Haghi *et al. Mathematical Sciences* 2013, 7:41
http://www.iaumath.com/content/7/1/41

Mathematical Sciences
a SpringerOpen Journal

ORIGINAL RESEARCH Open Access

# Fixed point results on subgraphs of directed graphs

SMA Aleomraninejad[1], Sh Rezapour[1] and N Shahzad[2*]

**Abstract**

In this paper, we obtain some fixed point results on subgraphs of directed graphs. We show that the Caristi fixed point theorem and a version of Knaster-Tarski fixed point theorem are special cases of our results.

**Keywords:** Directed graph; Fixed point; Self-path map

**2010 MSC:** 47H10; 05C20; 54H25

## Introduction

In 2005, Echenique started combining fixed point theory and graph theory by giving a short constructive proof for the Tarski fixed point theorem using graphs [1]. Afterwards, Espinola and Kirk applied fixed point results in graph theory [2]. A considerable contribution was made by Jachymski [3] and Beg et al. [4]. More recently, the authors, by providing a new notion of (P)-graphs and using arguments similar to those of Reich et al. [5-8], presented some iterative scheme results for $G$-contractive and $G$-nonexpansive maps on graphs [9]. In this paper, we obtain some fixed point results on subgraphs of directed graphs. As some consequences of our results, we obtain the Caristi fixed point theorem and Knaster-Tarski fixed point theorem.

Let $(X, d)$ be a metric space and $G$ a directed graph $G$ such that $V(G) = X$ and the set $E(G)$ of its edges contains all loops. We denote the conversion of a graph $G$ by $G^{-1}$, that is, the graph obtained from $G$ by reversing the direction of the edges. A mapping $f : X \to X$ preserves the edges of $G$ whenever $(x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ for all $x, y \in X$ [3]. Since $G$ is a directed graph, the direction of edge $(x, y)$ is the inverse of the direction of edge $(y, x)$, that is, $(x, y) \neq (y, x)$. Let $G$ be the directed graph. A finite path of length $n$ in $G$ from $x$ to $y$ is a sequence $\{x_i\}_{i=0}^{n}$ of distinct vertices such that $x_0 = x$, $x_n = y$, and $(x_i, x_{i+1}) \in E(G)$ for $i = 0, 1, ..., n-1$ [9]. In fixed point theory, we like to deal with infinite graphs (see [9]). For this reason, we consider infinite paths. In fact, $B \subseteq E(G)$ is an infinite path whenever there is a finite path between any of its two vertices. Throughout this paper, a path could be finite or infinite, and the vertices of the path are pairwise distinct. Also, we consider cycles as finite paths. We denote by $[x]_G$ the set of all vertices in $G$ wherein there is a (finite or infinite) path from those to $x$.

Let $G'$ be a subgraph of the directed graph $G$ and $x \in G'$. We emphasize that $[x]_G$ denotes the set of all vertices in $G$ wherein there is a path from those to $x$ via the edges in $G$. Also, we remind here that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. Let $G'$ be a subgraph of the directed graph $G$. We say that $b \in G$ is an upper bound for $G'$ whenever $g' \in [b]_G$ for all $g' \in G'$. Also, we say that $c \in G$ is a supremum of $G'$ whenever $c \in [b]_G$ for all upper bounds $b$. In fact, $c$ is a least upper bound in a sense.

*Example 1.1.* Let $G$ be the directed graph via the vertices $V(G) = \{a, b, c, d\}$ and the edges $E(G) = \{(a, b), (b, c), (c, d), (d, a)\}$. Suppose that $G'$ is a subgraph of $G$ denied by $V(G') = \{a, b, c\}$ and $E(G') = \{(a, b), (b, c)\}$, then $c, d$ are upper bounds of $G'$. Thus, an upper bound is not unique in a subgraph necessarily.

*Example 1.2.* Let $G$ be the directed graph via the vertices $V(G) = \{0, 2, \frac{1}{n} : n \geq 1\}$ and the edges $E(G) = \{(\frac{1}{n}, \frac{1}{n+1}) : n \geq 1\} \bigcup \{(\frac{1}{n}, 0)\} \bigcup (0, 2) \bigcup \{(\frac{1}{n}, 2) : n \geq 1\}$. If $V(G') = \{0, \frac{1}{n} : n \geq 1\}$ and $E(G') = \{(\frac{1}{n}, \frac{1}{n+1})\}$, then 0 and 2 are the supremum of $G'$. Thus, a supremum is not unique in a subgraph necessarily.

*Correspondence: nshahzad@kau.edu.sa
[2]Department of Mathematics, King AbdulAziz University, P.O. Box 80203, 21859 Jeddah, Saudi Arabia
Full list of author information is available at the end of the article

Let $G$ be a directed graph and $x_0 \in G$. We say that $x_0$ is an end point whenever there is no $x \in G$ such that $(x_0, x) \in G$ and $x \neq x_0$. There are many directed graphs via end points. In the following result, we give a class of directed graphs which have end points. The proof of this result is straightforward.

**Lemma 1.1.** *Let $G$ be the directed graph, $X = V(G)$, $\varphi : X \to \mathbb{R}$ a function, $E(G) = \{(x, y) : d(x, y) \leq \varphi(x) - \varphi(y)\}$, and $d$ a metric on $X$. If there exists $x_0 \in X$ such that $\varphi(x_0) = \inf_{x \in X} \varphi(x)$, then $x_0$ is an end point of $G$.*

## Main results

Now we are ready to state and prove our main results. Let $G$ be the directed graph and $\mathcal{M}$ the set of all paths in $G$. Then $\subseteq$ is a partial order on $\mathcal{M}$. By using Hausdorff's maximum principle, $\mathcal{M}$ has a maximal element. This means that $G$ has a maximal path. We use this subject in our results.

**Theorem 2.1.** *Let $G$ be a directed graph such that every path in $G$ has an upper bound. Then $G$ has an end point or a cycle.*

*Proof.* Suppose that $G$ has no cycle. Let $B$ be the maximal path in $G$ and $u$ an upper bond of $B$. If $u$ is not an end point, there exists $x \in G$ such that $x \neq u$ and $(u, x) \in E(G)$. Thus, $B \bigcup \{x\}$ is a path in $G$ and $B \subset B \bigcup \{x\}$. This contradiction shows that $u$ is an end point of $G$. □

Let $G$ be a directed graph and $T$ a selfmap on $G$. We say that $T$ is a self-path map whenever $x \in [Tx]_G$ for all $x \in G$.

**Theorem 2.2.** *Let $G$ be a directed graph. Then $G$ has an end point if and only if each self-path map on $G$ has a fixed point.*

*Proof.* Suppose that $G$ has an end point $x_0$ and $T$ is a self-path map. We prove that $x_0$ is a fixed point of $T$. Since $x_0 \in [Tx_0]_G$, there is a (finite or infinite) path $\{\lambda_i\}_{i \geq 0}$ between $x_0$ and $Tx_0$. Since $x_0$ is the end point of $G$ and $\lambda_0 = x_0$, we have $x_0 = \lambda_1$. By continuing this process, it is easy to see that $x_0 = \lambda_i$ for all $i$. Thus, $x_0 = Tx_0$. Now assume that $G$ is a directed graph and each self-path map on $G$ has a fixed point but has no end point. Then for each $x \in G$, there exists $y \in G$ such that $y \neq x$ and $(x, y) \in E(G)$. By using the selection principle, we can define a selfmap $T$ on $G$ by $Tx = y$. Note that $T$ is a self-path map which has no fixed point. □

*Example 2.1.* Let $G$ be the directed graph via the vertices $V(G) = \{0, \frac{1}{n} : n \geq 1\}$ and the edges $E(G) = \{(\frac{1}{n}, \frac{1}{n+1}) : n \geq 1\} \bigcup \{(\frac{1}{n}, 0)\} \bigcup (0, 1)$. Define the selfmap $T$ on $G$ by $T_0 = 1$ and $T\frac{1}{n} = \frac{1}{n+1}$ for all $n \geq 1$. Then $x \in [Tx]_G$ for all $x \in G$; $T$ has no fixed point and $G$ has no end point.

**Theorem 2.3.** *Let $G$ be a directed graph such that every path in $G$ has a supremum and $T$ a selfmap on $G$ such that $Tx \in [Ty]_G$ for all $x \in [y]_G$, $G' = \{x \in G : x \in [Tx]_G\} \neq \emptyset$, and $G'$ has no cycle. Then $T$ has a fixed point in $G'$.*

*Proof.* Suppose that $B$ is a path in $G$ and $b$ is the supremum of $B$ in $G$. Since $c \in [b]_G$ for all $c \in B$, $Tc \in [Tb]_G$ and so $c \in [Tb]_G$. It follows that $Tb$ is an upper bound for $B$. Since $b$ is the supremum, $b \in Tb$. Thus, $b \in G'$. By using Theorem 2.1, $G'$ has an end point. Since $x \in [Tx]$ for all $x \in G'$, $Tx \in [T^2x]_G$ and so $T$ is a self-path map on $G'$. Now by using Theorem 2.2, $T$ has a fixed point in $G'$. □

Now we show that a version of Knaster-Tarski fixed point theorem is a consequence of Theorem 2.3.

**Theorem 2.4.** *Let $(X, \preceq)$ be a partially ordered set such that each chain in $X$ has a supremum and $T$ a monotone selfmap on $X$. Assume that there exists $a \in X$ such that $a \preceq Ta$. Then $T$ has a fixed point.*

*Proof.* Define the graph $G$ by $V(G) = X$ and $E(G) = \{(x, y) : x \preceq y \text{ and } x \neq y\}$. Then $Tx \in [Ty]_G$ for all $x \in [y]_G$. Since $G' = \{x \in G : x \in [Tx]_G\} \neq \emptyset$ and $G'$ has no cycle, by using Theorem 2.3, $T$ has a fixed point. □

Let $X$ be a set and $\varphi : X \to (-\infty, \infty)$ a map. Suppose that $G$ is the directed graph defined by $V(G) = X$ and $E(G) = \{(x, y) : d(x, y) \leq \varphi(x) - \varphi(y)\}$. We say that $\varphi$ is lower semi-continuous whenever $\varphi(x) \leq \varphi(x_n)$ for all sequence $\{x_n\}$ in $X$ with $x_n \to x$.

**Lemma 2.5.** *Let $X$ be a complete metric space and $\varphi : X \to (-\infty, \infty)$ a map bounded from below. Suppose that $G$ is the directed graph defined by $V(G) = X$ and $E(G) = \{(x, y) : d(x, y) \leq \varphi(x) - \varphi(y)\}$. If $\varphi$ is lower semi-continuous, then $G$ has an end point.*

*Proof.* First we prove that $G$ has no cycle. If $G$ has a cycle, then there exists a path $\{\lambda_i\}_{i=1}^n$ in $G$ such that $\lambda_1 = \lambda_n$. It is easy to check that $d(\lambda_1, \lambda_i) \leq \varphi(\lambda_1) - \varphi(\lambda_i)$ and $d(\lambda_i, \lambda_n) \leq \varphi(\lambda_i) - \varphi(\lambda_n)$ for all $i = 2, 3, ..., n-1$, and so $\lambda_i = \lambda_1$ for $i \geq 2$. This contradiction shows that $G$ has no cycle. Now we prove that each path in $G$ has an upper bound. Let $\{x_\alpha\}_{\alpha \in \Omega}$ be a path in $G$. Then $\{\varphi(x_\alpha)\}_\Omega$ is a decreasing net of real numbers. Since $\varphi$ is bounded from below, there is an increasing sequence $\{\alpha_n\}_{n \geq 1}$ in $\Omega$ such that $\lim_{n \to \infty} \varphi(x_{\alpha_n}) = \inf_{\alpha \in \Omega} \varphi(x_\alpha)$. One can easily show that $\{x_{\alpha_n}\}_{n \geq 1}$ is a Cauchy sequence and so converges to some $x \in X$. Since $\varphi$ is lower semi-continuous, $x_{\alpha_n} \in [x]_G$ for all $n \geq 1$. Thus, $x$ is an upper bound for $\{x_{\alpha_n}\}_{n \geq 1}$.

Now we show that $x$ is an upper bound for $\{x_\alpha\}_{\alpha\in\Omega}$. If there exists $\beta \in \Omega$ such that $x_{\alpha_n} \in [x_\beta]_G$ for all $n \geq 1$, then $\varphi(x_\beta) \leq \varphi(x_{\alpha_n})$ for all $n \geq 1$ which implies that $\varphi(x_\beta) = \inf_{\alpha\in\Omega}\varphi(x_\alpha)$. Since $d(x_{\alpha_n}, x_\beta) \leq \varphi(x_{\alpha_n}) - \varphi(x_\beta)$, we get $x_{\alpha_n} \to x_\beta$ which implies that $x_\beta = x$. Hence, $\varphi(x) = \inf_{\alpha\in\Omega}\varphi(x_\alpha)$. Now we claim that $x_\alpha \in [x]_G$, and so $x$ is an upper bound for $\{x_{\alpha_n}\}_{n\geq 1}$. In fact if there is $\alpha \in \Omega$ such that $x \in [x_\alpha]_G$, then $d(x, x_\alpha) \leq \varphi(x) - \varphi(x_\alpha) \leq \varphi(x_\alpha) - \varphi(x_\alpha) = 0$, and so $x = x_\alpha$. Since $\{x_\alpha\}_{\alpha\in\Omega}$ is a path in $G$, if the last case does not hold, then for each $\alpha \in \Omega$ there exists $n \geq 1$ such that $x_\alpha \in [x_{\alpha_n}]_G$. Hence, $x_\alpha \in [x]_G$ for all $\alpha \in \Omega$. Thus, $x$ is an upper bound for $\{x_\alpha\}_{\alpha\in\Omega}$. Now by using Theorem 2.1, $G$ has an end point. □

Now we can consequent the Caristi fixed point theorem.

**Theorem 2.5.** *Let $X$ be a complete metric space, $\varphi : X \to (-\infty, \infty)$ a map bounded from below and lower semi-continuous, and $T : X \to X$ a selfmap satisfying $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$ for all $x \in X$. Then $T$ has a fixed point.*

*Proof.* Suppose that $G$ is the directed graph via the vertices $V(G) = X$ and the edges $E(G) = \{(x, y) : d(x, y) \leq \varphi(x) - \varphi(y)\}$. By using Lemma 2.5, $G$ has an end point. It is easy to see that $T$ is a self-path map on $G$. Now by using Theorem 2.2, $T$ has a fixed point. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
AB carried out in this manuscript. Also, all authors read and approved the final manuscript.

**Acknowledgements**
Research of the first and second authors was supported by Azarbaidjan Shahid Madani University. Also, the authors express their gratitude to the referees for their helpful suggestions which improved final version of this paper.

**Author details**
[1]Department of Mathematics, Azarbaidjan Shahid Madani University, Azarshahr, Tabriz, Iran. [2]Department of Mathematics, King AbdulAziz University, P.O. Box 80203, 21859 Jeddah, Saudi Arabia.

Received: 31 May 2013 Accepted: 15 July 2013
Published: 19 Aug 2013

**References**
1. Echenique, F: A short and constructive proof of Tarski's fixed point theorem. Internat. J. Game Theory. **33**(2), 215–218 (2005)
2. Espinola, R, Kirk, WA: Fixed point theorems in R-trees with applications to graph theory. Topology Appl. **153**, 1046–1055 (2006)
3. Jachymski, J: The contraction principle for mappings on a metric space with a graph. Proc. Amer. Math. Soc. **136**(4), 1359–1373 (2008)
4. Beg, I, Butt, AR, Radojevic, S: The contraction principle for set valued mappings on a metric space with a graph. Comput. Math. Appl. **60**, 1214–1219 (2010)
5. Reich, S, Zaslavski, AJ: Convergence of inexact iterative schemes for nonexpansive set-valued mappings. Fixed Point Theory Appl. **2010**, 518243 (2010). doi:10.1155/2010/518243
6. De Blasi, FS, Myjak, J, Reich, S, Zaslavski, AJ: Generic existence and approximation of fixed points for nonexpansive set-valued maps. Set-Valued Anal. **17**, 97–112 (2009)
7. Reich, S, Zaslavski, AJ: Approximating fixed points of contractive set-valued mappings. Commun. Math. Anal. **8**, 70–78 (2010)
8. Reich, S, Zaslavski, AJ: Existence and approximation of fixed points for set-valued mappings. Fixed Point Theory Appl. **2010**, 351531 (2010). doi:10.1155/2010/351531
9. Aleomraninejad, SMA, Rezapour, Sh, Shahzad, N: Some fixed point results on a metric space with a graph. Topology Appl. **159**, 659–663 (2012)

10.1186/2251-7456-7-41
**Cite this article as:** Haghi *et al.*: Fixed point results on subgraphs of directed graphs. *Mathematical Sciences* 2013, 7:41