gamma(n+\delta+1)\Gamma(\gamma+1)} L_3(xt) \right) \times \|\nu\| \right\} dx \\
&= p^* + q^* + r^* \|\nu\| \\
&\le M. \qquad (38)
\end{aligned}$$

Next, to prove that $\mathbf{T}$ is a contraction map, let $\nu_1, \nu_2 \in B$. Then, for $t \in [0,1]$, we obtain

$$\|u_1(t) - u_2(t)\| \le \frac{\|\nu_1 - \nu_2\| t^n}{\Gamma(n+1)},$$

$$\|\chi_1(t) - \chi_2(t)\| \le \frac{\Gamma(\sigma+\delta+\gamma+1) \|\nu_1 - \nu_2\| t^{n+\delta}}{\Gamma(n+\delta+1)\Gamma(\gamma+1)}.$$

and

$$\begin{aligned}
\|\mathbf{T}\nu_1(t) - \mathbf{T}\nu_2(t)\| &\le \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta} \\
&\quad \times {}_2F_1(\alpha+\beta,-\gamma-n;\alpha-n;1-x) \\
&\quad \times \left\| g(tx,u_1(tx),\chi_1(tx)) \right. \\
&\quad \left. - g(tx,u_2(tx),\chi_2(tx)) \right\| dx \\
&\le \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta} \\
&\quad \times {}_2F_1(\alpha+\beta,-\gamma-n;\alpha-n;1-x) \\
&\quad \times \{ L_2(tx) \|u_1 - u_2\| + L_3(tx) \|\chi_1 - \chi_2\| \} dx \\
&\le \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta} \\
&\quad \times {}_2F_1(\alpha+\beta,-\gamma-n;\alpha-n;1-x) \\
&\quad \times \left(\frac{(xt)^n}{\Gamma(n+1)} L_2(xt) \right. \\
&\quad \left. + \frac{\Gamma(\sigma+\delta+\gamma+1)(xt)^{n+\delta}}{\Gamma(n+\delta+1)\Gamma(\gamma+1)} L_3(xt) \right) \\
&\quad \times \|\nu_1 - \nu_2\| dx \\
&\le r^* \|\nu_1 - \nu_2\|_*,
\end{aligned}$$

where χ_1 and χ_2 are defined in (27). Thus,

$$\|\mathbf{T}\nu_1(t) - \mathbf{T}\nu_1(t)\|_* \le r^* \|\nu_1 - \nu_2\|_*.$$

By assumption (H6), $r^* < 1$; therefore, the operator $\mathbf{T}$ is a contraction in B. Hence, by Theorem 2.10 the operator $\mathbf{T}$ has a unique fixed point, which corresponds to the unique solution of the initial value problem (1) and (2). □

Next, theorems are based on the existence of the solution for the initial value problem (1) and (2).

Theorem 4.6. *Let us assume that the (H1), (H5), (H6) and (H7) holds. Then, the initial value problem* (1) *and* (2) *has at least one solution in the space B.*

Proof. Consider the operator defined in (35) and $F = \{\nu \in B : \|\nu\|_* \le \mathcal{K}\}$, where

$$\|\mathbf{T}\nu\|_* \le p^* + q^* + r^* R := \mathcal{K}.$$

In view of the proof of Theorem 4.5, it can be easily shown that the operator $\mathbf{T}$ maps F into itself. To prove that the operator $\mathbf{T}$ is compact and continuous, we shall divide the proof in the following steps:

Step 1: $\mathbf{T}$ is continuous.
Let $\{\nu_n\}$ be a sequence in F such that $\nu_n \to \nu$ as $n \to \infty$. Clearly, for $t \in [0,1]$, by using Lemma 2.3, we find

$$\|u_n(t) - u(t)\| \le \frac{\|\nu_n - \nu\| t^n}{\Gamma(n+1)},$$

$$\|\chi_n(t) - \chi(t)\| \le \frac{\Gamma(\sigma+\delta+\gamma+1) \|\nu_n - \nu\| t^{n+\delta}}{\Gamma(n+\delta+1)\Gamma(\gamma+1)}.$$

Then, by the assumption (H1), we have

$$\begin{aligned}
\|\mathbf{T}\nu_n(t) - \mathbf{T}\nu(t)\| &\le \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta} \\
&\quad \times {}_2F_1(\alpha+\beta,-\gamma-n;\alpha-n;1-x) \\
&\quad \times \left\| g(tx,u_n(tx),\chi_n(tx)) \right. \\
&\quad \left. - g(tx,u(tx),\chi(tx)) \right\| dx \\
&\le \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta} \\
&\quad \times {}_2F_1(\alpha+\beta,-\gamma-n;\alpha-n;1-x) \\
&\quad \times \left(\frac{(xt)^n}{\Gamma(n+1)} L_2(xt) \right. \\
&\quad \left. + \frac{\Gamma(\sigma+\delta+\gamma+1)(xt)^{n+\delta}}{\Gamma(n+\delta+1)\Gamma(\gamma+1)} L_3(xt) \right) \\
&\quad \times \|\nu_n - \nu\| dx \\
&\le r^* \|\nu_n - \nu\|_*.
\end{aligned}$$

This implies that
$\|\mathbf{T}\nu_n(t) - \mathbf{T}\nu(t)\|_* \le r^* \|\nu_n - \nu\|_*$.
Thus, $\|\nu_n - \nu\|_* \to 0$
$\Rightarrow \|\mathbf{T}\nu_n(t) - \mathbf{T}\nu(t)\|_* \to 0$ as $n \to \infty$.

Step 2: The operator $\mathbf{T}$ is bounded in F into itself. The proof is similar to the proof of Theorem 4.5.

Step 3: The operator $\mathbf{T}$ is equicontinuous on F.

Let $v \in F$ and $t_1, t_2 \in [0,1]$ such that $t_1 < t_2$. Then,

$$\begin{aligned}\|\mathbf{T}v(t_2) - \mathbf{T}v(t_1)\| &\le \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta} \\ &\times {}_2F_1\ (\alpha+\beta, -\gamma-n; \alpha-n; 1-x) \\ &\times \{\|g(t_2x, u(t_2x), \chi(t_2x)) - g(t_2x, 0, 0)\| \\ &+ \|g(t_2x, 0, 0) - g(t_1x, 0, 0)\| \\ &+ \|g(t_1x, 0, 0) - g(t_1x, u(t_1x), \chi(t_1x))\|\}\, dx \\ &\le \frac{1}{\Gamma(\alpha-n)} \int_0^1 (1-x)^{\alpha-n-1} x^{n+\beta} \\ &\times {}_2F_1\ (\alpha+\beta, -\gamma-n; \alpha-n; 1-x) \\ &\times \{\|L_1(t_2x) - L_1(t_1x)\| + (L_2(t_1x) \\ &+ L_2(t_2x))\|u\| \\ &+ (L_3(t_1x) + L_3(t_2x))\|\chi\|\}\, dx.\end{aligned}$$

By the assumption (H1), $L_1(t)$ is uniformly continuous in[0,1]. So, for the given $\varepsilon > 0$, we find $\rho > 0$ such that $\|t_2 - t_1\| < \rho$, then $\|L_1(t_2) - L_1(t_1)\| < \varepsilon = \frac{\rho}{\Xi}$. Hence,

$$\|\mathbf{T}v(t_2) - \mathbf{T}v(t_1)\|_* \le \rho + 2q^* + 2r^*\mathcal{K},$$

which is independent of v.

Thus, the operator $\mathbf{T}$ is relatively compact. Hence, by consequence of the Arzelà-Ascoli theorem, the operator $\mathbf{T}$ is compact and continuous. Using Theorem 2.11, we conclude that the operator $\mathbf{T}$ has at least one solution for the initial value problem (1) and (2). □

Theorem 4.7. *Let the assumptions (H1), (H5), (H6) and (H7) hold. Then, the initial value problem* (1) *and* (2) *has a solution.*

Proof. Consider the operator $\mathbf{T}$ defined in (35) and $\mathcal{H} = \{v \in B : \|v\|_* < \mathcal{P}\}$.

Then, by (H1) and the Arzelà-Ascoli theorem, it can be easily shown that the operator $\mathbf{T} : \bar{\mathcal{H}} \to \bar{\mathcal{H}}$ is compact and continuous.

Next, we show that $\mathcal{H}$ is a priori bound. If possible, suppose that there is a solution $v \in \partial\mathcal{H}$ such that

$$v = \lambda\, \mathbf{T}v \quad \text{with } \lambda \in (0,1). \tag{39}$$

Then for $\lambda \in (0,1)$, we obtain

$$\begin{aligned}\|v\|_* &\le \sup_{t\in[0,1]} \lambda\,|\Phi(t)| + \sup_{t\in[0,1]} \frac{\lambda}{\Gamma(\alpha-n)} \int_0^1 \\ &\times (1-x)^{\alpha-n-1} x^{n+\beta} \\ &\times {}_2F_1(\alpha+\beta, -\gamma-n; \alpha-n; 1-x) \\ &\|g(tx, u(tx), \chi(tx))\|\, dx \\ &\le \sup_{t\in[0,1]} \lambda\,|\Phi(t)| + \sup_{t\in[0,1]} \frac{\lambda}{\Gamma(\alpha-n)} \int_0^1 \\ &\times (1-x)^{\alpha-n-1} x^{n+\beta} \\ &\times {}_2F_1(\alpha+\beta, -\gamma-n; \alpha-n; 1-x) \\ &\times \left\{ L_1(xt) + \sum_{k=0}^{n-1} \frac{(xt)^k}{k!} |c_k|\, L_2(xt) \right. \\ &+ \left(\frac{(xt)^n}{\Gamma(n+1)} L_2(xt) \right. \\ &\left.\left. + \frac{\Gamma(\sigma+\delta+\gamma+1)(xt)^{n+\delta}}{\Gamma(n+\delta+1)\,\Gamma(\gamma+1)} L_3(xt) \right) \|v\| \right\} dx \\ &= \lambda\left(p^* + q^* + r^*\|v\|_*\right) \\ &< \mathcal{P}.\end{aligned}$$

Therefore $v \notin \partial\mathcal{H}$. Hence, by Theorem 2.12, $\mathbf{T}$ has a fixed point in $\bar{\mathcal{H}}$, which is a solution of initial value problem (1) and (2). □

Example 4.8. Consider the fractional differential equation

$${}^C\mathcal{D}_{0+}^{3.5,2.5,2.5} u(t) - t^{4.5}\, {}^C\mathcal{D}_{0+}^{1.5,0.5,2.5} u(t) - u(t) = t^{6.5}, \quad t \in [0,1], \tag{40}$$

and

$$u(0) = 2.5,\ u'(0) = 2,\ u''(0) = 6,\ u'''(0) = 7.05. \tag{41}$$

The above equation (40) can be written as

$${}^C\mathcal{D}_{0+}^{3.5,2.5,2.5} u(t) = t^{6.5} + u(t) + t^{4.5}\, {}^C\mathcal{D}_{0+}^{1.5,0.5,2.5} u(t). \tag{42}$$

Here, $3 < \alpha < 4$ and $1 < \sigma < 2$.
Set

$$g(t,u,v) \equiv t^2 + t^{-4.5} u(t) + v(t).$$

Clearly, $L_1(t) = t^2$, $L_2(t) = t^{-4.5}$ and $L_3(t) = 1$ satisfied the condition (H1).

Also, for each $u, \bar{u}, v, \bar{v} \in \mathbb{R}$, we have

$$\|g(t,u,v) - g(t,\bar{u},\bar{v})\| \le L_2(t)\,\|u - \bar{u}\| + L_3(t)\,\|v - \bar{v}\|,$$

Again

$$cr^* = \sup_{t\in[0,1]} \frac{t^{-2.5}}{\Gamma(3)\Gamma(1.5)} \int_0^1 x^{0.5}(1-x)^2 {}_2F_1(6,-4.5,1.5,x)dx$$
$$+ \sup_{t\in[0,1]} \frac{t^{2.5}\Gamma(5.5)}{\Gamma(3.5)\gamma(3.5)\Gamma(1.5)} \int_0^1 x^{0.5}(1-x)^7$$
$$\times {}_2F_1(6,-4.5,1.5,x)dx$$
$$\leq 2.0989\times 10^{-5} + 0.0058$$
$$= 0.0058 \quad \text{(approx.)}$$
$$<1.$$

Similarly, we can find $p^* \leq 13.0510$ and $q^* \leq 0.0133$.

Take $M > 2.2523 \times 10^3$. As a consequence of Theorem 4.5 and Theorem 4.6, the initial value problem (40) and (41) has at least one unique solution defined in $[0,1]$.

Conclusions

The existence and uniqueness of solution for the nonlinear fractional differential equations with initial conditions comprising the Caputo-type modification of Saigo's fractional derivatives have been discussed in $(\mathbf{C}[0,1],\mathbb{R})$. For our discussion, we have used the fixed point theorems and nonlinear alternative of Leray and Schauder. The existence and uniqueness theorem may be explored for other classes of fractional differential equations involving the Caputo-type modification of Saigo's fractional derivative. From the above discussion, it is expected that this may provide a new direction to the study of fractional differential equation, which may give higher degrees of freedom than the fractional differential equation available in literature.

Competing interests

Both authors declare that they have no competing interests.

Authors' contributions

Both the authors contributed equally in writing this manuscript. Both authors read and approved the final manuscript.

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References

1. Ross, B: Fractional calculus. Math. Mag. **50**(3), 115–122 (1977)
2. Sabatier, J, Agrawal, O, Machado, J: Advances in fractional calculus: Theoretical developments and applications in physics and engineering. Springer, London (2007)
3. Baleanu, D, Machado, JAT, Luo, ACJ: Fractional dynamics and control. Springer, New York (2011)
4. Diethelm, K: The analysis of fractional differential equations: An application-oriented exposition using differential operators of caputo type. Lecture Notes in Mathematics, Vol. 2004. Springer, New York (2010)
5. Herrmann, R: Fractional calculus: An introduction for physicists. World Scientific, Singapore (2011)
6. Kilbas, A, Srivastava, H, Trujillo, J: Theory and applications of fractional differential equations. North-Holland mathematics studies. Elsevier Science & Tech, Netherlands (2006)
7. Podlubny, I: Fractional differential equations: An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Mathematics in Science and Engineering. Academic Press, San Diego (1998)
8. Ahmad, B, Ntouyas, S, Assolami, A: Caputo type fractional differential equations with nonlocal Riemann-Liouville integral boundary conditions. J. Appl. Math. Comput. **41**, 339–350 (2013). doi:10.1007/s12190-012-0610-8
9. Abbas, S, Benchohra, M, Graef, J: Integro-differential equations of fractional order. Differential Equations Dynam. Sys. **20**(2), 139–148 (2012). doi:10.1007/s12591-012-0110-1
10. Daftardar-Gejji, V, Jafari, H: Analysis of a system of nonautonomous fractional differential equations involving Caputo derivatives. J. Math. Anal. Appl. **328**(2), 1026–1033 (2007). doi:10.1016/j.jmaa.2006.06.007
11. Deng, J, Ma, L: Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations. Appl. Math. Lett. **23**(6), 676–680 (2010). doi:10.1016/j.aml.2010.02.007
12. Ghanbari, K, Gholami, Y: Existence and nonexistence results of positive solution for nonlinear fractional eigenvalue problem. J. Frac. Calc. Appl. **4**(2), 1–12 (2013)
13. Khan, R, Rehman, M, Henderson, J: Existence and uniqueness of solutions for nonlinear fractional differential equations with integral boundary conditions. Fract. Differ. Calc. **1**, 29–43 (2011). doi:10.7153/fdc-01-02
14. Kosmatov, N: Integral equations and initial value problems for nonlinear differential equations of fractional order. Nonlinear Anal. **70**(7), 2521–2529 (2009). doi:10.1016/j.na.2008.03.037
15. Matar, M: On existence of solution to some nonlinear differential equations of fractional order $2 \leq \alpha \leq 3$. Int. J. Math. Anal. **6**(34), 1649–1657 (2012)
16. Wang, C, Zhang, H, Wang, S: Positive solution of a nonlinear fractional differential equation involving Caputo derivative. Discrete Dyn. Nat. Soc. **2012**, 16 (2012). doi:10.1155/2012/425408
17. Wang, J, Lv, L, Zhou, Y: Boundary value problems for fractional differential equations involving Caputo derivative in Banach spaces. J. Appl. Math. Comput. **38**, 209–224 (2012). doi:10.1007/s12190-011-0474-3
18. Wang, X, Guo, X, Tang, G: Anti-periodic fractional boundary value problems for nonlinear differential equations of fractional order. J. Appl. Math. Comput. **41**, 367–375 (2013). doi:10.1007/s12190-012-0613-5
19. Jiang, W, Huang, X, Guo, W, Zhang, Q: The existence of positive solutions for the singular fractional differential equation. J. Appl. Math. Comput. **41**, 171–182 (2013). doi:10.1007/s12190-012-0603-7
20. Zhang, S: Existence and uniqueness result of solutions to initial value problems of fractional differential equations of variable-order. J. Frac. Calc. Anal. **4**(1), 82–98 (2013)
21. Zhang, S: Existence of solution for a boundary value problem of fractional order. Acta. Math. Sinica. **26**(2), 220–228 (2006). doi:10.1016/S0252-9602(06)60044-1
22. Agarwal, R, Benchohra, M, Hamani, S: A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl Math. **109**(3), 973–1033 (2010). doi:10.1007/s10440-008-9356-6
23. Kiryakova, V: A brief story about the operators of the generalized fractional calculus. Frac. Calc. Appl. Anal. **11**(2), 203–220 (2008)
24. Gorenflo, R, Luchko, Y, Mainardi, F: Wright functions as scale-invariant solutions of the diffusion-wave equation. J. Comput. Appl. Math. **118**(2), 175–191 (2000). doi:10.1016/S0377-0427(00)00288-0
25. Luchko, Y, Trujillo, J: Caputo-type modification of the Erdélyi-Kober fractional derivative. Frac. Calc. Appl. Anal. **10**(3), 249–267 (2007)
26. Rao, A, Garg, M, Kalla, S: Caputo type fractional derivative of a hypergeometric fractional integral operator. Kuwait. J. Sci. Engg. **37**(1A), 15–29 (2010)
27. Saigo, M: A remark on integral operators involving the Gauss hypergeometric functions. Math. Rep. College General Ed. Kyushu Univ. **11**(2), 135–143 (1978)
28. Saigo, M: A certain boundary value problem for the Euler-Darboux equation I. Math. Japon. **24**(4), 377–385 (1979)
29. Saigo, M: A certain boundary value problem for the Euler-Darboux equation II. Math. Japon. **25**, 211–220 (1980)
30. Dutta, B, Arora, L: On existence and uniqueness solutions of a class of fractional differential equation. International conference on special

functions and their applications and symposium on works of Ramanujan 28–30 July 2011
31. Achar, B, Hanneken, J, Enck, T, Clarke, T: Dynamics of the fractional oscillator. Phys A. **297**, 361–367 (2001). doi:10.1016/S0378-4371(01)00200-X
32. Momani, S, Ibrahim, R: Analytical solutions of a fractional oscillator by the decomposition method. Int. J. Pure Appl. Math. **37**, 119–131 (2007)
33. Torvik, P, Bagley, R: On the appearance of the fractional derivative in the behavior of real materials. J. Appl. Mech. **51**(2), 294–298 (1984). doi:10.1115/1.3167615
34. Gradshtein, I, Ryzhik, I, Jeffrey, A, Zwillinger, D: Table of Integrals, Series and Products. 7th edn. Academic, New York (2007)
35. Dimovski, I: Operational calculus for a class of differential operators. Comptes-rendus de l'Académie Bulgare des Sciences. **19**, 1111–1114 (1966)
36. Agarwal, RP, Meehan, M, O'Regan, D: Fixed Point Theory and Applications. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (2001)
37. Zeidler, E: Nonlinear Functional Analysis and its Applications: I. Fixed-Point Theorems. Springer, New York (1986)
38. Jankov, D, Pogány, T: Andreev-Korkin identity, Saigo fractional integration operator and $Lip_L(a)$ functions. J. Math. Phys. Anal. Geom. **8**(2), 144–157 (2012)
39. Carlson, B: Some inequalities for hypergeometric functions. Proc. Amer. Math. Soc. **17**, 32–39 (1966). doi:10.1090/S0002-9939-1966-0188497-6

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