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A manual approach for calculating the root of square matrix of dimension $\leq \ni$

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Abstract

In this article, we are to find the root of a square matrix *A*. Specially, if matrix *A* has multiple eigenvalues, we present a manual solution so as to find the root of it. In other words, we focus on solving the equation $X^2 = A$ and find the solutions.

Keywords: Square root of matrix; Matrix equation; Eigenvalue

MSC: 15A24

Introduction

In recent years, several articles are written about the root of a matrix, and we can refer to [1-6] for instance. In 2007, Kh. Ikramov tried to solve the matrix equation $X\overline{X} = I$ and $\overline{X}X = I$. At first, he considered the problem as $X\overline{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and then generalized it to general case $X\overline{X} = I_n$ and $\overline{X}X = I_n$ [7]. If matrix A has different eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the root of A, i.e. solution of the equation $X^2 = A$ is achieved as the following:

$$X = VEV^{-1}, \quad E = \mathrm{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \cdots, \sqrt{\lambda_n}), \tag{1}$$

where V is a matrix, columns of which are eigenvectors of A. If matrix A is not diagonalizable, the solution is not so simple. In this article, first of all, we are to solve the following matrix equation in different cases and try to generalize the result for greater dimensions:

$$X\overline{X} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A. \tag{2}$$

In section "Solution of $\boldsymbol{X^2 = A}$", we solve the matrix equation (2). In section "Solution of $\boldsymbol{X^2 = A}$ in which $\boldsymbol{A}$ is of dimension 3", we solve the equation $X^2 = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ and ultimately in section "Second root of **3 × 3** matrix", generalizations of it are verified.

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Solution of $X^2 = A$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In this section, we are to solve the equation (2) in four cases as the following:

Case 1 - The matrix A has two different eigenvalues.
Case 2 - The matrix A has two nonzero different simple eigenvalues and $b \neq 0$.
Case 3 - The matrix A has two nonzero different simple eigenvalues and $b = 0$.
Case 4 - The matrix A has two different simple eigenvalues which are zero.

For the first case, as the right-hand side matrix is diagonalized, the solution of (2) is as (1).

Lemma 1. *(Case two) If A has a multiple nonzero eigenvalue, and* $b \neq 0$*, then solution of the equation (2) is as below:*

$$X = \begin{bmatrix} \frac{3a+d}{4\sqrt{\frac{a+d}{2}}} & \frac{b}{2\sqrt{\frac{a+d}{2}}} \\ \frac{c}{\sqrt{\frac{a+d}{2}}} & \frac{a+3d}{4\sqrt{\frac{a+d}{2}}} \end{bmatrix}.$$

Proof. Writing the characteristic equation of A, we have:

$$\lambda I - A = \lambda^2 - (a+d)\lambda + ad - bc = 0.$$

Consider its eigenvalues as follows:

$$\lambda_1 = \frac{a+d}{2}, \qquad \lambda_2 = \frac{a+d}{2} + t, \qquad (t \to 0).$$

In this case, as we see in (1) , we attempt to find the root of matrix A, setting

$$E = \begin{bmatrix} \sqrt{\frac{a+d}{2}} & 0 \\ 0 & \sqrt{\frac{a+d}{2}+t} \end{bmatrix}.$$

If so, the matrix of corresponding eigenvalues with these eigenvalues is as below:

$$V = \begin{bmatrix} 2b & 2b \\ d-a & d-a+2t \end{bmatrix} \Rightarrow V^{-1} = \begin{bmatrix} \frac{d-a+2t}{4} & \frac{-1}{2t} \\ \frac{a-d}{4bt} & \frac{1}{2t} \end{bmatrix}.$$

In this case,

$$X = \lim_{t\to 0} VEV^{-1} = \lim_{t\to 0} \times \begin{bmatrix} \frac{2b\sqrt{\frac{a+b}{2}}(b-a+2t)+(a-d)\left(2b\sqrt{\frac{a+d}{2}+t}\right)}{4bt} & \frac{-2b\sqrt{\frac{a+d}{2}}+2b\sqrt{\frac{a+d}{2}+t}}{2t} \\ \frac{(d-a)\sqrt{\frac{a+d}{2}}(d-a+2t)\sqrt{\frac{a+b}{2}+t(a-d)}}{4bt} & (a-b)\sqrt{\frac{a+b}{2}}+(d-a+2t)\sqrt{\frac{a+d}{2}+t} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{3a+d}{4\sqrt{\frac{a+d}{2}}} & \frac{b}{2\sqrt{\frac{a+d}{2}}} \\ \frac{c}{2\sqrt{\frac{a+d}{2}}} & \frac{a+3d}{4\sqrt{\frac{a+d}{2}}} \end{bmatrix}$$

is solution of the problem. $\square$

Theorem 1. *(Case 3) If A has two simple nonzero eigenvalues, $b=0$ and $c \neq 0$, then the matrices*

$$X = \begin{bmatrix} \sqrt{a} & c \\ \frac{c}{2\sqrt{a}} & \sqrt{a} \end{bmatrix}, X = \begin{bmatrix} -\sqrt{a} & c \\ \frac{-c}{2\sqrt{a}} & -\sqrt{a} \end{bmatrix}$$

are some solutions of (2).

If $A = \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$ and $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is the root of it, then setting $X^2 = A$ is the solution above.

Corollary 1. *If A has a zero multiple eigenvalue and $b = c = 0$, then in addition to the two solutions in (4), an infinite number of solutions exist as the following for (2):*

$$X = \begin{bmatrix} \sqrt{a} & c \\ a & -\sqrt{a} \end{bmatrix}, X = \begin{bmatrix} -\sqrt{a} & c \\ a & \sqrt{a} \end{bmatrix}, X = \begin{bmatrix} \beta & \gamma \\ \frac{\alpha-\beta^2}{\gamma} & -\beta \end{bmatrix},$$

in which $\alpha, \beta, \gamma \neq 0 \in C$.

Lemma 2. *(Case 4) If A has two zero eigenvalues, then one of the following cases occurs:*

a) If $b = c = 0$, then $X = 0$ is a solution of the equation.
b) If $b = 0$ and $c \neq 0$, then the equation has no solution.
c) If $c = 0$ and $b \neq 0$, then the equation has no solution.
d) If $bc \neq 0$, then the equation has no solution.

Assuming $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, we can verify all four cases presented before. Case (a) is trivial. For case (b), if $b = 0$ and $c \neq 0$, then from $X^2 = A$, we must have:

$$\begin{cases} w^2 + xy = 0 \\ x^2 + yz = 0 \\ (x+w)y = 0 \\ z(x+w) = c \end{cases} \Rightarrow \quad y = 0 \;\; or \;\; (x+w) = 0.$$

If $y = 0$, then $x = w = 0$ which is in contradiction with $c \neq 0$. If $x + w = 0$, then we must have $c = 0$, which is impossible. Therefore, in this case, the second root of A does not exist.

Case (c) is similar to case (b) and for case (d), according to the characteristic equation of matrix A, we have:

$$\lambda^2 - (a+d)\lambda + ab - bc = 0.$$

By our assumption, matrix A has two zero eigenvalues, then

$$\begin{cases} a + d = 0 \\ ad - bc = 0. \end{cases}$$

As a result, $a = -d$ and $bc = -a^2$ and as b and c are nonzero, so $a \neq 0$. Thus, in removing b and d, we have:

$$A = \begin{bmatrix} a & \frac{-a^2}{c} \\ c & -a \end{bmatrix}.$$

Now, we claim that this matrix has no square root. If $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is its root, then according to $X^2 = A$, we have:

$$w^2 + xy = -a^2, \tag{3}$$

$$x^2 + yz = a, \tag{4}$$

$$xy + yw = \frac{-a^2}{c}, \tag{5}$$

$$xz + wz = c, \tag{6}$$

from (5), we have:

$$y(x+w) = \frac{-a^2}{c}, \tag{7}$$

and also from (6), with $c \neq 0$, we have:

$$z(x+w) = c \Rightarrow x + w \neq 0 \,\text{and}\, z \neq 0. \tag{8}$$

According to both (5) and (6), we achieve $\frac{y}{z} = \frac{-a^2}{c^2} \Rightarrow z = \frac{-c^2}{a^2}y$, (4) implies that $x^2 - \frac{y^2c^2}{a^2} = a$ and regarding (6) and (8), conclude that

$$(x+w)(\frac{-c^2}{a^2}y) = c. \tag{9}$$

The relation (3) implies that

$$w^2 - \frac{c^2}{a^2}y^2 = -a$$

and ultimately considering (4) and (9), we have:

$$x^2 - w^2 = 2a \tag{10}$$

from (9) and (10), we have:

$$(x-w)\left(-\frac{a^2}{cy}\right) = 2a \Rightarrow x - w = -\frac{2cy}{a}; \tag{11}$$

from (9), we have:

$$x + w = -\frac{a^2}{cy}; \tag{12}$$

the relations (11) and (12) imply that

$$2x = -\frac{2c}{a}y - \frac{az}{cy} \Rightarrow x = -\frac{c}{a}y - \frac{a^2}{2cy};$$

from (7), we have:

$$\left(\frac{c}{a}y - \frac{a^2}{2cy}\right)^2 - \frac{c^2y^2}{az} = a \Rightarrow \frac{a^2}{4c^2yz} = 0 \Rightarrow a = 0,$$

and this is impossible and proves the lemma.

Solution of $X^2 = A$ in which A is of dimension 3

The second root of all square matrices that have different eigenvalues are achieved. In this article, the cases are studied in which eigenvalues are not different.

The second root of a matrix A is $X = VEV^{-1}$, where V is a matrix, rows of which are eigenvectors of A and also V is diagonal matrix, and the entries on its diagonal are square root of eigenvalues of A, because

$$X^2 = (VEV^{(-1)})^2 = VEV^{(-1)}VEV^{(-1)} = VE^2V^{(-1)} = A.$$

In this section, we solve the problem for the triangular matrix A. Next, we attempt to solve the problem for a more generalized case.

Theorem 2. *The second root of matrix*

$$A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & d \end{bmatrix},$$

in which $a \neq d$ and $d \neq 0$ equals:

$$X = \begin{bmatrix} \sqrt{a} & \frac{b}{\sqrt{a}+\sqrt{d}} & m \\ 0 & \sqrt{d} & \frac{e}{2\sqrt{d}} \\ 0 & 0 & \sqrt{d} \end{bmatrix},$$

in which

$$m = \frac{-be(\sqrt{d}-\sqrt{a}) + (d-a)(\frac{be}{2\sqrt{d}} + c\sqrt{d} - c\sqrt{a})}{(a-d)^2}.$$

Proof. $\lambda_1 = a, \lambda_2 = d, \lambda_3 = d + t\ (t \to 0) \Rightarrow E = \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{d} & 0 \\ 0 & 0 & \sqrt{d+t} \end{bmatrix}.$

$$\left.\begin{aligned} AX_1 = \lambda_1 X_1 \Rightarrow X_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ AX_2 = \lambda_2 X_2 \Rightarrow X_2 &= \begin{pmatrix} -b \\ a-d \\ 0 \end{pmatrix} \\ AX_3 = \lambda_3 X_3 \Rightarrow X_3 &= \begin{pmatrix} \frac{-be-tc}{t(a-d-t)} & \frac{e}{t} & 1 \end{pmatrix}^T \end{aligned}\right\} \Rightarrow$$

$$V = \begin{bmatrix} 1 & -b & \frac{be+tc}{td+t^2-ta} \\ 0 & a-d & \frac{e}{t} \\ 0 & 0 & 1 \end{bmatrix}.$$

$$X = \lim_{t\to 0} VEV^{-1} = \begin{bmatrix} \sqrt{a} & \frac{b}{\sqrt{a}+\sqrt{d}} & m \\ 0 & \sqrt{d} & \frac{e}{2\sqrt{d}} \\ 0 & 0 & \sqrt{d} \end{bmatrix}$$

$$m = \frac{-be(\sqrt{d}-\sqrt{a}) + (d-a)(\frac{be}{2\sqrt{d}} + c\sqrt{d} - c\sqrt{a})}{(a-d)^2}.$$

□

Example 1. *The second root of* $A = \begin{bmatrix} 1 & 4 & 9 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$ *equals:*

$$X = \begin{bmatrix} 1 & \frac{4}{3} & \frac{26}{9} \\ 0 & 2 & \frac{1}{4} \\ 0 & 0 & 2 \end{bmatrix}.$$

Theorem 3. *The second root of* $A = \begin{bmatrix} d & b & c \\ 0 & d & e \\ 0 & 0 & a \end{bmatrix}$, *in which $a \neq d$ and $d \neq 0$ equals X such that*

$$X = \begin{bmatrix} \sqrt{d} & \frac{b}{2\sqrt{d}} & \frac{(be+ca-cd)(\sqrt{a}-\sqrt{d}) + \frac{be(d-a)}{2\sqrt{d}}}{(a-d)^2} \\ 0 & \sqrt{d} & \frac{e}{\sqrt{a}+\sqrt{d}} \\ 0 & 0 & \sqrt{a} \end{bmatrix}.$$

$\lambda_1 = d,\ \lambda_2 = d + t\ (t \to 0),$

$\lambda_3 = a \Rightarrow E = \begin{bmatrix} \sqrt{d} & 0 & 0 \\ 0 & \sqrt{d+t} & 0 \\ 0 & 0 & \sqrt{a} \end{bmatrix}$

Proof. $AX_i=\lambda_i X_i, i=1,2,3 \Rightarrow V=\begin{bmatrix}1 & \frac{b}{t} & \frac{be+ca-cd}{(a-d)^2}\\ 0 & 1 & \frac{e}{a-d}\\ 0 & 0 & 1\end{bmatrix}$

$$\Rightarrow X=\lim_{t\to 0} VEV^{-1}=\begin{bmatrix}\sqrt{d} & \frac{b}{2\sqrt{d}} & \frac{(be+ca-cd)(\sqrt{a}-\sqrt{d})+\frac{be(d-a)}{2\sqrt{d}}}{(a-d)^2}\\ 0 & \sqrt{d} & \frac{e}{\sqrt{a}+\sqrt{d}}\\ 0 & 0 & \sqrt{a}\end{bmatrix}.$$

□

Example 2. *Second root of* $A=\begin{bmatrix}9 & -6 & 17\\ 0 & 9 & 14\\ 0 & 0 & 25\end{bmatrix}$ *is equal with* $X=\begin{bmatrix}3 & -1 & \frac{75}{32}\\ 0 & 3 & \frac{7}{4}\\ 0 & 0 & 5\end{bmatrix}$.

Theorem 4. *Second root of* $A=\begin{bmatrix}a & b & c\\ 0 & d & e\\ 0 & 0 & a\end{bmatrix}$ *in which* $a\neq d$ *and* $a\neq 0$ *equals X such that* $X=\begin{bmatrix}\sqrt{a} & \frac{b}{\sqrt{a}+\sqrt{d}} & n\\ 0 & \sqrt{d} & \frac{e}{\sqrt{a}+\sqrt{d}}\\ 0 & 0 & \sqrt{a}\end{bmatrix}$ *and n is as* $n=\frac{\frac{(-be-ca+cd)(a+d)}{2\sqrt{a}}+eb\sqrt{d}+c\sqrt{a}(a-d)}{(a-d)^2}$.

$$\lambda_1=a,\ \lambda_2=d,\ \lambda_3=a+t(t\to 0)$$

$$E=\begin{bmatrix}\sqrt{a} & 0 & 0\\ 0 & \sqrt{d} & 0\\ 0 & 0 & \sqrt{a+t}\end{bmatrix},\ AX_i=\lambda_i X_i,\quad i=1,2,3\Rightarrow$$

Proof. $V=\begin{bmatrix}1 & b & \frac{be+ct+ca-cd}{(a-d)^2}\\ 0 & d-a & \frac{e}{t+a-d}\\ 0 & 0 & 1\end{bmatrix}$

$$\Rightarrow X=\lim_{t\to 0} VEV^{-1}=\begin{bmatrix}\sqrt{a} & \frac{b}{\sqrt{a}+\sqrt{d}} & n\\ 0 & \sqrt{d} & \frac{e}{\sqrt{a}+\sqrt{d}}\\ 0 & 0 & \sqrt{a}\end{bmatrix}$$

and

$$n=\frac{\frac{(-be-ca+cd)(a+d)}{2\sqrt{a}}+eb\sqrt{d}+c\sqrt{a}(a-d)}{(a-d)^2}$$

□

Example 3. *Second root of* $A=\begin{bmatrix}9 & -7 & -5\\ 0 & 16 & 4\\ 0 & 0 & 9\end{bmatrix}$ *equals:*

$$X=\begin{bmatrix}3 & -1 & \frac{-31}{42}\\ 0 & 4 & \frac{4}{7}\\ 0 & 0 & 3\end{bmatrix}.$$

Theorem 5. *Second root of* $A=\begin{bmatrix}a & b & c\\ 0 & a & d\\ 0 & 0 & a\end{bmatrix}$, *that* $a\neq 0$ *is*

$$X=\begin{bmatrix}\sqrt{a} & \frac{b}{2\sqrt{a}} & \frac{4ca-bd}{8a\sqrt{a}}\\ 0 & \sqrt{a} & \frac{d}{2\sqrt{a}}\\ 0 & 0 & \sqrt{a}\end{bmatrix}.$$

$$\lambda_1=a,\ \lambda_2=a+t,(t\to 0),\ \lambda_3=a+2t,$$

$$(t\to 0)\Rightarrow E=\begin{bmatrix}\sqrt{a} & 0 & 0\\ 0 & \sqrt{a+t} & 0\\ 0 & 0 & \sqrt{a+2t}\end{bmatrix},$$

Proof. $AX_i=\lambda_i X_i, i=1,2,3\Rightarrow V=\begin{bmatrix}1 & \frac{bd+tc}{t^2} & \frac{bd+2ct}{4t^2}\\ 0 & \frac{d}{t} & \frac{d}{2t}\\ 0 & 1 & 1\end{bmatrix}$,

$$\Rightarrow X=\lim_{t\to 0} VEV^{-1}=\begin{bmatrix}\sqrt{a} & \frac{b}{2\sqrt{a}} & \frac{4ca-bd}{8a\sqrt{a}}\\ 0 & \sqrt{a} & \frac{d}{2\sqrt{a}}\\ 0 & 0 & \sqrt{a}\end{bmatrix}.$$

□

Example 4. *Calculate the second root of* $A=\begin{bmatrix}4 & 9 & -3\\ 0 & 4 & 11\\ 0 & 0 & 4\end{bmatrix}$,

$$X=\begin{bmatrix}2 & \frac{9}{4} & \frac{-147}{64}\\ 0 & 2 & \frac{11}{4}\\ 0 & 0 & 2\end{bmatrix}.$$

Second root of lower triangular matrix

In order to find the second root of the lower triangular matrix that has iterated eigenvalue (the four cases mentioned previously), we do as the upper triangular matrix since $(A^t)^m y=(A^m y)^t$.

Theorem 6. *If* $A=\begin{bmatrix}0 & 0 & 0\\ a & 0 & 0\\ b & c & d\end{bmatrix}$, $d\neq 0$ *then*

(a) If $a\neq 0$, *then A has no lower triangular second root.*
(b) If $a = 0$, *then A has infinite second roots as* $X=\begin{bmatrix}0 & 0 & 0\\ t & 0 & 0\\ \frac{b\sqrt{d}-tc}{d} & \frac{c}{\sqrt{d}} & \sqrt{d}\end{bmatrix}$, *where t is a free parameter.*

Example 5. *Find the second root of* $\begin{bmatrix}0 & 0 & 1,100\\ 0 & 0 & 1,200\\ 0 & 0 & 1,600\end{bmatrix}$ *such that sum of all entries is 100.*

$$X=\begin{bmatrix}0 & t & \frac{110-3t}{4}\\ 0 & 0 & 30\\ 0 & 0 & 40\end{bmatrix}\Rightarrow X=\begin{bmatrix}0 & 10 & 20\\ 0 & 0 & 30\\ 0 & 0 & 40\end{bmatrix}.$$

Theorem 7. *Find the second root of a lower triangular matrix which has three eigenvalues (all are 0);*

considering $A = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix}$, *with* $b \neq 0$, *we have the following:*

(a) If $a = c = 0$, *we can find infinite second roots of* A.
(b) If at least a *or* c *is not 0, then* A *has no lower triangular second root.*

Example 6. *Calculate the second root of* $A = \begin{bmatrix} 0 & 0 & 1,390 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *such that sum of all entries is 1,391.*

$$X = \begin{bmatrix} 0 & \frac{1,390}{t} & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1,390 \\ 0 & 0 & 0 \end{bmatrix} \quad or$$

$$X = \begin{bmatrix} 0 & 0 & 1,390 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Theorem 8. *The second root of the matrix* $A = \begin{bmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix}$, *when* $a \neq b$, *and* $a \neq 0$, *equals:*

$$\begin{bmatrix} \sqrt{a} & \frac{bc\left(\sqrt{d}-\sqrt{a}+\frac{a-d}{2\sqrt{a}}\right)}{(a-d)^2} & \frac{b}{\sqrt{a}+\sqrt{d}} \\ 0 & \sqrt{a} & 0 \\ 0 & 0 & \sqrt{d} \end{bmatrix}.$$

Example 7. *Calculate the second root of* $A = \begin{bmatrix} 9 & 0 & -19 \\ 0 & 9 & 0 \\ 0 & 5 & 16 \end{bmatrix}$.
The solution is $X = \begin{bmatrix} 3 & \frac{95}{294} & \frac{-19}{7} \\ 0 & 3 & 0 \\ 0 & \frac{5}{7} & 4 \end{bmatrix}$.

Second root of 3×3 matrix

In this section, we find the solution of equation $X^2 = A$, when A is special 3×3 non-triangular matrix.

Theorem 9. *The second root of matrix* $A = \begin{bmatrix} a & 0 & b \\ 0 & a & 0 \\ 0 & 0 & d \end{bmatrix}$, *when* $a \neq b$, *and* $a \neq 0$, *equals*

$$\begin{bmatrix} \sqrt{a} & \frac{bc\left(\sqrt{d}-\sqrt{a}+\frac{a-d}{2\sqrt{a}}\right)}{(a-d)^2} & \frac{b}{\sqrt{a}+\sqrt{d}} \\ 0 & \sqrt{a} & 0 \\ 0 & \frac{c}{\sqrt{a}+\sqrt{d}} & \sqrt{d} \end{bmatrix}.$$

Proof. At first, we find the eigenvalues of matrix A. From $\det(A - \lambda I) = 0$, we have $\lambda = a, a, d$. Thus,

$$E = \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{a+t} & 0 \\ 0 & & \sqrt{d} \end{bmatrix}.$$

The matrix eigenvector associated to these eigenvalues has the following form:

$$V = \begin{bmatrix} 1 & \frac{bc}{t(a-d+t)} & b \\ 0 & 1 & 0 \\ 0 & \frac{c}{a-d+t} & d-a \end{bmatrix},$$

where t is a free parameter. Then, we have:

$$V^{-1} = \frac{1}{\det A}\operatorname{adj}A = \frac{1}{d-a}\begin{bmatrix} d-a & 0 & 0 \\ \frac{bc}{t} & d-a & -\frac{c}{a-d+t} \\ -b & 0 & 1 \end{bmatrix}^T,$$

and then we have:

$$VEV^{-1} = \begin{bmatrix} \sqrt{a} & x_{12} & \frac{b}{\sqrt{a}+\sqrt{d}} \\ 0 & \sqrt{a+t} & \sqrt{d} \\ -0 & x_{23} & 1 \end{bmatrix}, \tag{13}$$

where

$$x_{12} = \frac{\sqrt{a}bc(a-d+t) + bc\sqrt{a+t}(d-a) - tbc\sqrt{d}}{t(d-a)(a-d+t)},$$

$$x_{32} = \frac{c\sqrt{a+t}(d-a) - c(d-a)\sqrt{d}}{(d-a)(a-d+t)}.$$

Now, we take limit of all elements of matrix (13) when t tends to zero; therefore,

$$X = \lim_{t\to 0} VEV^{-1} = \begin{bmatrix} \sqrt{a} & \frac{bc(\sqrt{d}-\sqrt{a}+\frac{a-d}{2\sqrt{a}})}{(a-d)^2} & \frac{b}{\sqrt{a}+\sqrt{d}} \\ 0 & \sqrt{a} & 0 \\ 0 & \frac{c}{\sqrt{a}+\sqrt{d}} & \sqrt{d} \end{bmatrix}.$$

□

Theorem 10. *The square root of* 3×3 *permutation matrices* $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ *and* $P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ *equals* $\begin{bmatrix} \frac{i+1}{2} & \frac{1-i}{2} & 0 \\ \frac{1-i}{2} & \frac{i+1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \frac{i+1}{2} & 0 & \frac{1-i}{2} \\ 0 & 1 & 0 \\ \frac{1-i}{2} & 0 & \frac{i+1}{2} \end{bmatrix}$ *and* $\begin{bmatrix} 1 & & 0 \\ 0 & \frac{i+1}{2} & \frac{1-i}{2} \\ 0 & \frac{1-i}{2} & \frac{i+1}{2} \end{bmatrix}$, *respectively, where* $i = \sqrt{-1}$.

Proof. We present proof only for P_{12}. From $\det(P_{12} - \lambda I) = 0$, we have $\lambda = 1, 1, -1$. Thus, the matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1+t} & 0 \\ 0 & 0 & i \end{bmatrix}$, and the matrix $V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{bmatrix}$. Then,

$$VEV^{-1} = \begin{bmatrix} \frac{\sqrt{1+t}+i}{2} & \frac{\sqrt{1+t}-i}{2} & 1-\sqrt{1+t} \\ \frac{\sqrt{1+t}-i}{2} & \frac{\sqrt{1+t}+i}{2} & 1-\sqrt{1+t} \\ 0 & 0 & 1 \end{bmatrix},$$

and by taking limit when t tends to zero, we obtain solution. $\square$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AN and HF carried out the proof. AN and MB read and approved the final manuscript.

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