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# The generalized $\chi^2$ sequence spaces over *p*- metric spaces defined by Musielak

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**Abstract**

In this paper, we introduce generalized $\chi^2$ sequence spaces over *p*- metric spaces defined by Musielak function $f = (f_{mn})$ and study some topological properties.

**Keywords:** Analytic sequence; Double sequences; $\chi^2$ space; Difference sequence space; Musielak-modulus function; *p*- metric space; Duals

**MSC:** 40A05; 40C05; 40D05



## Introduction

Throughout this paper, $w$, $\chi$, and $\Lambda$ denote the classes of all, gai, and analytic scalar valued single sequences, respectively. We write $w^2$ for the set of all complex sequences $(x_{mn})$, where $m, n \in \mathbb{N}$, the set of positive integers. Then, $w^2$ is a linear space under the coordinatewise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on, they were investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy [6], Turkmenoglu [7], and many others. We procure the following sets of double sequences:

$$\mathcal{M}_u(t) := \left\{ (x_{mn}) \in w^2 : sup_{m,n \in N} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_p(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn} - |^{t_{mn}} = 1 \text{ for some } \in \mathbb{C} \right\},$$

$$\mathcal{C}_{0p}(t) := \left\{ (x_{mn}) \in w^2 : p - lim_{m,n\to\infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

$$\mathcal{L}_u(t) := \left\{ (x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$\mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \bigcap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{0bp}(t) = \mathcal{C}_{0p}(t) \bigcap \mathcal{M}_u(t),$$

where $t = (t_{mn})$ is the sequence of strictly positive reals $t_{mn}$ for all $m, n \in \mathbb{N}$ and $p - lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case where $t_{mn} = 1$ for all $m, n \in \mathbb{N}$, $\mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$, and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$, and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [8,9] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zeltser [10] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [11], and Tripathy [6] have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and BaŞar [12] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$, and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums is in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$, and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$, and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Basar and Sever [13] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space $\ell_q$ of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently, Subramanian and Misra [14] have studied the space $\chi^2_M(p, q, u)$ of double sequences and gave some inclusion relations.

The class of sequences which is strongly Cesàro summable with respect to a modulus was introduced

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by Maddox [15] as an extension of the definition of strongly Cesàro summable sequences. Connor [16] further extended this definition to a definition of strong $A$-summability with respect to a modulus, where $A = \left(a_{n,k}\right)$ is a non-negative regular matrix, and established some connections between strong $A$- summability, strong $A$-summability with respect to a modulus, and $A$- statistical convergence. In [17], the notion of convergence of double sequences was presented by Pringsheim. Also, in [18,19], and [20], the four-dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{k\ell}^{mn} x_{mn}$ was studied extensively by Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p. \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence $(s_{mn})$ is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$. A sequence $x = (x_{mn})$ is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by $\Lambda^2$. A sequence $x = (x_{mn})$ is called double gai sequence if$((m+n)! \, |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by $\chi^2$. Let $\phi =$ {all finite sequences}.

Consider a double sequence $x = (x_{ij})$. The $(m,n)^{\text{th}}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \Im_{ij}$ for all $m, n \in \mathbb{N}$, where $\Im_{ij}$ denotes the double sequence whose only non-zero term is a $\frac{1}{(i+j)!}$ in the $\left(i,j\right)$th place for each $i, j \in \mathbb{N}$.

A Fréchet coordinate space (FK-space or a metric space) $X$ is said to have an AK property if $(\Im_{mn})$ is a Schauder basis for $X$, or equivalently $x^{[m,n]} \to x$. An FDK-space is a double sequence space endowed with a complete metrizable space, locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn}) (m, n \in \mathbb{N})$ are also continuous.

Let $M$ and $\Phi$ be mutually complementary modulus functions. Then, we have

(1) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), \text{(Young's inequality; see [21]).} \tag{2}$$

(2) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi\left(\eta(u)\right). \tag{3}$$

(3) For all $u \geq 0$ and $0 < \lambda < 1$,

$$M\left(\lambda u\right) \leq \lambda M(u). \tag{4}$$

Lindenstrauss and Tzafriri [22] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for some } \rho > 0 \right\}.$$

The space $\ell_M$ with the norm

$$\|x\| = inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p \, (1 \leq p < \infty)$, the spaces $\ell_M$ coincide with the classical sequence space $\ell_p$.

A sequence $f = \left(f_{mn}\right)$ of modulus function is called a Musielak-modulus function. A sequence $g = \left(g_{mn}\right)$ defined by

$$g_{mn}(v) = sup \left\{ |v| \, u - \left(f_{mn}\right)(u) : u \geq 0 \right\}, m, n = 1, 2, \cdots$$

is called the complementary function of a Musielak-modulus function $f$. For a given Musielak modulus function $f$, the Musielak-modulus sequence space $t_f$ and its subspace $h_f$ are defined, respectively, as follows:

$$t_f = \left\{ x \in w^2 : I_f \left(|x_{mn}|\right)^{1/m+n} \to 0 \text{ as } m, n \to \infty \right\}$$

and

$$h_f = \left\{ x \in w^2 : I_f \left(|x_{mn}|\right)^{1/m+n} \to 0 \text{ as } m, n \to \infty \right\},$$

where $I_f$ is a convex modular defined by

$$I_f(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \left(|x_{mn}|\right)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider that $t_f$ is equipped with the Luxemburg metric

$$d\left(x, y\right) = sup_{mn} \left\{ inf \left( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f_{mn} \times \left( \frac{|x_{mn}|^{1/m+n}}{mn} \right) \right) \leq 1 \right\}.$$

If $X$ is a sequence space, we give the following definitions:

(1) $X'$ = the continuous dual of $X$;
(2) $X^{\alpha} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn} x_{mn}| < \infty, \right.$ for each $x \in X \}$;
(3) $X^{\beta} = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn} \right.$ is convergent, for each $x \in X \}$;
(4) $X^{\gamma} = \left\{ a = (a_{mn}) : sup_{mn} \geq 1 \left| \sum_{m,n=1}^{M,N} a_{mn} x_{mn} \right| < \infty, \right.$ for each $x \in X \}$;
(5) let $X$ be an FK-space $\supset \phi$, then $X^f = \left\{ f(\Im_{mn}) : f \in X' \right\}$;
(6) $X^{\delta} = \left\{ a = (a_{mn}) : sup_{mn} |a_{mn} x_{mn}|^{1/m+n} < \infty, \right.$ for each $x \in X \}$,

where $X^{\alpha}$, $X^{\beta}$, and $X^{\gamma}$ are called $\alpha-$ (or Köthe-Toeplitz) dual of $X$, $\beta-$ (or generalized Köthe-Toeplitz) dual of $X$, $\gamma-$ dual of $X$, and $\delta-$ dual of $X$, respectively. $X^{\alpha}$ is defined by Kantham and Gupta [21]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold since the sequence of partial sums of a double convergent series needs not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [23] as follows:

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and $\ell_\infty$, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here, $c$, $c_0$, and $\ell_\infty$ denote the classes of convergent, null, and bounded scalar valued single sequences, respectively. The difference sequence space $bv_p$ of the classical space $\ell_p$ is introduced and studied in the case $1 \leq p \leq \infty$ and in the case $0 < p < 1$ by Altay and BaŞar in [12]. The spaces $c(\Delta)$, $c_0(\Delta)$, $\ell_\infty(\Delta)$, and $bv_p$ are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on, the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\},$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

## Definition and preliminaries

Let $n \in \mathbb{N}$ and $X$ be a real vector space of dimension $w$, where $n \leq w$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1), \ldots, d_n(x_n))\|_p$ on $X$ satisfying the following four conditions:

(1) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p = 0$ if and and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,
(2) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p$ is invariant under permutation,
(3) $\|(\alpha d_1(x_1), \ldots, d_n(x_n))\|_p = |\alpha| \, \|(d_1(x_1), \ldots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$
(4) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p} for 1 \leq p < \infty$; (or)
(5) $d((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)) :=$ $\sup\left\{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\right\}$, for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ which is called the $p$ product metric of the Cartesian product of $n$ metric spaces is the $p$ norm of the $n$-vector of the norms of the $n$ subspaces.

A trivial example of the $p$ product metric of the $n$ metric space is the $p$ norm space which is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space:

$$\|(d_1(x_1), \ldots, d_n(x_n))\|_E = sup\left(|det(d_{mn}(x_{mn}))|\right) = sup\left(\left|\begin{matrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \cdots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \cdots & d_{2n}(x_{1n}) \\ \vdots \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \cdots & d_{nn}(x_{nn}) \end{matrix}\right|\right),$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.

If every Cauchy sequence in $X$ converges to some $L \in X$, then $X$ is said to be complete with respect to the $p$- metric. Any complete $p$- metric space is said to be $p$- Banach metric space.

Let $X$ be a linear metric space. A function $w : X \to \mathbb{R}$ is called paranorm if

(1) $w(x) \geq 0$ for all $x \in X$;
(2) $w(-x) = w(x)$ for all $x \in X$,
(3) $w(x + y) \leq w(x) + w(y)$ for all $x, y \in X$;
(4) If $(\sigma_{mn})$ is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n \to \infty$, and $(x_{mn})$ is a sequence of vectors with $w(x_{mn} - x) \to 0$ as $m, n \to \infty$, then $w(\sigma_{mn} x_{mn} - \sigma x) \to 0$ as $m, n \to \infty$.

A paranorm $w$ for which $w(x) = 0$ implies $x = 0$ is called a total paranorm, and the pair $(X, w)$ is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [24], Theorem 10.4.2, p.183).

The notion of $\lambda-$ double gai and double analytic sequences is as follows: Let $\lambda = (\lambda_{mn})_{m,n=0}^{\infty}$ be a strictly increasing sequence of positive real numbers tending to infinity, that is,

$$0 < \lambda_0 < \lambda_1 < \cdots \text{ and } \lambda_{mn} \to \infty \text{ as } m, n \to \infty$$

and that a sequence $x = (x_{mn}) \in w^2$ is $\lambda-$ convergent to 0, called a the $\lambda-$ limit of $x$, if $\mu_{mn}(x) \to 0$ as $m, n \to \infty$, where

$$\mu_{mn}(x) = \frac{1}{\varphi_{rs}} \sum_{m \in \sigma, \sigma \in P_{rs}} \sum_{n \in \sigma, \sigma \in P_{rs}} \times \left(\lambda_{m,n} - \lambda_{m,n+1} - \lambda_{m+1,n} + \lambda_{m+1,n+1}\right) |x_{mn}|^{1/m+n}.$$

The sequence $x = (x_{mn}) \in w^2$ is $\lambda-$ double analytic if $sup_{uv} |\mu_{mn}(x)| < \infty$. If $lim_{mn} x_{mn} = 0$ in the ordinary sense of convergence, then

$$lim_{mn} \left(\frac{1}{\varphi_{rs}} \sum_{m \in \sigma, \sigma \in P_{rs}} \sum_{n \in \sigma, \sigma \in P_{rs}} \left(\lambda_{m,n} - \lambda_{m,n+1} - \lambda_{m+1,n} + \lambda_{m+1,n+1}\right) \left((m+n)! \, |x_{mn} - 0|\right)^{1/m+n}\right) = 0.$$

This implies that it yields $lim_{uv} \mu_{mn}(x) = 0$, and hence, $x = (x_{mn}) \in w^2$ is $\lambda-$ convergent to 0. Let $f = (f_{mn})$ be a Musielak-modulus function, $\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$ be a $p$-metric space, and $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. By $w^2(p - X)$, we denote the space of all sequences as $\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$. The following inequality will be used throughout the paper. If $0 \leq q_{mn} \leq supq_{mn} = H, K = max\left(1, 2^{H-1}\right)$, then

$$|a_{mn} + b_{mn}|^{q_{mn}} \leq K\left\{|a_{mn}|^{q_{mn}} + |b_{mn}|^{q_{mn}}\right\} \quad (5)$$

for all $m, n$ and $a_{mn}, b_{mn} \in \mathbb{C}$. Also, $|a|^{q_{mn}} \leq max\left(1, |a|^H\right)$ for all $a \in \mathbb{C}$.

In the present paper, we define the following sequence spaces:

$$\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = \lim_{mn}\left\{\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} = 0\right\},$$

$$\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = sup_{mn}\left\{\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} < \infty\right\}.$$

If we take $f_{mn}(x) = x$, we get

$$\left[\chi^{2q}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = \lim_{mn}\left\{\left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} = 0\right\},$$

$$\left[\Lambda^{2q}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = sup_{mn}\left\{\left[\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} < \infty\right\}.$$

If we take $q = (q_{mn}) = 1$

$$\left[\chi^{2}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = \lim_{mn}\left\{\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] = 0\right\},$$

$$\left[\Lambda^{2}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = sup_{mn}\left\{\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] < \infty\right\}.$$

In the present paper, we plan to study some topological properties and inclusion relation between the above defined sequence spaces, $\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ and $\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$, which we shall discuss in this paper.

## Main results

**Theorem 1.** *Let $f = (f_{mn})$ be a Musielak-modulus function and $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers; the sequence spaces $\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ and $\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ are linear spaces.*

*Proof.* It is routine verification. Therefore, the proof is omitted. □

**Theorem 2.** *Let $f = (f_{mn})$ be a Musielak-modulus function and $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers; the sequence space $\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ is a paranormed space with respect to the paranorm defined by*

$$g(x) = inf\left\{\left(\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\},$$

*where $H = max\,(1, sup_{mn}q_{mn} < \infty)$.*

*Proof.* Clearly, $g(x) \geq 0$ for $x = (x_{mn}) \in \left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{V_2}_p\right]$. Since $f_{mn}(0) = 0$, we get $g(0) = 0$.

Conversely, suppose that $g(x) = 0$, then

$$inf\left\{\left(\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H}\right\} \leq 1 = 0.$$

Suppose that $\mu_{mn}(x) \neq 0$ for each $m, n \in \mathbb{N}$. Then, $\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p \rightarrow \infty$. It follows that $\left(\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{V_2}_p\right)\right]^{q_{mn}}\right)^{1/H} \rightarrow \infty$ which is a contradiction. Therefore, $\mu_{mn}(x) = 0$. Let

$$\left(\left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1$$

and

$$\left(\left[f_{mn}\left(\|\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right)\right]^{q_{mn}}\right)^{1/H} \leq 1.$$

Then, by using Minkowski's inequality, we have

$$\left(\left[f_{mn}\left(\|\mu_{mn}(x+y),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}$$
$$\leq\left(\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}$$
$$+\left(\left[f_{mn}\left(\|\mu_{mn}(y),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}.$$

So, we have

$$g(x+y)=inf\left\{\left(\left[f_{mn}\left(\|\mu_{mn}(x+y),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right\}$$
$$\leq inf\left\{\left(\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right\}$$
$$+inf\left\{\left(\left[f_{mn}\left(\|\mu_{mn}(y),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right\}.$$

Therefore,

$$g(x+y)\leq g(x)+g(y).$$

Finally, to prove that the scalar multiplication is continuous, let $\lambda$ be any complex number. By definition,

$$g(\lambda x)=inf\left\{\left(\left[f_{mn}\left(\|\mu_{mn}(\lambda x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right\}.$$

Then,

$$g(\lambda x)=inf\left\{((|\lambda|\,t)^{q_{mn}/H}:\left(\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right\},$$

where $t=\frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}}\leq max\left(1,|\lambda|^{supp_{mn}}\right)$, we have

$$g(\lambda x)\leq max\left(1,|\lambda|^{supp_{mn}}\right)$$
$$\times inf\left\{t^{q_{mn}/H}:\left(\left[f_{mn}\left(\|\mu_{mn}(\lambda x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right)\right]^{q_{mn}}\right)^{1/H}\leq 1\right\}.$$

□

**Theorem 3.** *The $\beta-$ dual space of* $\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}=\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$.

*Proof.* First, we observe that

$$\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}\subset\left[\Gamma^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right].$$

Therefore,

$$\left[\Gamma^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}\subset\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}.$$

But

$$\left[\Gamma^{2q}_{f\mu}\right]^{\beta}\subsetneq\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right].$$

Hence,

$$\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]\subset\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}. \quad (6)$$

Next, we show that

$$\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}\subset\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right].$$

Let $y=(y_{mn})\in\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}$. Consider $f(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}x_{mn}y_{mn}$ with

$$x=(x_{mn})\in\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$$
$$x=[(\lambda_{mn}-\lambda_{mn+1})-(\lambda_{m+1n}-\lambda_{m+1n+1})]$$
$$=\begin{pmatrix}0&0&\ldots 0&0&\ldots&0\\0&0&\ldots 0&0&\ldots&0\\\vdots&&&&&\\0&0\ldots&\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}&\frac{-\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}&\ldots&0\\0&0&\ldots 0&0&\ldots&0\end{pmatrix}$$
$$-\begin{pmatrix}0&0&\ldots 0&0&\ldots&0\\0&0&\ldots 0&0&\ldots&0\\\vdots&&&&&\\0&0\ldots&\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}&\frac{-\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}&\ldots&0\\0&0&\ldots 0&0&\ldots&0\end{pmatrix}$$

$$\left[f_{mn}\left(\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right)\right]$$
$$=\begin{pmatrix} 0 & 0 & \ldots 0 & 0 & \ldots 0\\ 0 & 0 & \ldots 0 & 0, & \ldots 0\\ \vdots & & & & \\ 0 & 0 & \ldots f_{mn}\left(\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right) & f_{mn}\left(\frac{-\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right) & \ldots 0\\ 0 & 0 & \ldots f_{mn}\left(\frac{-\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right) & f_{mn}\left(\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right) & \ldots 0\\ 0 & 0 & \ldots 0 & 0, & \ldots 0 \end{pmatrix}.$$

Hence, it converges to zero.

Therefore,

$$[(\lambda_{mn}-\lambda_{mn+1})-(\lambda_{m+1n}-\lambda_{m+1n+1})] \in \left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right].$$

Hence, $d((\lambda_{mn}-\lambda_{mn+1})-(\lambda_{m+1n}-\lambda_{m+1n+1}),0)=1$. But

$$|y_{mn}| \le \|f\|\, d((\lambda_{mn}-\lambda_{mn+1})-(\lambda_{m+1n}-\lambda_{m+1n+1}),0) \le \|f\|\cdot 1<\infty$$

for each $m,n$. Thus, $(y_{mn})$ is a $p$- metric paranormed space of double analytic sequence and, hence, an $p$- metric double analytic sequence.

In other words. $y\in\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$. But $y=(y_{mn})$ is arbitrary in $\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta}$.Therefore,

$$\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta} \subset \left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]. \quad (7)$$

From (6) and (7), we get

$$\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{\beta} = \left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right].$$

□

**Theorem 4.** *The dual space of* $\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$ *is* $\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$. *In other words.* $\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{*} = \left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$.

*Proof.* We recall that

$$\lambda_{mn}=\begin{pmatrix} 0 & 0 & \ldots 0 & 0 & \ldots\\ 0 & 0 & \ldots 0 & 0 & \ldots\\ \vdots & & & & \\ 0 & 0 & \ldots \frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!} & 0 & \ldots\\ 0 & 0 & \ldots 0 & 0 & \ldots \end{pmatrix}$$

with $\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}$ in the $(m,n)$th position and zeros elsewhere,

$$\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$$
$$=\begin{pmatrix} 0. & . & .\ 0\\ . & & \\ . & & \\ . & & \\ 0 & f\left(\frac{\varphi_{rs}}{\Delta\lambda_{mn}(m+n)!}\right)^{1/m+n} & .\ 0\\ & (m,n)^{\text{th}} & \\ 0 & . & .\ 0 \end{pmatrix}$$

which is a $p$- metric of double gai sequence. Hence,

$$x\in\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right].f(x) = \sum_{m,n=1}^{\infty} x_{mn}y_{mn}$$

with $x\in\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$ and $f\in\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{*}$, where $\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{*}$ is the dual space of $\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$.

Take $x=(x_{mn})\in\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$. Then,

$$|y_{mn}|\le\|f\|\,d(\varphi_{rs},0)<\infty\forall m,n. \quad (8)$$

Thus, $(y_{mn})$ is a $p$- metric of the double analytic sequence and an $p$- metric of double analytic sequence. In other words, $y\in\left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]$. Therefore,

$$\left[\chi^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right]^{*} = \left[\Lambda^{2q}_{f\mu},\|\mu_{mn}(x),(d(x_1),d(x_2),\cdots,d(x_{n-1}))\|_p^{\varphi}\right].$$

This completes the proof. □

**Theorem 5.** *(1) If the sequence $(f_{mn})$ satisfies uniform $\Delta_2-$ condition, then*

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha} = \left[\chi^{2q\mu}_g, \|\mu_{uv}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

*(2) If the sequence $(g_{mn})$ satisfies uniform $\Delta_2-$ condition, then*

$$\left[\chi^{2q\mu}_g, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha} = \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

*Proof.* Let the sequence $(f_{mn})$ satisfies uniform $\Delta_2-$ condition; we get

$$\left[\chi^{2q\mu}_g, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \subset \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha}. \quad (9)$$

To prove the inclusion

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha} \subset \left[\chi^{2q\mu}_g, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right],$$

let $a \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha}$. Then, for all $\{x_{mn}\}$ with $(x_{mn}) \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |x_{mn}a_{mn}| < \infty. \quad (10)$$

Since the sequence $(f_{mn})$ satisfies the uniform $\Delta_2-$ condition and then

$$(y_{mn}) \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right],$$

we get $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left|\frac{\varphi_{rs} y_{mn} a_{mn}}{\Delta\lambda_{mn}(m+n)!}\right| < \infty$. by (10). Thus, $(\varphi_{rs}a_{mn}) \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = \left[\chi^{2q\mu}_g, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$, and hence, $(a_{mn}) \in \left[\chi^{2q\mu}_g, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$. This gives that

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha} \subset \left[\chi^{2q\mu}_g, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]. \quad (11)$$

We are granted with (9) and (11) that

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha} = \left[\chi^{2q\mu}_g, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

(3) Similarly, one can prove that

$$\left[\chi^{2q\mu}_g, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]^{\alpha} \subset \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$$

if the sequence $(g_{mn})$ satisfies the uniform $\Delta_2-$ condition. □

**Proposition 1.** *If $0 < q_{mn} < p_{mn} < \infty$ for each $m$ and $m$, then*

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \subseteq \left[\Lambda^{2p}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

*Proof.* Let $x = (x_{mn}) \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$. We have

$$sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] < \infty.$$

This implies that

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] < 1$$

for sufficiently large value of $m$ and $n$. Since $f_{mn}$s are non-decreasing, we get

$$sup_{mn}\left[\Lambda^{2p}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \leq sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

Thus, $x = (x_{mn}) \in \left[\Lambda^{2p}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$. □

**Proposition 2.** *(1) If $0 < infq_{mn} \leq q_{mn} < 1$, then*

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \subset \left[\Lambda^{2}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

*(2) If $1 \leq q_{mn} \leq supq_{mn} < \infty$, then* $\left[\Lambda^{2}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] \subset \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$.

*Proof.* Let $x = (x_{mn}) \in \left[\Lambda^{2}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$. Since $0 < \inf q_{mn} \leq 1$, we have

$$sup_{uv}\left[\Lambda^{2}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right] \leq \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right],$$

and hence

$$x = (x_{mn}) \in \left[\Lambda^{2}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

(3) Let $q_{mn}$ for each $(m, n)$ and $sup_{mn} q_{mn} < \infty$.

Let $x = (x_{mn}) \in \left[\Lambda^{2}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$. Then, for each $0 < \epsilon < 1$, there exists a positive integer $\mathbb{N}$ such that

$$sup_{uv}\left[\Lambda^{2}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right] \leq \epsilon < 1,$$

for all $m, n \geq N$. This implies that

$$sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right] \leq sup_{mn}\left[\Lambda^{2}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

Thus, $x = (x_{mn}) \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$. □

**Proposition 3.** *Let $f' = (f'_{mn})$ and $f'' = (f''_{mn})$ be sequences of Musielak functions; we have*

$$\left[\Lambda^{2q}_{f'\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right] \times \bigcap \left[\Lambda^{2q}_{f''\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right] \times \subseteq \left[\Lambda^{2q}_{f'+f''\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

*Proof.* The proof is easy, so we omit it. □

**Proposition 4.** *For any sequence of Musielak functions $f = (f_{mn})$ and $q = (q_{mn})$ be double analytic sequence of strictly positive real numbers. Then,*

$$\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right] \subset \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

*Proof.* The proof is easy, so we omit it. □

**Proposition 5.** *The sequence space $\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$ is solid.*

*Proof.* Let $x = (x_{mn}) \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$, i.e.,

$$sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right] < \infty.$$

Let $(\alpha_{mn})$ be double sequence of scalars such that $|\alpha_{mn}| \leq 1$ for all $m, n \in N \times N$. Then, we get

$$sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(\alpha x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right] \leq sup_{mn}\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

□

**Proposition 6.** *The sequence space $\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$ is monotone.*

*Proof.* The proof follows from Proposition 5. □

**Proposition 7.** *If $f = (f_{mn})$ is any Musielak function, then*

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right] \subset \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_{p}\right]$$

*if and only if $sup_{r,s\geq 1} \frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty$.*

*Proof.* Let $x \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right]$ and $N = sup_{r,s\geq 1} \frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty$. Then, we get

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}_{rs}}_{p}\right] = N\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{*}_{rs}}_{p}\right] = 0.$$

Thus, $x \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_{p}\right]$. Conversely, suppose that

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right] \subset \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_{p}\right]$$

and $x \in \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right]$.

Then, $\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_{p}\right] < \epsilon$ for every $\epsilon > 0$. Suppose that $sup_{r,s\geq 1} \frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} = \infty$, then there exists a sequence of members $(rs_{jk})$ such that $lim_{j,k\to\infty} \frac{\varphi^*_{jk}}{\varphi^{**}_{jk}} = \infty$. Hence, we have $\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*_{rs}}_{p}\right] = \infty$. Therefore, $x \notin \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_{p}\right]$, which is a contradiction. □

**Proposition 8.** *If $f = (f_{mn})$ is any Musielak function, then*

$$\left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^*}_p\right] = \left[\Lambda^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi^{**}}_p\right]$$

*if and only if* $sup_{r,s\geq 1}\frac{\varphi^*_{rs}}{\varphi^{**}_{rs}} < \infty, sup_{r,s\geq 1}\frac{\varphi^{**}_{rs}}{\varphi^*_{rs}} > \infty.$

*Proof.* It is easy to prove, so we omit it. □

**Proposition 9.** *The sequence space* $\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ *is not solid.*

*Proof.* The result follows from the following example. Consider

$$x = (x_{mn}) = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & & & \\ 1 & 1 & \dots & 1 \end{pmatrix} \in \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right].$$

Let

$$\alpha_{mn} = \begin{pmatrix} -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \\ -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \\ \vdots & & & \\ -1^{m+n} & -1^{m+n} & \dots & -1^{m+n} \end{pmatrix},$$

for all $m, n \in \mathbb{N}$. Then, $\alpha_{mn}x_{mn} \notin \left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$. Hence, $\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ is not solid. □

**Proposition 10.** *The sequence space* $\left[\chi^{2q}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ *is not monotone.*

*Proof.* The proof follows from Proposition 9. □

## Generalized four-dimensional infinite matrix sequence spaces

Let $A = (a^{mn}_{k\ell})$ be a four-dimensional infinite matrix of complex numbers. Then, we have $A(x) = (Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a^{mn}_{k\ell}x_{mn}$ which converges for each $k, \ell$.

In this section, we introduce the following sequence spaces:

$$\left[\chi^{2qA}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = lim_{mn}\left\{\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} = 0\right\},$$

$$\left[\Lambda^{2qA}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = sup_{mn}\left\{\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} < \infty\right\}.$$

If we take $f_{mn}(x) = x$, we get

$$\left[\chi^{2qA}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = lim_{mn}\left\{\left[\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} = 0\right\},$$

$$\left[\Lambda^{2qA}_{\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = sup_{mn}\left\{\left[\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} < \infty\right\}.$$

If we take $q = (q_{mn}) = 1$,

$$\left[\chi^{2A}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = lim_{mn}\left\{\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] = 0\right\},$$

$$\left[\Lambda^{2A}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right] = sup_{mn}\left\{\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right] < \infty\right\}.$$

**Theorem 6.** *For a Musielak-modulus function,* $f = (f_{mn})$*. Then, the sequence spaces* $\left[\chi^{2qA}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ *and* $\left[\Lambda^{2qA}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_p\right]$ *are linear spaces over the set of complex numbers* $\mathbb{C}$.

*Proof.* It is routine verification. Therefore, the proof is omitted. □

**Theorem 7.** *For any Musielak-modulus function $f = (f_{mn})$ and a double analytic sequence $q = (q_{mn})$ of strictly positive real numbers, the space $\left[\chi^{2qA}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$ is a topological linear space paranormed by*

$$g(x) = inf\left\{\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\},$$

*where $H = max(1, sup_{mn}q_{mn} < \infty)$.*

*Proof.* Clearly, $g(x) \geq 0$ for $x = (x_{mn}) \in \left[\chi^{2qA}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{V_2}_{p}\right]$. Since $f_{mn}(0) = 0$, we get $g(0) = 0$. Conversely, suppose that $g(x) = 0$, then $inf\left\{\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]^{q_{mn}}\right)^{1/H}\right\} \leq 1 = 0$. Suppose that $A_{mn}\mu_{mn}(x) \neq 0$ for each $m, n \in \mathbb{N}$, then

$$\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p} \to \infty. \tag{12}$$

It follows that $\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{V_2}_{p}\right)\right]^{q_{mn}}\right)^{1/H} \to \infty$ which is a contradiction. Therefore, $A_{mn}\mu_{mn}(x) = 0$. Let $\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]^{q_{mn}}\right)^{1/H} \leq 1$ and $\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]^{q_{mn}}\right)^{1/H} \leq 1$.

Then, by using Minkowski's inequality, we have

$$\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x+y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]^{q_{mn}}\right)^{1/H}$$
$$\leq \left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]^{q_{mn}}\right)^{1/H}$$
$$+ \left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]^{q_{mn}}\right)^{1/H}.$$

So, we have

$$g(x+y) = inf\left\{\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x+y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\}$$
$$\leq inf\left\{\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\}$$
$$+ inf\left\{\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\}.$$

Therefore,

$$g(x+y) \leq g(x) + g(y).$$

Finally, to prove that the scalar multiplication is continuous, let $\lambda$ be any complex number. By definition,

$$g(\lambda x) = inf\left\{\left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(\lambda x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\}.$$

Then,

$$g(\lambda x) = inf\left\{((|\lambda| t)^{q_{mn}/H} : \left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\},$$

where $t = \frac{1}{|\lambda|}$. Since $|\lambda|^{q_{mn}} \leq max\left(1, |\lambda|^{supp_{mn}}\right)$, we have

$$g(\lambda x) \leq max\left(1, |\lambda|^{supp_{mn}}\right) inf\left\{t^{q_{mn}/H} : \left(\left[f_{mn}\left(\|A_{mn}\mu_{mn}(\lambda x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]^{q_{mn}}\right)^{1/H} \leq 1\right\}.$$

□

**Theorem 8.** *The* $\beta-$ *dual space of* $\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta} = \left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$.

*Proof.* First, we observe that

$$\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta} \subset \left[\Gamma^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

Therefore,

$$\left[\Gamma^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta} \subset \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta}.$$

But

$$\left[\Gamma^{2qA}_{f\mu}\right]^{\beta} \overset{\subset}{\neq} \left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

Hence,

$$\left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right] \subset \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta}.$$

Next, we show that

$$\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta} \subset \left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

Let $y = (y_{mn}) \in \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta}$. Consider $f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}y_{mn}$ with

$$x = (x_{mn}) \in \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$$

$$x = [(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})]$$

$$= \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ \cdots \\ 0 & 0 & \cdots \frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & \frac{-\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & \ldots & 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots \frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & \frac{-\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!} & \ldots & 0 \\ 0 & 0 & \cdots 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\left[f_{mn}\left(\|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right)\right]$$

$$= \begin{pmatrix} 0 & 0 & \ldots 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots 0 & 0, & \ldots & 0 \\ . \\ . \\ . \\ 0 & 0 & \ldots f_{mn}\left(\frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right) & f_{mn}\left(\frac{-\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right) & \ldots & 0 \\ 0 & 0 & \ldots f_{mn}\left(\frac{-\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right) & f_{mn}\left(\frac{\varphi_{rs}}{a^{mn}_{k\ell}\Delta\lambda_{mn}(m+n)!}\right) & \ldots & 0 \\ 0 & 0 & \ldots 0 & 0, & \ldots & 0 \end{pmatrix}.$$

Hence, converges to zero.

Therefore,

$$[(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1})] \times \in \left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right].$$

Hence, $d\left(a^{mn}_{k\ell}(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0\right) = 1$. However, $|y_{mn}| \leq \|f\| d\left(a^{mn}_{k\ell}(\lambda_{mn} - \lambda_{mn+1}) - (\lambda_{m+1n} - \lambda_{m+1n+1}), 0\right) \leq \|f\| \cdot 1 < \infty$ for each $m, n$. Thus, $(y_{mn})$ is a $p$- metric paranormed space of double analytic sequence and, hence, an $p$- metric double analytic sequence.

In other words, $y \in \left[\Lambda^{2qA}_{f\mu}, \|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]$. However, $y = (y_{mn})$ is arbitrary in $\left[\chi^{2q}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta}$. Therefore,

$$\left[\chi^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]^{\beta} \subset \left[\Lambda^{2qA}_{f\mu}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|^{\varphi}_{p}\right]. \quad (13)$$

From (12) and (13), we get

$$\left[\chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^{\beta} = \left[\Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right].$$

□

**Theorem 9.** *The dual space of* $\left[\chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]$ *is* $\left[\Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]$*. In other words,*

$$\left[\chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^{*} = \left[\Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right].$$

*Proof.* We recall that

$$\lambda_{mn} = \begin{pmatrix} 0 & 0 & \cdots 0 & 0 & \cdots \\ 0 & 0 & \cdots 0 & 0 & \cdots \\ \vdots & & & & \\ 0 & 0 & \cdots \frac{\varphi_{rs}}{a_{k\ell}^{mn}\Delta\lambda_{mn}(m+n)!} & 0 & \cdots \\ 0 & 0 & \cdots 0 & 0 & \cdots \end{pmatrix}$$

with $\frac{\varphi_{rs}}{a_{k\ell}^{mn}\Delta\lambda_{mn}(m+n)!}$ in the $(m,n)$th position and zero elsewhere,

$$\left[\chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right] = \begin{pmatrix} 0. & . & .\ 0 \\ \vdots & & \\ 0 & f\left(\frac{\varphi_{rs}}{a_{k\ell}^{mn}\Delta\lambda_{mn}(m+n)!}\right)^{1/m+n} & .\ 0 \\ & (m,n)^{\text{th}} & \\ 0 & . & .\ 0 \end{pmatrix}$$

which is a $p$- metric of double gai sequence. Hence,

$$x \in \left[\chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right] f(x) = \sum_{m,n=1}^{\infty} x_{mn}y_{mn}$$

with

$$x \in \left[\chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]$$

and

$$f \in \left[\chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^{*},$$

where

$$\left[\chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^{*}$$

is the dual space of $\left[\chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]$.

Take $x = (x_{mn}) \in \left[\chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]$. Then,

$$\left|y_{mn}\right| \le \|f\|\, d(\varphi_{rs}, 0) < \infty \forall m, n. \quad (14)$$

Thus, $(y_{mn})$ is a $p$- metric of double analytic sequence and, hence, an $p$- metric of double analytic sequence. In other words, $y \in \left[\Lambda_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]$. Therefore,

$$\left[\chi_{f\mu}^{2qA}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right]^{*} = \left[\Lambda_{f\mu}^{2q}, \|A_{mn}\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p^{\varphi}\right].$$

□

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally in introducing gai-2 sequence spaces generalized over $p$-metric defined by Musielak modulus function and in studying topological properties. All authors read and approved the final manuscript.

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**References**

1. Bromwich, TJI'A: An Introduction to the Theory of Infinite Series. Macmillan, New York (1965)
2. Hardy, GH: On the convergence of certain multiple series. Proc. Camb. Phil. Soc. **19**, 86–95 (1917)
3. Moricz, F: Extentions of the spaces $c$ and $c_0$ from single to double sequences. Acta. Math. Hung. **57**(1-2), 129–136 (1991)
4. Moricz, F, Rhoades, BE: Almost convergence of double sequences and strong regularity of summability matrices. Math. Proc. Camb. Phil. Soc. **104**, 283–294 (1988)
5. Basarir, M, Solancan, O: On some double sequence spaces. J. Indian Acad. Math. **21**(2), 193–200 (1999)
6. Tripathy, BC: On statistically convergent double sequences. Tamkang J. Math. **34**(3), 231–237 (2003)
7. Turkmenoglu, A: Matrix transformation between some classes of double sequences. J. Inst. Math. Comp. Sci. Math. Ser. **12**(1), 23–31 (1999)
8. Gökhan, A, Çolak, R: The double sequence spaces $c_2^P(p)$ and $c_2^{PB}(p)$. Appl. Math. Comput. **157**(2), 491–501 (2004)
9. Gökhan, A, Çolak, R: Double sequence spaces $\ell_2^{\infty}$. ibid. **160**(1), 147–153 (2005)
10. Zeltser, M: Investigation of double sequence spaces by soft and hard analytical methods. Dissertationes Mathematicae Universitatis Tartuensis 25, Tartu University Press, University of Tartu (2001)
11. Mursaleen, M, Edely, OHH: Statistical convergence of double sequences. J. Math. Anal. Appl. **288**(1), 223–231 (2003)
12. Altay, B, BaŞar, F: Some new spaces of double sequences. J. Math. Anal. Appl. **309**(1), 70–90 (2005)
13. BaŞar, F, Sever, Y: The space $\mathcal{L}_p$ of double sequences. Math. J. Okayama Univ. **51**, 149–157 (2009)
14. Subramanian, N, Misra, UK: The semi normed space defined by a double gai sequence of modulus function. Fasciculi Math. **45**, 111–120 (2010)
15. Maddox, IJ: Sequence spaces defined by a modulus. Math. Proc. Cambridge Philos. Soc. **100**(1), 161–166 (1986)

16. Cannor, J: On strong matrix summability with respect to a modulus and statistical convergence. Canad. Math. Bull. **32**(2), 194–198 (1989)
17. Pringsheim, A: Zurtheorie derzweifach unendlichen zahlenfolgen. Math. Ann. **53**, 289–321 (1900)
18. Hamilton, HJ: Transformations of multiple sequences. Duke Math. J. **2**, 29–60 (1936)
19. Hamilton, HJ: A generalization of multiple sequences transformation. Duke Math. J. **4**, 343–358 (1938)
20. Hamilton, HJ: Preservation of partial limits in multiple sequence transformations. Duke Math. J. **4**, 293–297 (1939)
21. Kamthan, PK, Gupta, M: Sequence Spaces and Series, Lecture notes, Pure and Applied Mathematics. 65 Marcel Dekker, Inc., New York (1981)
22. Lindenstrauss, J, Tzafriri, L: On Orlicz sequence spaces. Israel J. Math. **10**, 379–390 (1971)
23. Kizmaz, H: On certain sequence spaces. Cand. Math. Bull. **24**(2), 169–176 (1981)
24. Wilansky, A: Summability Through Functional Analysis, Vol. 85. North-Holland Mathematical Studies, North-Holland Publishing, Amsterdam (1984)

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