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Iterative approximation of a common solution of a split generalized equilibrium problem and a fixed point problem for nonexpansive semigroup

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Abstract

Purpose: In this paper, we introduce and study an iterative method to approximate a common solution of a split generalized equilibrium problem and a fixed point problem for a nonexpansive semigroup in real Hilbert spaces.

Methods: We prove a strong convergence theorem of the iterative algorithm in Hilbert spaces under certain mild conditions.

Results: We obtain a strong convergence result for approximating a common solution of a split generalized equilibrium problem and a fixed point problem for a nonexpansive semigroup in real Hilbert spaces, which is a unique solution of a variational inequality problem. Further, we obtain some consequences of our main result.

Conclusions: The results presented in this paper are the supplement, extension, and generalization of results in the study of Plubtieng and Punpaeng and that of Cianciaruso et al. The approach of the proof given in this paper is also different.

Keywords: Split generalized equilibrium problem, Fixed point problem, Nonexpansive semigroup

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Introduction

Throughout the paper, unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively.

A mapping $f : C \to C$ is said to be a contraction if there exists a constant $\alpha \in (0,1)$ such that $\|fx - fy\| \le \alpha\|x-y\|$, $\forall x, y \in C$. A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x-y\|$, $\forall x, y \in C$. $\mathrm{Fix}(T)$ denotes the fixed point set of the nonexpansive mapping $T : C \to C$.

Let $B : H_1 \to H_1$ be a strongly positive linear bounded operator, i.e., if there exists a constant $\bar{\gamma} > 0$ such that

$$\langle Bx, x\rangle \ge \bar{\gamma}\|x\|^2, \quad \forall x \in H_1.$$

A typical problem is to minimize a quadratic function over the set of fixed points of nonexpansive mapping T:

$$\min_{x\in \mathrm{Fix}(T)} \frac{1}{2}\langle Bx, x\rangle - \langle x, b\rangle,$$

where b is a given point in H_1.

In 2006, Marino and Xu [1] considered the following iterative method:

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n B)Tx_n, \quad \forall n \ge 0,$$

with $0 < \gamma < \frac{\bar{\gamma}}{\alpha}$ and proved that the sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)z, x - z\rangle, \quad \forall x \in \mathrm{Fix}(T)$$

which is the optimality condition for the minimization problem

$$\min_{x\in \mathrm{Fix}(T)} \frac{1}{2}\langle Bx, x\rangle - h(x),$$

where h is the potential function for γf.

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A family $S := \{T(s) : 0 \leq s < \infty\}$ of mappings from C into itself is called a *nonexpansive semigroup* on C if it satisfies the following conditions:

(i) $T(0)x = x$ for all $x \in C$.
(ii) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$.
(iii) $\|T(s)x - T(s)y\| \leq \|x-y\|$ for all $x, y \in C$ and $s \geq 0$.
(iv) For all $x \in C$, $s \mapsto T(s)x$ is continuous.

The set of all the common fixed points of a family S is denoted by $\mathrm{Fix}(S)$, i.e.,

$$\begin{aligned}\mathrm{Fix}(S) :&= \{x \in C : T(s)x = x, 0 \leq s < \infty\}\\ &= \bigcap_{0\leq s<\infty} \mathrm{Fix}(T(s)),\end{aligned}$$

where $\mathrm{Fix}(T(s))$ is the set of fixed points of $T(s)$. It is well known that $\mathrm{Fix}(S)$ is closed and convex.

The *fixed point problem* (FPP) for a nonexpansive semigroup S is:

$$\text{Find } x \in C \text{ such that } x \in \mathrm{Fix}(S). \tag{1}$$

In 1997, Shimizu and Takahashi [2] introduced and studied the following iterative method to prove a strong convergence theorem for FPP (1) in a real Hilbert space:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \qquad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{s_n\}$ is a sequence of positive real numbers which diverges to $+\infty$. Later, Chen and Song [3] introduced and studied the following iterative method to prove a strong convergence theorem for FPP (1) in a real Hilbert space:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \qquad \forall n \in \mathbb{N},$$

where f is a contraction mapping. Recently, Plubtieng and Punpaeng [4] introduced and studied the following iterative method to prove a strong convergence theorem for FPP (1) in a real Hilbert space:

$$\begin{aligned}x_{n+1} =& \alpha_n f(x_n) + \beta_n x_n\\ &+ (1-\alpha_n-\beta_n)\frac{1}{s_n}\int_0^{s_n} T(s)x_n ds, \qquad \forall n \in \mathbb{N},\end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$ and $\{s_n\}$ is a positive real divergent sequence.

The *equilibrium problem* (EP) [5] is of finding $x \in C$ such that

$$F(x,y) \geq 0, \ \ \forall y \in C, \tag{2}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction. The solution set of EP (2) is denoted by $\mathrm{EP}(F)$.

Cianciaruso et al. [6] introduced and studied the following iterative method to prove a strong convergence theorem for FPP (1) and EP (2) in a real Hilbert space: $x_0 \in H_1$:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1-\alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \qquad \forall n \in \mathbb{N},$$

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \ \forall y \in H_1,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{s_n\}$ is a positive real divergent sequence.

Recently, Moudafi [7] introduced the following *split equilibrium problem* (SEP):

Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the SEP is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \ \ \forall x \in C, \tag{3}$$

and such that

$$y^* = Ax^* \in Q \ \text{ solves } F_2(y^*, y) \geq 0, \ \ \forall y \in Q. \tag{4}$$

When looked separately, (3) is the classical EP, and we denoted its solution set by $\mathrm{EP}(F_1)$. SEP (3)-(4) constitutes a pair of equilibrium problems which have to be solved so that the image $y^* = Ax^*$, under a given bounded linear operator A, of the solution x^* of EP (3) in H_1 is the solution of another EP (4) in another space H_2, and we denote the solution set of EP (4) by $\mathrm{EP}(F_2)$.

The solution set of SEP (3)-(4) is denoted by $\Omega = \{p \in \mathrm{EP}(F_1) : Ap \in \mathrm{EP}(F_2)\}$. SEP (3)-(4) includes the split variational inequality problem, split zero problem, and split feasibility problem (see, for instance, [7-12]).

In this paper, we consider a *split generalized equilibrium problem* (SGEP): Find $x^* \in C$ such that

$$F_1(x^*, x) + h_1(x^*, x) \geq 0, \ \ \forall x \in C, \tag{5}$$

and such that

$$y^* = Ax^* \in Q \ \text{ solves } F_2(y^*, y) + h_2(y^*, y) \geq 0, \ \ \forall y \in Q, \tag{6}$$

where $F_1, h_1 : C \times C \to \mathbb{R}$ and $F_2, h_2 : Q \times Q \to \mathbb{R}$ are nonlinear bifunctions and $A : H_1 \to H_2$ is a bounded linear operator.

We denote the solution set of generalized equilibrium problem (GEP) (5) and GEP (6) by $\mathrm{GEP}(F_1, h_1)$ and $\mathrm{GEP}(F_2, h_2)$, respectively. The solution set of SGEP (5)-(6) is denoted by $\Gamma = \{p \in \mathrm{GEP}(F_1, h_1) : Ap \in \mathrm{GEP}(F_2, h_2)\}$.

If $h_1 = 0$ and $h_2 = 0$, then SGEP (5)-(6) reduces to SEP (3)-(4). If $h_2 = 0$ and $F_2 = 0$, then SGEP (5)-(6) reduces to the equilibrium problem considered by Cianciaruso et al. [13].

Motivated by the works of Moudafi [7], Marino and Xu [1], Shimizu and Takahashi [2], Chen and Song [3], Plubtieng and Punpaeng [4], and Cianciaruso et al. [6,13] and by the ongoing research in this direction, we introduce and study an iterative method for approximating a common solution of SGEP (5)-(6) and FPP (6) for a nonexpansive semigroup in real Hilbert spaces. The results presented in this paper extend and generalize the works of Shimizu and Takahashi [2], Chen and Song [3], Plubtieng and Punpaeng [4], and Cianciaruso et al. [6].

Now, we recall some concepts and results which are needed in sequel.

For every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \ \ \forall y \in C. \tag{7}$$

P_C is called the *metric projection* of H_1 onto C. It is well known that P_C is a nonexpansive mapping and is characterized by the following property:

$$\langle x - P_C x, y - P_C x\rangle \leq 0. \tag{8}$$

Further, it is well known that every nonexpansive operator $T : H_1 \to H_1$ satisfies, for all $(x, y) \in H_1 \times H_1$, the inequality

$$\begin{aligned} &\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \\ &\qquad \leq (1/2)\|(T(x) - x) - (T(y) - y\|^2, \end{aligned} \tag{9}$$

and therefore, we get, for all $(x, y) \in H_1 \times \mathrm{Fix}(T)$,

$$\langle x - T(x), y - T(x)\rangle \leq (1/2)\|T(x) - x\|^2 \tag{10}$$

(see, e.g., Theorem 3 in [14] and Theorem 1 in [15]).

It is also known that H_1 satisfies Opial's condition [16], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{11}$$

holds for every $y \in H_1$ with $y \neq x$.

Lemma 1. [17] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$, for all integers $n \geq 0$ and $\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2. [2] *Let C be a nonempty bounded closed convex subset of a Hilbert space H_1 and let $S := \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C, for each $x \in C$ and $t > 0$. Then, for any $0 \leq h < \infty$,*

$$\lim_{t\to\infty} \sup_{x\in C} \left\| \frac{1}{t}\int_0^t T(s)x ds - T(h)\left(\frac{1}{t}\int_0^t T(s)x ds\right)\right\| = 0.$$

Lemma 3. [18] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$.

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} | \delta_n |< \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 4. [1] *Assume that B is a strong positive linear bounded operator on a Hilbert space H_1 with coefficient $\bar{\gamma} > 0$ and $0 < \rho < \|B\|^{-1}$. Then, $\|I - \rho B\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 5. *The following inequality holds in a real Hilbert space H_1:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \ \ \forall x, y \in H_1.$$

Assumption 1 [19] Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying the following assumptions:

(i) $F(x, x) \geq 0, \ \ \forall x \in C$,
(ii) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \ \ \forall x \in C$,
(iii) F is upper hemicontinuous, i.e., for each $x, y, z \in C$, $\limsup_{t\to 0} F(tz + (1 - t)x, y) \leq F(x, y)$,
(iv) For each $x \in C$ fixed, the function $y \to F(x, y)$ is convex and lower semicontinuous;

let $h : C \times C \longrightarrow \mathbb{R}$ such that

(i) $h(x, x) \geq 0, \ \ \forall x \in C$,
(ii) For each $y \in C$ fixed, the function $x \to h(x, y)$ is upper semicontinuous,
(iii) For each $x \in C$ fixed, the function $y \to h(y, x)$ is convex and lower semicontinuous,

and assume that for fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of H_1 and $x \in C \cap K$ such that

$$F(y, x) + h(y, x) + \frac{1}{r}\langle y - x, x - z\rangle \ < \ 0, \ \ \forall y \in C\backslash K.$$

The proof of the following lemma is similar to the proof of Lemma 2.13 in [19] and hence omitted.

Lemma 6. *Assume that $F_1, h_1 : C \times C \longrightarrow \mathbb{R}$ satisfying Assumption 1. Let $r > 0$ and $x \in H_1$. Then, there exists $z \in C$ such that*

$$F_1(z,y) + h_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 7. [12] *Assume that the bifunctions $F_1, h_1 : C \times C \longrightarrow \mathbb{R}$ satisfy Assumption 1 and h_1 is monotone. For $r > 0$ and for all $x \in H_1$, define a mapping $T_r^{(F_1,h_1)} : H_1 \to C$ as follows:*

$$T_r^{(F_1,h_1)}(x) = \Big\{ z \in C : F_1(z,y) + h_1(z,y) + \frac{1}{r_n}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \Big\}.$$

Then, the following hold:

(i) $T_r^{(F_1,h_1)}$ is single-valued.
(ii) $T_r^{(F_1,h_1)}$ is firmly nonexpansive, i.e.,

$$\|T_r^{(F_1,h_1)}x - T_r^{(F_1,h_1)}y\|^2 \leq \langle T_r^{(F_1,h_1)}x - T_r^{(F_1,h_1)}y, x - y\rangle, \quad \forall x, y \in H_1.$$

(iii) $\mathrm{Fix}(T_r^{(F_1,h_1)}) = \mathrm{GEP}(F_1, h_1)$.
(iv) $\mathrm{GEP}(F_1, h_1)$ is compact and convex.

Further, assume that $F_2, h_2 : Q \times Q \longrightarrow \mathbb{R}$ satisfying Assumption 1. For $s > 0$ and for all $w \in H_2$, define a mapping $T_s^{(F_2,h_2)} : H_2 \to Q$ as follows:

$$T_s^{(F_2,h_2)}(w) = \Big\{ d \in Q : F_2(d,e) + h_2(d,e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0, \ \forall e \in Q \Big\}.$$

Then, we easily observe that $T_s^{(F_2,h_2)}$ is single-valued and firmly nonexpansive, $\mathrm{GEP}(F_2, h_2, Q)$ is compact and convex, and $\mathrm{Fix}(T_s^{(F_2,h_2)}) = \mathrm{GEP}(F_2, h_2, Q)$, where $\mathrm{GEP}(F_2, h_2, Q)$ is the solution set of the following generalized equilibrium problem:

Find $y^* \in Q$ such that $F_2(y^*, y) + h_2(y^*, y) \geq 0$, $\forall y \in Q$.

We observe that $\mathrm{GEP}(F_2, h_2) \subset \mathrm{GEP}(F_2, h_2, Q)$. Further, it is easy to prove that Γ is a closed and convex set.

Remark 1. *Lemmas 6 and 7 are slight generalizations of Lemma 3.5 in [13] where the equilibrium condition $F_1(x,x) = h_1(x,x) = 0$ has been relaxed to $F_1(x,x) \geq 0$ and $h_1(x,x) \geq 0$ for all $x \in C$. Further, the monotonicity of h_1 in Lemma 6 is not required.*

Lemma 8. [13] *Let $F_1 : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfying Assumption 1 hold and let $T_r^{F_1}$ be defined as in Lemma 4 for $r > 0$. Let $x, y \in H_1$ and $r_1, r_2 > 0$. Then,*

$$\|T_{r_2}^{F_1}y - T_{r_1}^{F_1}x\| \leq \|y - x\| + \left|\frac{r_2 - r_1}{r_2}\right| \|T_{r_2}^{F_1}y - y\|.$$

Notation. Let $\{x_n\}$ be a sequence in H_1, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) denotes strong (respectively, weak) convergence of the sequence $\{x_n\}$ to a point $x \in H_1$.

Methods

In this section, we prove a strong convergence theorem based on the proposed iterative method for computing the approximate common solution of SGEP (5)-(6) and FPP (1) for a nonexpansive semigroup in real Hilbert spaces.

We assume that $\Gamma \neq \emptyset$.

Theorem 1. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1, h_1 : C \times C \to \mathbb{R}$ and $F_2, h_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption 1; h_1, h_2 are monotone and F_2 is upper semicontinuous in the first argument. Let $S = \{T(s) : 0 \leq s < \infty\}$ be a nonexpansive semigroup on C such that $\mathrm{Fix}(S) \cap \Gamma \neq \emptyset$. Let $f : C \to C$ be a contraction mapping with constant $\alpha \in (0,1)$ and B be a strongly positive linear bounded self-adjoint operator on H_1 with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma + \frac{1}{\alpha}$. Let $\{s_n\}$ is a positive real sequence which diverges to $+\infty$. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by iterative algorithm:*

$$\begin{aligned} u_n &= T_{r_n}^{(F_1,h_1)}(x_n + \delta A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n); \\ x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n \\ &\quad + ((1-\beta_n)I - \alpha_n B)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds, \end{aligned} \tag{12}$$

where $r_n \subset (0,\infty)$ and $\delta \in (0, 1/L)$, L is the spectral radius of the operator A^*A, and A^* is the adjoint of A, and $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.
(iii) $\liminf r_n > 0$ and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.
(iv) $\lim_{n\to\infty} \frac{|s_{n+1} - s_n|}{s_{n+1}} = 0$.

Then, the sequence $\{x_n\}$ converges strongly to $z \in \mathrm{Fix}(S) \cap \Gamma$, where $z = P_{\mathrm{Fix}(S)\cap\Gamma}(I - B + \gamma f)z$.

Proof. We note that from condition (i), we may assume without loss of generality that $\alpha_n \leq (1-\beta_n)\|B\|^{-1}$ for all n. From Lemma 4, we know that if $0 < \rho \leq \|B\|^{-1}$, then $\|I-\rho B\| \leq 1-\rho\bar{\gamma}$. We will assume that $\|I-B\| \leq 1-\bar{\gamma}$. $\square$

Since B is a positive linear bounded self-adjoint operator on H_1, then

$$\|B\| = \sup\{|\langle Bu, u\rangle| : u \in H_1, \|u\| = 1\}.$$

Observe that

$$\begin{aligned} \langle((1-\beta_n)I-\alpha_n B)u, u\rangle &= 1-\beta_n-\alpha_n\langle Bu, u\rangle \\ &\geq 1-\beta_n-\alpha_n\|B\| \\ &\geq 0, \end{aligned}$$

which implies that $(1-\beta_n)I-\alpha_n B$ is positive. It follows that

$$\begin{aligned} \|(1-\beta_n)I-\alpha_n B\| &= \sup\{\langle((1-\beta_n)I-\alpha_n B)u, u\rangle : \\ &\quad u \in H_1, \|u\| = 1\} \\ &= \sup\{1-\beta_n-\alpha_n\langle Bu, u\rangle : \\ &\quad u \in H_1, \|u\| = 1\} \\ &\leq 1-\beta_n-\alpha_n\bar{\gamma}. \end{aligned}$$

Let $q = P_{\mathrm{Fix}(S)\cap\Gamma}$. Since f is a contraction mapping with constant $\alpha \in (0,1)$, it follows that

$$\begin{aligned} \|q(I-B+\gamma f)(x) &- q(I-B+\gamma f)(y)\| \\ &\leq \|(I-B+\gamma f)(x)-(I-B+\gamma f)(x)\| \\ &\leq \|I-B\|\|x-y\| + \gamma\|f(x)-f(y)\| \\ &\leq (1-\bar{\gamma})\|x-y\| + \gamma\alpha\|x-y\| \\ &\leq (1-(\bar{\gamma}-\gamma\alpha))\|x-y\|, \end{aligned}$$

for all $x, y \in H_1$. Therefore, the mapping $q(I-B+\gamma f)$ is a contraction mapping from H_1 into itself. It follows from the Banach contraction principle that there exists an element $z \in H_1$ such that $z = q(I-B+\gamma f)z = P_{\mathrm{Fix}(S)\cap\Gamma}(I-B+\gamma f)(z)$.

Let $p \in \mathrm{Fix}(S)\cap\Gamma$, i.e., $p \in \Gamma$, and we have $p = T_{r_n}^{(F_1,h_1)}p$ and $Ap = T_{r_n}^{(F_2,h_2)}(Ap)$.

We estimate

$$\begin{aligned} \|u_n-p\|^2 &= \|T_{r_n}^{(F_1,h_1)}(x_n+\delta A^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n)-p\|^2 \\ &= \|T_{r_n}^{(F_1,h_1)}(x_n+\delta A^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n) \\ &\quad -T_{r_n}^{(F_1,h_1)}p\|^2 \\ &\leq \|x_n+\delta A^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n-p\|^2 \\ &\leq \|x_n-p\|^2+\delta^2\|A^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n\|^2 \\ &\quad +2\delta\langle x_n-p, A^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n\rangle. \end{aligned} \tag{13}$$

Thus, we have

$$\begin{aligned} \|u_n-p\|^2 &\leq \|x_n-p\|^2 \\ &\quad +\delta^2\langle(T_{r_n}^{(F_2,h_2)}-I)Ax_n, AA^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n\rangle \\ &\quad +2\delta\langle x_n-p, A^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n\rangle. \end{aligned} \tag{14}$$

Now, we have

$$\begin{aligned} &\delta^2\langle(T_{r_n}^{(F_2,h_2)}-I)Ax_n, AA^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n\rangle \\ &\quad \leq L\delta^2\langle(T_{r_n}^{(F_2,h_2)}-I)Ax_n, (T_{r_n}^{(F_2,h_2)}-I)Ax_n\rangle \\ &\quad = L\delta^2\|(T_{r_n}^{(F_2,h_2)}-I)Ax_n\|^2. \end{aligned} \tag{15}$$

Denoting $\Lambda = 2\delta\langle x_n-p, A^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n\rangle$ and using (10), we have

$$\begin{aligned} \Lambda &= 2\delta\langle x_n-p, A^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n\rangle \\ &= 2\delta\langle A(x_n-p), (T_{r_n}^{(F_2,h_2)}-I)Ax_n\rangle \\ &= 2\delta\langle A(x_n-p)+(T_{r_n}^{(F_2,h_2)}-I)Ax_n \\ &\quad -(T_r^{(F_2,h_2)}-I)Ax_n, (T_{r_n}^{(F_2,h_2)}-I)Ax_n\rangle \\ &= 2\delta\left\{\langle T_{r_n}^{(F_2,h_2)}Ax_n-Ap, (T_{r_n}^{(F_2,h_2)}-I)Ax_n\rangle\right. \\ &\quad \left.-\|(T_{r_n}^{(F_2,h_2)}-I)Ax_n\|^2\right\} \\ &\leq 2\delta\left\{\frac{1}{2}\|(T_{r_n}^{(F_2,h_2)}-I)Ax_n\|^2-\|(T_{r_n}^{(F_2,h_2)}-I)Ax_n\|^2\right\} \\ &\leq -\delta\|(T_{r_n}^{(F_2,h_2)}-I)Ax_n\|^2. \end{aligned} \tag{16}$$

Using (14), (15), and (16), we obtain

$$\|u_n-p\|^2 \leq \|x_n-p\|^2+\delta(L\delta-1)\|(T_{r_n}^{(F_2,h_2)}-I)Ax_n\|^2. \tag{17}$$

Since $\delta \in (0, \frac{1}{L})$, we obtain

$$\|u_n-p\|^2 \leq \|x_n-p\|^2. \tag{18}$$

Now, setting $t_n := \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds$ and since $p \in \mathrm{Fix}(S)\cap\Gamma$, we obtain

$$\begin{aligned} \|t_n-p\| &= \left\|\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - p\right\| \\ &\leq \frac{1}{s_n}\int_0^{s_n}\|T(s)u_n-T(s)p\|ds \\ &\leq \|u_n-p\| \\ &\leq \|x_n-p\|. \end{aligned} \tag{19}$$

Further, we estimate

$$\begin{aligned} \|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n B)t_n-p\| \\ &= \|\alpha_n(\gamma f(x_n)-Bp)+\beta_n(x_n-p) \\ &\quad +((1-\beta_n)I-\alpha_n B)(t_n-p)\| \\ &\leq \alpha_n\|\gamma f(x_n)-Bp\|+\beta_n\|x_n-p\| \\ &\quad +(1-\beta_n-\alpha_n\bar{\gamma})\|t_n-p\| \\ &\leq \alpha_n\gamma\|f(x_n)-f(p)\|+\alpha_n\|\gamma f(p)-Bp\| \\ &\quad +\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\bar{\gamma})\|x_n-p\| \\ &\leq \alpha_n\gamma\alpha\|x_n-p\|+\alpha_n\|\gamma f(p)-Bp\| \\ &\quad +(1-\alpha_n\bar{\gamma})\|x_n-p\| \\ &= (1-(\bar{\gamma}-\gamma\alpha)\alpha_n)\|x_n-p\|+\alpha_n\|\gamma f(p)-Bp\| \\ &\leq \max\left\{\|x_n-p\|, \frac{1}{\bar{\gamma}-\gamma\alpha}\|\gamma f(p)-Bp\|\right\}, \quad n \geq 0 \\ &\vdots \\ &\leq \max\left\{\|x_0-p\|, \frac{1}{\bar{\gamma}-\gamma\alpha}\|\gamma f(p)-Bp\|\right\}. \end{aligned} \tag{20}$$

Hence, $\{x_n\}$ is bounded, and consequently, we deduce that $\{u_n\}$, $\{t_n\}$, and $\{f(x_n)\}$ are bounded.

Next, we estimate

$$\begin{aligned}\|t_{n+1}-t_n\| &= \left\|\frac{1}{s_{n+1}}\int_0^{s_{n+1}} T(s)u_{n+1}ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds\right\|\\ &= \left\|\frac{1}{s_{n+1}}\int_0^{s_{n+1}}[T(s)u_{n+1}-T(s)u_n]\,ds\right.\\ &\quad+\left(\frac{1}{s_{n+1}}-\frac{1}{s_n}\right)\times\int_0^{s_n} T(s)u_n ds\\ &\quad\left.+\frac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}} T(s)u_n ds\right\|\\ &= \left\|\frac{1}{s_{n+1}}\int_0^{s_{n+1}}[T(s)u_{n+1}-T(s)u_n]\,ds\right.\\ &\quad+\left(\frac{1}{s_{n+1}}-\frac{1}{s_n}\right)\times\int_0^{s_n}[T(s)u_n-T(s)p]\,ds\\ &\quad\left.+\frac{1}{s_{n+1}}\int_{s_n}^{s_{n+1}}[T(s)u_n-T(s)p]\,ds\right\|\\ &\le \|u_{n+1}-u_n\|+\frac{|s_{n+1}-s_n|s_n}{(s_{n+1})s_n}\|u_n-p\|\\ &\quad+\frac{|s_{n+1}-s_n|}{s_{n+1}}\|u_n-p\|\\ &\le \|u_{n+1}-u_n\|+2\frac{|s_{n+1}-s_n|}{s_{n+1}}\|u_n-p\|.\end{aligned} \tag{21}$$

Since $T_{r_{n+1}}^{(F_1,h_1)}$ and $T_{r_{n+1}}^{(F_2,h_2)}$ both are firmly nonexpansive, for $\delta\epsilon(0,\frac{1}{L})$, the mapping $T_{r_{n+1}}^{(F_1,h_1)}(I+\delta A^*(T_{r_{n+1}}^{(F_2,h_2)}-I)A)$ is nonexpensive, see [7,10]. Further, since $u_n=T_{r_n}^{(F_1,h_1)}(x_n+\delta A^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n)$ and $u_{n+1}=T_{r_{n+1}}^{(F_1,h_1)}(x_{n+1}+\delta A^*(T_{r_{n+1}}^{(F_2,h_2)}-I)Ax_{n+1})$, it follows from Lemma 8 that

$$\begin{aligned}&\|u_{n+1}-u_n\|\\ &\le \|T_{r_{n+1}}^{(F_1,h_1)}\left(x_{n+1}+\delta A^*\left(T_{r_{n+1}}^{(F_2,h_2)}-I\right)Ax_{n+1}\right)\\ &\quad-T_{r_{n+1}}^{(F_1,h_1)}\left(x_n+\delta A^*\left(T_{r_{n+1}}^{(F_2,h_2)}-I\right)Ax_n\right)\|\\ &\quad+\|T_{r_{n+1}}^{(F_1,h_1)}\left(x_n+\delta A^*\left(T_{r_{n+1}}^{(F_2,h_2)}-I\right)Ax_n\right)\\ &\quad-T_{r_n}^{(F_1,h_1)}\left(x_n+\delta A^*\left(T_{r_n}^{(F_2,h_2)}-I\right)Ax_n\right)\|\\ &\le \|x_{n+1}-x_n\|+\|\left(x_n+\delta A^*\left(T_{r_{n+1}}^{(F_2,h_2)}-I\right)Ax_n\right)\\ &\quad-\left(x_n+\delta A^*\left(T_{r_n}^{(F_2,h_2)}-I\right)Ax_n\right)\|\\ &\quad+\left|1-\frac{r_n}{r_{n+1}}\right|\|T_{r_{n+1}}^{(F_1,h_1)}\left(x_n+\delta A^*\left(T_{r_n}^{(F_2,h_2)}-I\right)Ax_n\right)\\ &\quad-\left(x_n+\delta A^*\left(T_{r_{n+1}}^{(F_2,h_2)}-I\right)Ax_n\right)\|\\ &\le \|x_{n+1}-x_n\|+\delta\|A\|\|T_{r_{n+1}}^{(F_2,h_2)}Ax_n-T_{r_n}^{(F_2,h_2)}Ax_n\|+\delta_n\\ &\le \|x_{n+1}-x_n\|+\delta\|A\|\left|1-\frac{r_n}{r_{n+1}}\right|\|T_{r_{n+1}}^{(F_2,h_2)}Ax_n-Ax_n\|+\delta_n\\ &= \|x_{n+1}-x_n\|+\delta\|A\|\sigma_n+\delta_n\end{aligned} \tag{22}$$

where

$$\sigma_n=\left|1-\frac{r_{n+1}}{r_n}\right|\|T_{r_n}^{(F_2,h_2)}Ax_n-Ax_n\|$$

and

$$\begin{aligned}\delta_n=\left|1-\frac{r_{n+1}}{r_n}\right|&\|T_{r_n}^{(F_1,h_1)}(x_n+\delta A^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n)\\ &-(x_n+\delta A^*(T_{r_n}^{(F_2,h_2)}-I)Ax_n)\|.\end{aligned}$$

Using (21) and (22), we have

$$\begin{aligned}\|t_{n+1}-t_n\|&\le \|x_{n+1}-x_n\|+\delta\|A\|\sigma_n+\delta_n\\ &\quad+2\frac{|s_{n+1}-s_n|}{s_{n+1}}\|u_n-p\|.\end{aligned} \tag{23}$$

Setting $x_{n+1}=\beta_n x_n+(1-\beta_n)e_n$ implies from (12) that $e_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}=\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_n B)t_n}{1-\beta_n}$.

Further, it follows that

$$\begin{aligned}e_{n+1}-e_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1})+((1-\beta_{n+1})I-\alpha_{n+1}B)t_{n+1}}{1-\beta_{n+1}}\\ &\quad-\frac{\alpha_n\gamma f(x_n)+((1-\beta_n)I-\alpha_n B)t_n}{1-\beta_n}\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}\gamma f(x_{n+1})+\frac{(1-\beta_{n+1})t_{n+1}}{1-\beta_{n+1}}-\frac{\alpha_{n+1}Bt_{n+1}}{1-\beta_{n+1}}\\ &\quad-\frac{\alpha_n}{1-\beta_n}\gamma f(x_n)-\frac{(1-\beta_n)t_n}{1-\beta_n}+\frac{\alpha_n Bt_n}{1-\beta_n}\\ &= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1})+Bt_{n+1})+t_{n+1}-t_n\\ &\quad+\frac{\alpha_n}{1-\beta_n}(Bt_n-\gamma f(x_n)).\end{aligned}$$

Using (23), we have

$$\begin{aligned}\|e_{n+1}-e_n\| &= \left\|\frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma f(x_{n+1})+Bt_{n+1})+t_{n+1}\right.\\ &\quad\left.-t_n+\frac{\alpha_n}{1-\beta_n}(Bt_n-\gamma f(x_n))\right\|\\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(x_{n+1})\|+\|Bt_{n+1}\|\right)\\ &\quad+\frac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n)\|+\|Bt_n\|)+\|t_{n+1}-t_n\|\\ &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\|\gamma f(x_{n+1})\|+\|Bt_{n+1}\|\right)\\ &\quad+\frac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n)\|+\|Bt_n\|)+\|x_{n+1}-x_n\|\\ &\quad+\gamma\|A\|\sigma_n+\delta_n+2\left(\frac{|s_{n+1}-s_n|}{s_{n+1}}\right)\|u_n-p\|\end{aligned}$$

which implies that

$$\begin{aligned}&\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\|\\ &\quad\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|f(x_{n+1})\|+\|Bt_{n+1}\|)+\gamma\|A\|\sigma_n+\delta_n\\ &\quad+\frac{\alpha_n}{1-\beta_n}(\|\gamma f(x_n)\|+\|Bt_n\|)\\ &\quad+2\left(\frac{|s_{n+1}-s_n|}{s_{n+1}}\right)\|u_n-p\|.\end{aligned}$$

Hence, it follows by conditions (i), (iii), and (iv) that

$$\limsup_{n\to\infty}\,[\|e_{n+1}-e_n\|-\|x_{n+1}-x_n\|]\le 0. \tag{24}$$

From Lemma 1 and (24), we get $\lim_{n\to\infty}\|e_n-x_n\|=0$ and

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|e_n-x_n\|=0. \tag{25}$$

Now,

$$\begin{aligned}x_{n+1}-x_n &= \alpha_n\gamma f(x_n)+\beta_n x_n+((1-\beta_n)I-\alpha_n B)t_n-x_n\\ &= \alpha_n(\gamma f(x_n)-x_n)+((1-\beta_n)I-\alpha_n B)(t_n-x_n).\end{aligned}$$

Since $\|x_{n+1}-x_n\|\to 0$ and $\alpha_n\to 0$ as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\|t_n-x_n\|=0. \tag{26}$$

Next, we have

$$\begin{aligned}\|T(s)x_n - x_n\| &= \Big\| T(s)x_n - T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds \\ &\quad + T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds \\ &\quad - \tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds + \tfrac{1}{s_n}\int_0^{s_n} T(s)u_n ds - x_n \Big\| \\ &\leq \Big\| T(s)x_n - T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds \Big\| \\ &\quad + \Big\| T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)u_n ds \Big\| \\ &\quad + \Big\| \tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds - x_n \Big\| \\ &\leq \Big\| x_n - \tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds \Big\| \\ &\quad + \Big\| T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)u_n ds \Big\| \\ &\quad + \Big\| \tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds - x_n \Big\| \\ &\leq 2\Big\| x_n - \tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds \Big\| \\ &\quad + \Big\| T(s)\tfrac{1}{s_n}\textstyle\int_0^{s_n} T(s)u_n ds - \tfrac{1}{s_n}\int_0^{s_n} T(s)u_n ds \Big\|.\end{aligned} \tag{27}$$

Since $\{x_n\}$ and $\{f(x_n)\}$ are bounded, let $K := \Big\{ w \in C : \|w - p\| \leq \|x_0 - p\|, \frac{1}{\bar{\gamma} - \gamma\alpha}\|\gamma f(p) - Bp\| \Big\}$, then K is a nonempty bounded closed convex subset of C which is $T(s)$-invariant for each $0 \leq s < \infty$ and contains $\{x_n\}$. So, without loss of generality, we may assume that $S := \{T(s) : 0 \leq s < \infty\}$ is a nonexpansive semigroup on K. By Lemma 2, we have

$$\lim_{n\to\infty} \Big\| T(s)\frac{1}{s_n}\int_0^{s_n} T(s)u_n ds - \frac{1}{s_n}\int_0^{s_n} T(s)u_n ds \Big\| = 0. \tag{28}$$

Using (21) to (23), we obtain

$$\lim_{n\to\infty} \|T(s)x_n - x_n\| = 0. \tag{29}$$

It follows from (17) and Lemma 5 that

$$\begin{aligned}&\|x_{n+1} - p\|^2 \\ &= \|\alpha_n\gamma f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n B]\, t_n - p\|^2 \\ &= \|\alpha_n(\gamma f(x_n) - Bp) + \beta_n(x_n - t_n) + (I - \alpha_n B)(t_n - p)\|^2 \\ &\leq \|(I - \alpha_n B)(t_n - p) + \beta_n(x_n - t_n)\|^2 + 2\alpha_n\langle \gamma f(x_n) \\ &\quad - Bp, x_{n+1} - p\rangle \\ &\leq [\|(I - \alpha_n B)(t_n - p)\| + \beta_n\|x_n - t_n\|]^2 + 2\alpha_n\|\gamma f(x_n) \\ &\quad - Bp\|\|x_{n+1} - p\| \\ &\leq [\|I - \alpha_n B\|\|u_n - p\| + \beta_n\|x_n - t_n\|]^2 + 2\alpha_n\|\gamma f(x_n) \\ &\quad - Bp\|\|x_{n+1} - p\| \\ &= (I - \alpha_n\bar{\gamma})^2\|u_n - p\|^2 + \beta_n^2\|x_n - t_n\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n \\ &\quad - p\|\|x_n - t_n\| + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\| \\ &\leq (1 - \alpha_n\bar{\gamma})^2\Big[\|x_n - p\|^2 + \delta(L\delta - 1)\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2\Big] \\ &\quad + \beta_n^2\|x_n - t_n\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n - p\|\|x_n - t_n\| \\ &\quad + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\| \\ &\leq \Big[1 - 2\alpha_n\bar{\gamma} + (\alpha_n\bar{\gamma})^2\Big]\|x_n - p\|^2 \\ &\quad + (1 - \alpha_n\bar{\gamma})^2\delta(L\delta - 1)\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2 \\ &\quad + \beta_n^2\|x_n - t_n\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n - p\|\|x_n - t_n\| \\ &\quad + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\| \\ &\leq \|x_n - p\|^2 + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 \\ &\quad + (1 - \alpha_n\bar{\gamma})^2\delta(L\delta - 1)\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2 \\ &\quad + \beta_n^2\|x_n - t_n\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n - p\|\|x_n - t_n\| \\ &\quad + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\|.\end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned}&(1 - \alpha_n\bar{\gamma})^2\delta(1 - L\delta)\|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2 \\ &\quad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \beta_n^2\|x_n - t_n\|^2 \\ &\qquad + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n \\ &\qquad - p\|\|x_n - t_n\| + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\| \\ &\quad \leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \beta_n^2\|x_n - t_n\|^2 \\ &\qquad + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n - p\|\|x_n - t_n\| \\ &\qquad + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\|.\end{aligned}$$

Since $\delta(1 - L\delta) > 0$, $\alpha_n \to 0$, $\|x_n - t_n\| \to 0$ and $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \|(T_{r_n}^{(F_2,h_2)} - I)Ax_n\| = 0. \tag{31}$$

Next, we show that $\|x_n - u_n\| \to 0$ as $n \to \infty$. Since $p \in \mathrm{Fix}(S) \cap \Gamma$, we obtain

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}^{(F_1,h_1)}(x_n + \delta A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n) - p\|^2 \\ &= \|T_{r_n}^{(F_1,h_1)}(x_n + \delta A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n) - T_{r_n}^{(F_1,h_1)}p\|^2 \\ &\leq \langle u_n - p, x_n + \delta A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n - p\rangle \\ &= \tfrac{1}{2}\Big\{\|u_n - p\|^2 + \|x_n + \delta A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n - p\|^2 \\ &\quad - \|(u_n - p) - [\, x_n + \delta A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n - p]\,\|^2\Big\} \\ &= \tfrac{1}{2}\Big\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n \\ &\quad - \delta A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2\Big\} \\ &= \tfrac{1}{2}\Big\{\|u_n - p\|^2 + \|x_n - p\|^2 - \Big[\|u_n - x_n\|^2 \\ &\quad + \delta^2\|A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\|^2 \\ &\quad - 2\delta\langle u_n - x_n, A^*(T_{r_n}^{(F_2,h_2)} - I)Ax_n\rangle\Big]\Big\}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\|u_n - p\|^2 \leq& \|x_n - p\|^2 - \|u_n - x_n\|^2 \\ &+ 2\delta\|A(u_n - x_n)\|\|T_{r_n}^{(F_2,h_2)} - I)Ax_n\|.\end{aligned}$$

It follows from (30) and (31) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1 - \alpha_n\bar{\gamma})^2\Big[\|x_n - p\|^2 - \|u_n - x_n\|^2 \\ &\quad + 2\delta\|A(u_n - x_n)\|\|\, T_{r_n}^{(F_2,h_2)} - I\Big)Ax_n\|\Big] \\ &\quad + \beta_n^2\|x_n - t_n\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n - p\|\|x_n \\ &\quad - t_n\| + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\| \\ &\leq \Big[1 - 2\alpha_n\bar{\gamma} + (\alpha_n\bar{\gamma})^2\Big]\|x_n - p\|^2 \\ &\quad - (1 - \alpha_n\bar{\gamma})^2\|u_n - x_n\|^2 + 2(1 - \alpha_n\bar{\gamma})^2\delta\| \\ &\quad A(u_n - x_n)\|\|\, T_{r_n}^{(F_2,h_2)} - I\Big)Ax_n\| \\ &\quad + \beta_n^2\|x_n - t_n\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n - p\|\|x_n \\ &\quad - t_n\| + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\| \\ &\leq \|x_n - p\|^2 + \alpha_n\bar{\gamma}^2\|x_n - p\|^2 - (1 - \alpha_n\bar{\gamma})^2\|u_n \\ &\quad - x_n\|^2 + 2(1 - \alpha_n\bar{\gamma})^2\delta\|A(u_n - x_n)\|\|T_{r_n}^{(F_2,h_2)} \\ &\quad - I\,)Ax_n\| + \beta_n^2\|x_n - t_n\|^2 + 2(1 - \alpha_n\bar{\gamma})\beta_n\|u_n \\ &\quad - p\|\|x_n - t_n\| + 2\alpha_n\|\gamma f(x_n) - Bp\|\|x_{n+1} - p\|.\end{aligned}$$

Therefore,

$$\begin{aligned}(1-\alpha_n\bar{\gamma})^2\|u_n-x_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+\beta_n^2\|x_n-t_n\|^2\\ &\quad+2(1-\alpha_n\bar{\gamma})^2\delta\|A(u_n-x_n)\|\|\ T_{r_n}^{(F_2,h_2)}-I\Big)Ax_n\|\\ &\quad+\alpha_n\bar{\gamma}^2\|x_n-p\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-p\|\|x_n-t_n\|\\ &\quad+2\alpha_n\|\gamma f(x_n)-Bp\|\|x_{n+1}-p\|\\ &\le(\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+\beta_n^2\|x_n-t_n\|^2\\ &\quad+2(1-\alpha_n\bar{\gamma})^2\delta\|A(u_n-x_n)\|\|\ T_{r_n}^{(F_2,h_2)}-I)Ax_n\|\\ &\quad+\alpha_n\bar{\gamma}^2\|x_n-p\|^2+2(1-\alpha_n\bar{\gamma})\beta_n\|u_n-p\|\|x_n-t_n\|\\ &\quad+2\alpha_n\|\gamma f(x_n)-Bp\|\|x_{n+1}-p\|.\end{aligned}$$

Since $\alpha_n\to 0$, $\|x_n-t_n\|\to 0$, $\|T_{r_n}^{(F_2,h_2)}-I)Ax_n\|\to 0$ and $\|x_{n+1}-x_n\|\to 0$ as $n\to\infty$, we obtain

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{32}$$

Thus, we can write

$$\begin{aligned}\|T(s)t_n-x_n\| &\le \|T(s)t_n-T(s)x_n\|+\|T(s)x_n-x_n\|\\ &\le \|t_n-x_n\|+\|T(s)x_n-x_n\|\\ &\quad\to 0 \text{ as } n\to\infty.\end{aligned}$$

Also, we have

$$\begin{aligned}\|T(s)t_n-t_n\| &\le \|T(s)t_n-T(s)x_n\|\\ &\quad+\|T(s)x_n-x_n\|+\|x_n-t_n\|\\ &\le \|t_n-x_n\|+\|T(s)x_n-x_n\|\\ &\quad+\|x_n-t_n\|\\ &\quad\to 0 \text{ as } n\to\infty.\end{aligned}$$

Next, we show that $\limsup_{n\to\infty}\langle(B-\gamma f)z,x_n-z\rangle\le 0$, where $z=P_{\mathrm{Fix}(S)\cap\Gamma}(I-B+\gamma f)z$. To show this inequality, we choose a subsequence $\{t_{n_i}\}$ of $\{t_n\}\subseteq K$ such that

$$\limsup_{n\to\infty}\langle(B-\gamma f)z,t_n-z\rangle=\lim_{i\to\infty}\langle(B-\gamma f)z,t_{n_i}-z\rangle.$$

Since $\{t_{n_i}\}$ is bounded, there exists a subsequence $\{t_{n_{i_j}}\}$ of $\{t_{n_i}\}$ which converges weakly to some $w\in C$. Without loss of generality, we can assume that $t_{n_i}\rightharpoonup w$.

Now, we prove that $w\in\mathrm{Fix}(S)\cap\Gamma$. Let us first show that $w\in\mathrm{Fix}(S)$. Assume that $w\notin\mathrm{Fix}(S)$. Since $t_{n_i}\rightharpoonup w$ and $T(s)w\ne w$, from Opial's condition (11), we have

$$\begin{aligned}\liminf_{i\to\infty}\|t_{n_i}-w\| &< \liminf_{i\to\infty}\|t_{n_i}-T(s)w\|\\ &\le \liminf_{i\to\infty}\left\{\|t_{n_i}-T(s)t_{n_i}\|\right.\\ &\quad\left.+\|T(s)t_{n_i}-T(s)w\|\right\}\\ &\le \liminf_{i\to\infty}\|t_{n_i}-w\|,\end{aligned}$$

which is a contradiction. Thus, we obtain $w\in\mathrm{Fix}(S)$.

Next, we show that $w\in\mathrm{GEP}(F_1,h_1)$. Since $u_n=T_{r_n}^{(F_1,h_1)}x_n$, we have

$$F_1(u_n,y)+h_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\ \ \forall y\in C.$$

It follows from the monotonicity of F_1 that

$$h_1(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge F_1(y,u_n),$$

and hence,

$$h_1(u_{n_i},y)+\left\langle y-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r_n}\right\rangle\ge F_1(y,u_{n_i}).$$

Since $\|u_n-x_n\|\to 0$, we get $u_{n_i}\rightharpoonup w$ and $\frac{u_{n_i}-x_{n_i}}{r_n}\to 0$. It follows by Assumption 1 (iv) that $0\ge F_1(y,w)$, $\forall w\in C$. For t with $0<t\le 1$ and $y\in C$, let $y_t=ty+(1-t)w$. Since $y\in C$, $w\in C$, we get $y_t\in C$, and hence, $F_1(y_t,w)\le 0$. So, from Assumption 1 (i) and (iv), we have

$$\begin{aligned}0&=F_1(y_t,y_t)+h_1(y_t,y_t)\le t[F_1(y_t,y)+h_1(y_t,y)]\\ &\quad+(1-t)[F_1(y_t,w)+h_1(y_t,w)]\le t[F_1(y_t,y)+h_1(y_t,y)]\\ &\quad+(1-t)[F_1(w,y_t)+h_1(w,y_t)]\le[F_1(y_t,y)+h_1(y_t,y)].\end{aligned}$$

Therefore, $0\le F_1(y_t,y)+h_1(y_t,y)$. From Assumption 1 (iii), we have $0\le F_1(w,y)+h_1(w,y)$. This implies that $w\in\mathrm{GEP}(F_1,h_1)$.

Next, we show that $Aw\in\mathrm{GEP}(F_2,h_2)$. Since $\|u_n-x_n\|\to 0$, $u_n\rightharpoonup w$ as $n\to\infty$ and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup w$, and since A is a bounded linear operator, so $Ax_{n_k}\rightharpoonup Aw$.

Now, setting $v_{n_k}=Ax_{n_k}-T_{r_{n_k}}^{F_2}Ax_{n_k}$. It follows from (31) that $\lim_{k\to\infty}v_{n_k}=0$ and $Ax_{n_k}-v_{n_k}=T_{r_{n_k}}^{F_2}Ax_{n_k}$.

Therefore, from Lemma 7, we have

$$\begin{aligned}&F_2(Ax_{n_k}-v_{n_k},z)+h_2(Ax_{n_k}-v_{n_k},z)\\ &\qquad+\frac{1}{r_{n_k}}\langle z-(Ax_{n_k}-v_{n_k}),(Ax_{n_k}-v_{n_k})\\ &\qquad-Ax_{n_k}\rangle\ge 0,\ \forall z\in Q.\end{aligned}$$

Since F_2 and h_2 are upper semicontinuous in the first argument, taking lim sup to above inequality as $k\to\infty$ and using condition (iii), we obtain

$$F_2(Aw,z)+h_2(Aw,z)\ge 0,\ \forall z\in Q,$$

which means that $Aw\in\mathrm{GEP}(F_2,h_2)$, and hence, $w\in\Gamma$.

Next, we claim that $\limsup_{n\to\infty}\langle f(z)-z,x_n-z\rangle\le 0$, where $z=P_{\mathrm{Fix}(S)\cap\Gamma}(I-B+\gamma f)z$. Now, from (8), we have

$$\begin{aligned}&\limsup_{n\to\infty}\langle(B-\gamma f)z-z,x_n-z\rangle\\ &\quad=\limsup_{n\to\infty}\langle(B-\gamma f)z-z,t_n-z\rangle\\ &\quad\le\limsup_{i\to\infty}\langle(B-\gamma f)z-z,t_{n_i}-z\rangle\\ &\quad=\langle(B-\gamma f)z-z,w-z\rangle\\ &\quad\le 0.\end{aligned} \tag{33}$$

Finally, we show that $x_n \to z$:

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\alpha_n\gamma f(x_n)+\beta_n x_n \\
&\quad +[\,(1-\beta_n)I-\alpha_n B]\,t_n - z\|^2 \\
&= \|\alpha_n(\gamma f(x_n)-Bz)+\beta_n(x_n-z) \\
&\quad +[\,(1-\beta_n)I-\alpha_n B]\,(t_n-z)\|^2 \\
&\le \|\beta_n(x_n-z)+[\,(1-\beta_n)I-\alpha_n B]\,(t_n-z)\|^2 \\
&\quad +2\alpha_n\langle\gamma f(x_n)-Bz, x_{n+1}-z\rangle \\
&\le [\|((1-\beta_n)I-\alpha_n B)(t_n-z)\| \\
&\quad +\|\beta_n(x_n-z)\|]^2+2\alpha_n\gamma\langle f(x_n) \\
&\quad -f(z), x_{n+1}-z\rangle+2\alpha_n\langle\gamma f(z) \\
&\quad -Bz, x_{n+1}-z\rangle \\
&\le [\,(1-\beta_n)-\alpha_n\bar{\gamma})\,\|x_n-z\|+\beta_n\|x_n-z\|]^2 \\
&\quad +2\alpha_n\gamma\alpha\|x_n-z\|\,\|x_{n+1}-z\| \\
&\quad +2\alpha_n\langle\gamma f(z)-Bz, x_{n+1}-z\rangle \\
&\le (1-\alpha_n\bar{\gamma})^2\|x_n-z\|^2+\alpha_n\gamma\alpha\{\|x_n-z\|^2 \\
&\quad +\|x_{n+1}-z\|^2\}+2\alpha_n\langle\gamma f(z)-Bz, x_{n+1}-z\rangle \\
&\le (1-\alpha_n\bar{\gamma})^2\|x_n-z\|^2+\alpha_n\gamma\alpha\|x_n-z\|^2 \\
&\quad +\gamma\alpha\,\|x_{n+1}-z\|^2+2\alpha_n\langle\gamma f(z) \\
&\quad -Bz, x_{n+1}-z\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \tfrac{1-2\alpha_n\bar{\gamma}+(\alpha_n\bar{\gamma})^2+\alpha_n\gamma\alpha}{1-\gamma\alpha}\|x_n-z\|^2 \\
&\quad +\tfrac{2\alpha_n}{1-\gamma\alpha}\langle\gamma f(z)-Bz, x_{n+1}-z\rangle \\
&= \left[1-\tfrac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\gamma\alpha}\right]\|x_n-z\|^2+\tfrac{(\alpha_n\bar{\gamma})^2}{1-\gamma\alpha}\|x_n-z\|^2 \\
&\quad +\tfrac{2\alpha_n}{1-\gamma\alpha}\langle\gamma f(z)-Bz, x_{n+1}-z\rangle \\
&\le \left[1-\tfrac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\gamma\alpha}\right]\|x_n-z\|^2+\tfrac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\gamma\alpha} \\
&\quad \times\left\{\tfrac{(\alpha_n\bar{\gamma}^2)M}{2(\bar{\gamma}-\gamma\alpha)}+\tfrac{1}{\bar{\gamma}-\gamma\alpha}\langle\gamma f(z)-Bz, x_{n+1}-z\rangle\right\} \\
&= (1-\delta_n)\|x_n-z\|^2+\delta_n\sigma_n,
\end{aligned} \tag{34}$$

where $M := \sup\{\|x_n-z\|^2 : n \ge 1\}$, $\delta_n = \frac{2(\bar{\gamma}-\gamma\alpha)\alpha_n}{1-\gamma\alpha}$, and $\sigma_n = \frac{(\alpha_n\bar{\gamma}^2)M}{2(\bar{\gamma}-\gamma\alpha)}+\frac{1}{\bar{\gamma}-\gamma\alpha}\langle\gamma f(z)-Bz, x_{n+1}-z\rangle$. Since $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, it is easy to see that $\lim_{n\to\infty}\delta_n = 0$, $\sum_{n=0}^{\infty}\delta_n = \infty$, and $\lim\sup_{n\to\infty}\sigma_n \le 0$. Hence, from (33), (34), and Lemma 3, we deduce that $x_n \to z$. This completes the proof.

We have the following consequences of Theorem 1.

Corollary 1. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C\times C \to \mathbb{R}$ and $F_2 : Q\times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption 1 and F_2 is upper semicontinuous in the first argument. Let $S = \{T(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C such that $Fix(S)\cap\Omega \neq \emptyset$. Let $f : C \to C$ be a contraction mapping with constant $\alpha \in (0,1)$ and B be a strongly positive linear bounded self-adjoint operator on H_1 with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma+\frac{1}{\alpha}$. Let $\{s_n\}$ is a positive real sequence which diverges to $+\infty$. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$\begin{aligned}
u_n &= T_{r_n}^{F_1}(x_n+\delta A^*(T_{r_n}^{F_2}-I)Ax_n); \\
x_{n+1} &= \alpha_n\gamma f(x_n)+\beta_n x_n \\
&\quad +((1-\beta_n)I-\alpha_n B)\frac{1}{s_n}\int_0^{s_n}T(s)u_n ds,
\end{aligned}$$

where $r_n \subset (0,\infty)$ and $\delta \in (0,1/L)$, L is the spectral radius of the operator A^*A, and A^* is the adjoint of A, and $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$.

(ii) $0 < \lim\inf_{n\to\infty}\beta_n \le \lim\sup_{n\to\infty}\beta_n < 1$.

(iii) $\lim\inf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$.

(iv) $\lim_{n\to\infty}\frac{|s_{n+1}-s_n|}{s_{n+1}} = 0$.

Then, the sequence $\{x_n\}$ converges strongly to $z \in Fix(S)\cap\Omega$, where $z = P_{Fix(S)\cap\Omega}(I-B+\gamma f)z$.

Proof. Taking $h_1 = h_2 = 0$ in Theorem 1, then the conclusion of Corollary 1 is obtained. □

Corollary 2. [6] *Let H be a real Hilbert space and let $C \subseteq H$ be a nonempty closed convex subset. Assume that $F : C\times C \to \mathbb{R}$ is a bifunction satisfying Assumption 1 for F only. Let $S = \{T(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C such that $Fix(S)\cap EP(F) = \emptyset$. Let $f : C \to C$ be a contraction mapping with constant $\alpha \in (0,1)$ and B be a strongly positive linear bounded self-adjoint operator on H with constant $\bar{\gamma} > 0$ such that $0 < \gamma < \frac{\bar{\gamma}}{\alpha} < \gamma+\frac{1}{\alpha}$. Let $\{s_n\}$ is a positive real sequence which diverges to $+\infty$. For a given $x_0 \in C$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$u_n = T_{r_n}^{F}x_n;$$

$$x_{n+1} = \alpha_n\gamma f(x_n)+(1-\alpha_n B)\tfrac{1}{s_n}\int_0^{s_n}T(s)u_n ds,$$

where $r_n \subset (0,\infty)$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$.

(ii) $\lim\inf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$.

(iii) $\lim_{n\to\infty}\frac{|s_{n+1}-s_n|}{s_{n+1}} = 0$.

Then, the sequence $\{x_n\}$ converges strongly to $z \in PFix(S)\cap EP(F)$, where $z = P_{Fix(S)\cap EP(F)}\,(I-B+\gamma f)z$.

Proof. Taking $F_1 = F_2 = F$, $H_1 = H_2 = H$, $h_1 = h_2 = 0$, $\{\beta_n\} = 0$, and $A = 0$ in Theorem 1, then the conclusion of Corollary 2 is obtained. □

Corollary 3. [4] *Let H be a real Hilbert space and let $C \subseteq H$ be a nonempty closed convex subset. Let $S = \{T(s) : 0 \le s < \infty\}$ be a nonexpansive semigroup on C such that $Fix(S) \neq \emptyset$. Let $f : C \to C$ be a contraction mapping with constant $\alpha \in (0, 1)$. Let $\{s_n\}$ be a positive real sequence which diverges to $+\infty$. For a given $x_0 \in C$ arbitrarily, let the iterative sequence $\{x_n\}$ be generated by*

$$\begin{aligned} x_{n+1} &= \alpha_n \gamma f(x_n) + \beta_n x_n \\ &\quad + (1 - \alpha_n - \beta_n) \frac{1}{s_n} \int_0^{s_n} T(s) x_n ds, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.
(ii) $0 < \lim \inf_{n\to\infty} \beta_n \le \lim \sup_{n\to\infty} \beta_n < 1$.
(iii) $\lim_{n\to\infty} \frac{|s_{n+1} - s_n|}{s_{n+1}} = 0$.

Then, the sequence $\{x_n\}$ converges strongly to $z \in Fix(S)$, where $z = P_{Fix(S)} f(z)$.

Proof. Taking $H_1 = H_2 = H$, $u_n = x_n$, $F_1 = F_2 = h_1 = h_2 = 0$, and $B = I$ in Theorem 1, then the conclusion of Corollary 3 is obtained. □

Results and discussion

We introduce and study an iterative method for approximating a common solution of split generalized equilibrium problem and fixed point problem for a nonexpansive semigroup in real Hilbert spaces. We obtain a strong convergence result for approximating a common solution of split generalized equilibrium problem and fixed point problem for a nonexpansive semigroup in real Hilbert spaces. Further, we obtain some consequences of our main result.

Conclusions

The results presented in this paper extend and generalize the works of Shimizu and Takahashi [2], Chen and Song [3], Plubtieng and Punpaeng [4], and Cianciaruso et al. [6]. The algorithm considered in Theorem 1 is different from those considered in [7-10] in the sense that the variable sequence $\{r_n\}$ has been taken in place of fixed r. Further, the approach of the proof presented in this paper is also different. The use of the iterative method presented in this paper for the split monotone variational inclusions considered in Moudafi [9] needs further research effort.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors contributed equally and significantly in writing this article. Both authors read and approved the final manuscript.

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