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Global attractor for a nonlinear Timoshenko equation with source terms

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Abstract
We study weak solutions of the Timoshenko equation in a bounded domain. We consider a nonlinear dissipation and a nonlinear source term. We obtain boundedness of the solutions as well as their asymptotic behavior. In particular, the source term does not produce a blowup, and the global attractor is the set of all equilibria.

Keywords: Timoshenko equation; Global solutions; Boundedness; Asymptotic behavior; Global attractor

MSC: 35L70; 35B35; 35B40

Introduction

In this work, we shall study the dynamics of the following equation:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|_2^2)\Delta u + g(u_t) = f(u) \text{ in } \Omega, \quad (1)$$

with one set of the following boundary conditions:

$$u = 0 \text{ and } \Delta u = 0 \text{ on } \partial\Omega$$

or

$$u = 0 \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega,$$

and the following initial conditions:

$$u(x,0) = u_0, u_t(x,0) = v_0, x \in \Omega.$$

Here, $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary, $\|\cdot\|_2$ is the norm in $L^2(\Omega)$, and the nonlinearities considered are defined by:

$$M(s^2) = \alpha + \beta s^{2\gamma}, \quad \alpha \geq 0, \quad \beta \geq 0, \quad \alpha + \beta > 0, \quad \gamma \geq 1, \quad (2)$$

$$g(u_t) = \delta u_t |u_t|^{\lambda - 2}, \delta > 0, \lambda \geq 2, \quad (3)$$

and

$$f(u) = \mu u |u|^{r-2}, \mu > 0, r > 2. \quad (4)$$

For $n = 1$, Timoshenko equation is an approximate model describing the transversal motion of a rod. See the work of Antman [1] for a general and rigorous framework of models in the theory of elasticity, in particular, of Equation 1. Here, we are interested in the qualitative behavior of solutions of the Timoshenko equation for any n. The dynamics of second-order equations in time has been widely studied by Alves and Cavalcanti [2], Barbu et al. [3], Cavalcanti et al. [4-9], Rammaha and Sakuntasathien [10,11], Todorova and Vitillaro [12,13]. There are a number of papers studying the dynamics of Equation 1, when $sf(s) \leq 0, s \in \mathbb{R}$; see for instance the books of Hale [14] and Haraux [15] and references therein. For a destabilizing source term, $sf(s) > 0$, $s \in \mathbb{R} \setminus \{0\}$, there are several results studying the effect of this force in nonlinear wave equations; see the papers of Payne and Sattinger [16], Georgiev and Todorova [17], Ikehata [18], and Esquivel-Avila [19,20]. For the undamped Timoshenko equation, Bainov and Minchev [21] gave sufficient conditions for the nonexistence of smooth solutions of (1), with negative initial energy, and gave an upper bound of the maximal time of existence. For positive and sufficiently small initial energy, blowup and globality properties are characterized in the study of Esquivel-Avila [22]. For damped Timoshenko equation, see another study of Esquivel-Avila [23]; we proved blowup in finite time, globality and unboundedness, globality and convergence to the equilibria, and rates of such convergence for the zero equilibrium. All these results were obtained by means of the potential well theory under the assumption that $r \geq 2(\gamma + 1)$. To the knowledge of the author, the behavior of the solutions is still unknown when $2 < r < 2(\gamma + 1)$. Here, we prove that, in this case, there is no blowup; all

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of the solutions are global, are uniformly bounded, and converge to the equilibria set.

Preliminaries

We present an existent, unique, and continued theorem for Equation 1 (see [23]).

Theorem 1. *Assume that $r > 2$ and $r \leq 2(n-2)/(n-4)$ if $n \geq 5$. For every initial data $(u_0, v_0) \in H \equiv B \times L^2(\Omega)$, where B is defined either by $B \equiv H^2(\Omega) \cap H_0^1(\Omega)$ or by $B \equiv H_0^2(\Omega)$, there exists a unique (local) weak solution $(u(t), v(t))$ of problem (1), that is,*

$$\frac{d}{dt}(v(t), w)_2 + (\Delta u(t), \Delta w)_2 + M(\|\nabla u(t)\|_2^2)(\nabla u(t), \nabla w)_2 + (g(v(t)), w)_2 = (f(u(t)), w)_2, \tag{5}$$

almost everywhere (a.e.) in (0,T) and for every $w \in B \cap L^\lambda(\Omega)$, such that

$$u \in C([0,T]; B) \cap C^1([0,T]; L^2(\Omega)),$$
$$v \equiv u_t \in L^\lambda((0,T) \times \Omega).$$

Here, $(\cdot,\cdot)_2$ denotes the inner product in $L^2(\Omega)$.
The following energy equation holds:

$$E_0 = E(t) + \int_0^t \delta \|v(\tau)\|_\lambda^\lambda d\tau, \tag{6}$$

where

$$E(t) \equiv E(u(t), v(t)) \equiv \frac{1}{2}\|v(t)\|_2^2 + J(u(t)) \tag{7}$$

and

$$J(u) \equiv \frac{1}{2}a(u) + \frac{1}{2(\gamma+1)}c(u) - \frac{1}{r}b(u), \tag{8}$$

with

$$a(u) \equiv \|u\|_B^2 \equiv \|\Delta u\|_2^2 + \alpha\|\nabla u\|_2^2, b(u) \equiv \mu\|u\|_r^r, \quad c(u) \equiv \beta\|\nabla u\|_2^{2(\gamma+1)}. \tag{9}$$

Here, $E_0 \equiv E(u_0, v_0)$ is the initial energy, and $\|\cdot\|_q$ denotes the norm in the $L^q(\Omega)$ space.
If the maximal time of existence $T_M < \infty$, then $(u(t), v(t)) \to \infty$ as $t \nearrow T_M$, in the norm of H:

$$\|(u,v)\|_H^2 \equiv \|u\|_B^2 + \|v\|_2^2. \tag{10}$$

In that case, from (6) to (9), $\|u(t)\|_r \to \infty$ as $t \nearrow T_M$.

We define the set of equilibria of Equation 1 by:

$$\mathcal{E} \equiv \{u_e \in B : \Delta^2 u_e - M(\|u\|_2^2)\Delta u_e = f(u_e)\}. \tag{11}$$

We notice that, in particular, $0 \in \mathcal{E}$.

Main result

In this section, we prove that all of the solutions are global and uniformly bounded and that the global attractor is $\mathcal{E}$.

Theorem 2. *Let $(u(t), v(t))$ be a solution of problem (1), given by Theorem 1. Assume that $r < 2(\gamma+1)$ and $r \leq 2n/(n-2)$ if $n \geq 3$. Then, $(u(t), v(t))$ is global and uniformly bounded, and $(u(t), v(t)) \to \mathcal{E}_\infty$ is strongly in H as $t \to \infty$, where $\mathcal{E}_\infty \equiv \{(u_e, 0) : u_e \in \mathcal{E}, J(u_e) = E_\infty \equiv \lim_{t\to\infty} E(u(t), v(t))\}$.*

Proof. Since the proof is long, we shall divide it into five steps as follows. First, we prove that the solution is global and bounded. Next, we show that this implies weak convergence to the equilibria set. In order to conclude strong convergence, we have to prove that the orbit is precompact in the phase space. In order to do that, we show that the solution is uniformly continuous. We then prove the precompactness of the orbit.

Globality and boundedness

Notice that, from the continuous injection $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$,

$$c(u)^{\frac{1}{2(\gamma+1)}} = \beta^{\frac{1}{2(\gamma+1)}}\|\nabla u\|_2 \geq \beta^{\frac{1}{2(\gamma+1)}} C(\Omega)\|u\|_r = \beta^{\frac{1}{2(\gamma+1)}} C(\Omega)\mu^{-\frac{1}{r}} b(u)^{\frac{1}{r}} \equiv \kappa^{\frac{1}{2(\gamma+1)}} b(u)^{\frac{1}{r}},$$

and $C(\Omega) > 0$ is a Sobolev constant. Then, along the solution, for any $t \geq 0$,

$$\frac{1}{2(\gamma+1)}c(u(t)) \geq \frac{\kappa}{2(\gamma+1)}b(u(t))^{\frac{2(\gamma+1)}{r}}$$

and

$$\frac{1}{2(\gamma+1)}c(u(t)) - \frac{1}{r}b(u(t)) \geq \frac{\kappa}{2(\gamma+1)}b(u(t))^{\frac{2(\gamma+1)}{r}} \times \left(1 - \nu b(u(t))^{-\left(\frac{2(\gamma+1)-r}{r}\right)}\right), \tag{12}$$

where $\kappa\nu \equiv \frac{2(\gamma+1)}{r}$. Here, we have either

$$\text{(i)}\ \ b(u(t)) \leq \nu^{\frac{r}{2(\gamma+1)-r}} \quad \text{or} \quad \text{(ii)}\ \ b(u(t)) \geq \nu^{\frac{r}{2(\gamma+1)-r}}.$$

We use the energy equation in both cases. For the first one, we get the following from (12):

$$\|(u(t),v(t))\|_H^2 \le \|v(t)\|_2^2 + a(u(t)) + \frac{1}{\gamma+1}c(u(t)) \\ \le 2E_0 + \frac{2}{r}\nu^{\frac{r}{2(\gamma+1)-r}}.$$

In the second case,

$$\|(u(t),v(t))\|_H^2 = \|v(t)\|_2^2 + a(u(t)) \le 2E(t) \le 2E_0.$$

Therefore, for any $t \ge 0$,

$$\|(u(t),v(t))\|_H^2 \le 2E_0 + \frac{2}{r}\nu^{\frac{r}{2(\gamma+1)-r}} \equiv \hat{E}_0. \tag{13}$$

Weak convergence to $\mathcal{E}$

Since the solution is global and uniformly bounded in the norm of H, there exists a sequence, $\{t_n\}$, such that if $n \to \infty$, then $t_n \to \infty$, and $(u(t_n), v(t_n)) \to (\hat{u}, \hat{v})$ weakly in H. Moreover, $b(u(t_n)) \to b(\hat{u})$, because of the compact injection $B \hookrightarrow L^r(\Omega)$. On the other hand, the energy is uniformly bounded and nonincreasing; consequently,

$$-\infty < E_\infty \equiv \lim_{t\to\infty} E(t) = \inf_{t\ge 0} E(t) \le E(t) \le E_0 < \infty. \tag{14}$$

From the energy equation and the continuous injection $L^\lambda(\Omega) \hookrightarrow L^2(\Omega)$, this implies that

$$\lim_{t\to\infty} \int_t^{t+1} \|v(\tau)\|_2^\lambda \, d\tau = 0.$$

In particular, for any sequence $\{s_n\}$ such that $s_n \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} \int_0^1 h_n(\tau)\, d\tau = 0,$$

where $h_n(\tau) \equiv \|v(s_n+\tau)\|_2^\lambda$, for $\tau \in [0,1]$. By Fatou Lemma,

$$\liminf_{n\to\infty} \|v(s_n+\tau)\|_2^\lambda = \liminf_{n\to\infty} h_n(\tau) = 0,$$

for a.e. $\tau \in [0,1]$, and by the weak convergence to $\hat{v}$,

$$\|\hat{v}\|_2 \le \liminf_{n\to\infty} \|v(t_n)\|_2 = 0,$$

where we choose $\{s_n\}$ such that $t_n = s_n + \tau_0$, for some $\tau_0 \in [0,1]$. Then, the weak limit set of the orbit is such that

$$\omega_w(u_0,v_0) = \{(\hat{u},0) : (u(t_n),v(t_n)) \to (\hat{u},0), \text{ weakly in } H\}.$$

The weak limit set is positive invariant (see Ball [24]), that is,

$$(u(0),v(0)) \in \omega_w(u_0,v_0) \Rightarrow (u(t),v(t)) \in \omega_w(u_0,v_0), \forall t > 0.$$

Consequently $\omega_w(u_0,v_0) \subset \mathcal{E}$, that is, there exists some $u_e \in \mathcal{E}$, such that, along a sequence of times, the solution converges weakly to $(u_e, 0)$.

Strong convergence to $\mathcal{E}$

If the convergence is strong in H,

$$\lim_{n\to\infty} \|(u(t_n) - u_e, v(t_n))\|_H = 0,$$

then

$$J(u_e) = \lim_{n\to\infty} J(u(t_n)),$$

and by (14),

$$E_\infty \equiv \lim_{n\to\infty} E(t_n) = \lim_{t\to\infty} J(u(t_n)).$$

Consequently,

$$E_\infty \equiv \lim_{t\to\infty} E(t) = J(u_e),$$

and the assertion of the theorem holds.

Now, strong convergence follows if the orbit

$$\{(u(t),v(t))\}_{t\ge 0} \tag{15}$$

is a precompact subset of H.

To show this, we shall use a technique due to Haraux [15], and we shall extend it to handle the nonlinearities of Equation 1 as follows.

Uniform continuity

We shall prove that $t \mapsto (u(t),v(t)) \in H$ is uniformly continuous. In order to do that, we define, for every $\epsilon > 0$ and $t \ge 0$,

$$u_\epsilon(t) \equiv u(t+\epsilon) - u(t), v_\epsilon(t) \equiv v(t+\epsilon) - v(t), \\ 2w_\epsilon(t) \equiv \|(u_\epsilon(t), v_\epsilon(t))\|_H^2.$$

Hence, the uniform continuity of the solution holds if for any $\eta > 0$, there exists $\epsilon(\eta) > 0$, such that

$$w_\epsilon(t) \le \eta, \tag{16}$$

for every $t \ge 0$, and $\epsilon \in (0, \epsilon(\eta))$. To get that estimate, we need the energy equation for $(u_\epsilon(t), v_\epsilon(t))$. Then, from (5), we obtain

$$w_\epsilon(0) = w_\epsilon(t) + \int_0^t (g_\epsilon(\tau) - f_\epsilon(\tau) - \widehat{m}_\epsilon(\tau), v_\epsilon(\tau))_2 \, d\tau, \tag{17}$$

where

$$g_\epsilon(t) \equiv g(v(t+\epsilon)) - g(v(t)), f_\epsilon(t) \equiv f(u(t+\epsilon)) - f(u(t)), \\ \widehat{m}_\epsilon(t) \equiv m(\|\nabla u(t+\epsilon)\|_2^2)\Delta u(t+\epsilon) - m(\|\nabla u(t)\|_2^2)\Delta u(t),$$

and

$$m(\|\nabla u(t)\|_2^2) \equiv \beta \|\nabla u(t)\|_2^{2\gamma}.$$

However, we cannot obtain (16) through the energy equation (17) alone because of the form of the nonlinearity $\widehat{m}_\epsilon(t)$. Hence, we have to work with the auxiliary function:

$$W_\epsilon(t) \equiv w_\epsilon(t) + \frac{1}{2} m(\|\nabla u(t+\epsilon)\|_2^2)\|\nabla u_\epsilon(t)\|_2^2. \tag{18}$$

The corresponding energy equation for $W_\epsilon(t)$ is

$$W_\epsilon(0) = W_\epsilon(t) + \int_0^t (g_\epsilon(\tau) - f_\epsilon(\tau) - m_\epsilon(\tau), v_\epsilon(\tau))_2 d\tau - \int_0^t (n_\epsilon(\tau), u_\epsilon(\tau))_2 \, d\tau, \tag{19}$$

where

$$m_\epsilon(t) \equiv (m(\|\nabla u(t+\epsilon)\|_2^2) - m(\|\nabla u(t)\|_2^2))\Delta u(t)$$

and

$$n_\epsilon(t) \equiv m'(\|\nabla u(t+\epsilon)\|_2^2)(\Delta u(t+\epsilon), v(t+\epsilon))_2 \Delta u_\epsilon(t).$$

Notice that since the solution is uniformly bounded by $\hat{E}_0$, there exists a constant $\tilde{E}_0 > 0$, depending on $\hat{E}_0$, such that

$$w_\epsilon(t) \le W_\epsilon(t) \le \tilde{E}_0 w_\epsilon(t), \tag{20}$$

that is, W_ϵ is an equivalent norm of the solution in H. We shall show the uniform continuity property for W_ϵ.

For every $t \ge 0$, we have either

$$W_\epsilon(t+1) \le W_\epsilon(t) \tag{21}$$

or

$$W_\epsilon(t+1) > W_\epsilon(t). \tag{22}$$

If (22) holds,

$$0 > W_\epsilon(t) - W_\epsilon(t+1) = \int_t^{t+1} (g_\epsilon(\tau) - f_\epsilon(\tau) - m_\epsilon(\tau), v_\epsilon(\tau))_2 d\tau - \int_t^{t+1} (n_\epsilon(\tau), u_\epsilon(\tau))_2 d\tau. \tag{23}$$

A well-known inequality can be applied to the monotone form of the damping term:

$$(g_\epsilon(t), v_\epsilon(t))_2 \ge 2^{2-\lambda}\delta \|v_\epsilon(t)\|_\lambda^\lambda. \tag{24}$$

Therefore, (23) yields

$$2^{2-\lambda}\delta \int_t^{t+1} \|v_\epsilon(\tau)\|_\lambda^\lambda d\tau < \int_t^{t+1} (f_\epsilon(\tau) + m_\epsilon(\tau), v_\epsilon(\tau))_2 d\tau + \int_t^{t+1} (n_\epsilon(\tau), u_\epsilon(\tau))_2 d\tau. \tag{25}$$

On the other hand, we apply another inequality for the source term:

$$\left|f_\epsilon(t)\right| \le \sigma(r)\mu \left(|u(t+\epsilon)|^{r-2} + |u(t)|^{r-2}\right) |u_\epsilon(t)|, \tag{26}$$

where $\sigma(r) = 1$ if $r \in [2,3]$ and $\sigma(r) = (r-1)/2$ if $r > 3$.

Now, we apply Hölder inequality to obtain

$$\left(\int_t^{t+1} \|f_\epsilon(\tau)\|_2^2 d\tau\right)^{\frac{1}{2}} \le \hat{C} \sup_{t \ge 0} a(u(t))^{(r-2)/2} \times \left(\int_t^{t+1} \|u_\epsilon(\tau)\|_{2(r-1)}^{2(r-1)} d\tau\right)^{\frac{1}{2(r-1)}}, \tag{27}$$

where $\hat{C} \equiv 2\sigma(r)\mu C(\Omega)$, and $C(\Omega) > 0$ is a Sobolev constant of the injection $B \hookrightarrow L^{2(r-1)}(\Omega)$.

We claim that $t \mapsto u(t) \in L^{2(r-1)}(\Omega)$ must be uniformly continuous. Otherwise, there exists some $\eta_0 > 0$ and sequences $\{\epsilon_n\}_{n\ge1}\{t_n\}_{n\ge1}$, such that $\epsilon_n \to 0$ and $t_n \to \infty$, as $n \to \infty$, and

$$\|u_{\epsilon_n}(t_n)\|_{2(r-1)} > \eta_0, \tag{28}$$

for every $n \ge 1$. By assumption, $B \hookrightarrow L^{2(r-1)}(\Omega)$ is compact, and since $\{u(t)\}_{t\ge0}$ is bounded in B, then $\{u(t_n + \epsilon_n)\}_{n\ge1}, \{u(t_n)\}_{n\ge1}$ are precompact in $L^{2(r-1)}(\Omega)$. Therefore, we can extract subsequences $\{u(t_n' + \epsilon_n')\}_{n\ge1}, \{u(t_n')\}_{n\ge1}$, such that for some fixed n_0, which is sufficiently big, and every $n \ge n_0$,

$$\begin{aligned}\|u(t_n' + \epsilon_n') - u(t_n')\|_{2(r-1)} &\le \|u(t_n' + \epsilon_n') - u(t_{n_0} + \epsilon_{n_0})\|_{2(r-1)} \\ &\quad + \|u(t_{n_0} + \epsilon_{n_0}) - u(_{n_0})\|_{2(r-1)} \\ &\quad + \|u(t_{n_0}) - u(t_n')\|_{2(r-1)} \\ &\le \frac{\eta_0}{3} + \frac{\eta_0}{3} + \frac{\eta_0}{3} = \eta_0.\end{aligned}$$

This contradicts (28). Hence, for any $\eta > 0$, there exists some $\hat{\epsilon}(\eta) > 0$, such that for every $t \ge 0$ and every $\epsilon \in (0, \hat{\epsilon}(\eta))$,

$$\left(\int_t^{t+1} \|u_\epsilon(\tau)\|_{2(r-1)}^{2(r-1)} \, d\tau\right)^{\frac{1}{2(r-1)}} \le \eta^{4(\lambda-1)}. \tag{29}$$

Consequently, from (27), (29), and Hölder inequality, we get

$$\int_t^{t+1} |(f_\epsilon(\tau), v_\epsilon(\tau))_2| d\tau \le C\eta^{4(\lambda-1)} \times \left(\int_t^{t+1} \|v_\epsilon(\tau)\|_\lambda^\lambda d\tau\right)^{\frac{1}{\lambda}}, \tag{30}$$

where $C > 0$ depends on $\hat{C}, \hat{E}_0$, and the inclusion $L^\lambda(\Omega) \hookrightarrow L^2(\Omega)$.

Now notice that

$$\begin{aligned}\|m_\epsilon(t)\|_2 &\le \sup_{t\ge0}\{m'(\|\nabla u(t)\|_2^2)\} \left|\|\nabla u(t+\epsilon)\|_2^2 - \|\nabla u(t)\|_2^2\right| \|\Delta u(t)\|_2 = \sup_{t\ge0}\{m'(\|\nabla u(t)\|_2^2)\} | \\ &\quad (\nabla u(t+\epsilon) + \nabla u(t), \nabla u(t+\epsilon) - \nabla u(t))_2| \|\Delta u(t)\|_2 \\ &\le \sup_{t\ge0}\{m'(\|\nabla u(t)\|_2^2) \|\Delta u(t+\epsilon) + \Delta u(t)\|_2 \|\Delta u(t)\|_2\} \\ &\quad \|u_\epsilon(t)\|_2 \le C(\hat{E}_0)\|u_\epsilon(t)\|_2,\end{aligned}$$

then

$$|(m_\epsilon(t), v_\epsilon(t))_2| \leq C(\hat{E}_0)\|u_\epsilon(t)\|_2\|v_\epsilon(t)\|_2. \tag{31}$$

Also,

$$\begin{aligned}|(n_\epsilon(t), u_\epsilon(t))_2| \leq & m'(\|\nabla u(t+\epsilon)\|_2^2)\|\Delta u(t+\epsilon)\|_2 \\ & \|v(t+\epsilon)\|_2\|\Delta u_\epsilon(t)\|_2\|u_\epsilon(t)\|_2 \\ \leq & C(\hat{E}_0)\|u_\epsilon(t)\|_2.\end{aligned} \tag{32}$$

Here, $C(\hat{E}_0) > 0$ is a constant.

Since $B \hookrightarrow L^2(\Omega)$ is compact, we show like in (29) that

$$\left(\int_t^{t+1} \|u_\epsilon(\tau)\|_2^2 d\tau\right)^{\frac{1}{2}} \leq \eta^{4(\lambda-1)}. \tag{33}$$

Consequently, from (31) to (33) and Hölder inequality, we obtain

$$\begin{aligned}\int_t^{t+1}\{|(m_\epsilon(\tau), v_\epsilon(\tau))_2| + |(n_\epsilon(\tau), u_\epsilon(\tau))_2|\}d\tau \leq \widehat{C}\eta^{4(\lambda-1)} \\ \times\left\{\left(\int_t^{t+1}\|v_\epsilon(\tau)\|_\lambda^\lambda d\tau\right)^{\frac{1}{\lambda}} + 1\right\},\end{aligned} \tag{34}$$

where $\widehat{C} > 0$ depends on $C(\hat{E}_0)$ and the inclusion $L^\lambda(\Omega) \hookrightarrow L^2(\Omega)$.

Taking into account (30) and (34) in (25), we have

$$\begin{aligned}\int_t^{t+1}\|v_\epsilon(\tau)\|_\lambda^\lambda d\tau \leq & \widetilde{C}\eta^{4(\lambda-1)} \\ & \times\left\{\left(\int_t^{t+1}\|v_\epsilon(\tau)\|_\lambda^\lambda d\tau\right)^{\frac{1}{\lambda}} + 1\right\},\end{aligned}$$

where $\widetilde{C} > 0$ is a constant. Consequently, for η that is sufficiently small, we obtain

$$\int_t^{t+1} \|v_\epsilon(\tau)\|_\lambda^\lambda \, d\tau \leq \eta^{3\lambda} \tag{35}$$

and

$$\int_t^{t+1} \|v_\epsilon(\tau)\|_2^2 \, d\tau \leq \frac{\eta^2}{5}. \tag{36}$$

We want to get a similar estimate for u_ϵ in the B norm. To this end, from (5), we get

$$\begin{aligned}\frac{d}{dt}(v_\epsilon(t), u_\epsilon(t))_2 - \|v_\epsilon(t)\|_2^2 + \|\Delta u_\epsilon(t)\|_2^2 + \alpha\|\nabla u_\epsilon(t)\|_2^2 \\ + (g_\epsilon(t), u_\epsilon(t))_2 \\ = (\widehat{m}_\epsilon(t), u_\epsilon(t))_2 + (f_\epsilon(t), u_\epsilon(t))_2.\end{aligned} \tag{37}$$

We apply inequality (26) to g_ϵ, and by Hölder inequality, we get

$$\begin{aligned}\int_t^{t+1}(g_\epsilon(\tau), u_\epsilon(\tau))_2 d\tau \leq & \sigma(\lambda)\delta\left(\int_t^{t+1}\|v_\epsilon(\tau)\|_\lambda^\lambda d\tau\right)^{\frac{1}{\lambda}} \\ & \times\left(\int_t^{t+1}\|u_\epsilon(\tau)\|_\lambda^\lambda d\tau\right)^{\frac{1}{\lambda}}\left[\left(\int_t^{t+1}\|v(\tau+\epsilon)\|_\lambda^\lambda d\tau\right)^{\frac{\lambda-2}{\lambda}}\right. \\ & \left. + \left(\int_t^{t+1}\|v(\tau)\|_\lambda^\lambda d\tau\right)^{\frac{\lambda-2}{\lambda}}\right].\end{aligned} \tag{38}$$

Notice that by (6) and (14),

$$\int_0^\infty \|v(t)\|_\lambda^\lambda \, dt \leq \frac{E_0 - E_\infty}{\delta}. \tag{39}$$

By assumption, $B \hookrightarrow L^\lambda(\Omega)$, then

$$\left(\int_t^{t+1} \|u_\epsilon(\tau)\|_\lambda^\lambda \, d\tau\right)^{\frac{1}{\lambda}} \leq 2C(\Omega)\sup_{t\geq 0}\|u(t)\|_B \leq C, \tag{40}$$

where $C > 0$ depends on the embedding constant $C(\Omega)$ and $\hat{E}_0$.

Therefore, from (35), (39), and (40) and for η that is sufficiently small, (38) becomes

$$\begin{aligned}\left|\int_t^{t+1}(g_\epsilon(\tau), u_\epsilon(\tau))_2 d\tau\right| \leq & 2\sigma(\lambda)\delta C \\ & \times\left(\frac{E_0 - E_\infty}{\delta}\right)^{\frac{\lambda-2}{\lambda}}\eta^3 \leq \frac{\eta^2}{5}.\end{aligned} \tag{41}$$

Now, by (27), (29), and (33) and for small η, we obtain

$$\int_t^{t+1} (f_\epsilon(\tau), u_\epsilon(\tau))_2 \, d\tau \leq C\eta^{8(\lambda-1)} \leq \frac{\eta^2}{5}. \tag{42}$$

Next, we have the estimate

$$\begin{aligned}(\widehat{m}_\epsilon(t), u_\epsilon(t))_2 = & -m(\|\nabla u(t+\epsilon)\|_2^2)\|\nabla u_\epsilon(t)\|_2^2 \\ & + \{m(\|\nabla u(t+\epsilon)\|_2^2) - m(\|\nabla u(t)\|_2^2)\} \\ & \times (\Delta u(t), u_\epsilon(t))_2 \\ \leq & \; C(\hat{E}_0)\|u_\epsilon(t)\|_2.\end{aligned}$$

Hence, by Hölder inequality, (33) and small η,

$$\int_t^{t+1} (\widehat{m}_\epsilon(\tau), u_\epsilon(\tau))_2 \, d\tau \leq \frac{\eta^2}{5}. \tag{43}$$

Therefore, (36), (41), (42), and (43) in (37), yield

$$\int_t^{t+1} \|u_\epsilon(\tau)\|_B^2 d\tau \leq 2\sup_{t\geq 0}\{\|v_\epsilon(t)\|_2\|u_\epsilon(t)\|_2\} + \frac{4\eta^2}{5}. \tag{44}$$

Notice that for every $t \geq 0$, we have

$$\|u_\epsilon(t)\|_2 \leq \epsilon \sup_{s\in[t,t+\epsilon]} \|v(s)\|_2 \leq \epsilon\sqrt{2\hat{E}_0}$$

and

$$\|v_\epsilon(t)\|_2 \leq 2\sup_{t\geq 0} \|v(t)\|_2 \leq 2\sqrt{2\hat{E}_0}.$$

Consequently, for $\epsilon \in (0, \hat{\epsilon}(\eta))$, with $\hat{\epsilon}(\eta)$ sufficiently small, (44) is

$$\int_t^{t+1} \|u_\epsilon(\tau)\|_B^2 \, d\tau \leq \eta^2. \tag{45}$$

Hence, in case (22), from (36) and (45) we conclude that

$$\int_t^{t+1} w_\epsilon(\tau)\, d\tau \leq \frac{3\eta^2}{5}. \tag{46}$$

Also, from (20) with η small,

$$\int_t^{t+1} W_\epsilon(\tau)\, d\tau \leq \frac{3\eta}{5}. \tag{47}$$

From (19) and by (24), we have for any $s \in [t, t+1]$, $t \geq 0$ that

$$\begin{aligned} W_\epsilon(t+1) \leq\ & W_\epsilon(s) + \int_s^{t+1} (m_\epsilon(\tau) + f_\epsilon(\tau), v_\epsilon(\tau))_2\, d\tau \\ & + \int_s^{t+1} (n_\epsilon(\tau), u_\epsilon(\tau))_2\, d\tau. \\ \leq\ & W_\epsilon(s) + \int_t^{t+1} \{|(f_\epsilon(\tau), v_\epsilon(\tau))_2| + |(m_\epsilon(\tau), \\ & v_\epsilon(\tau))_2| + |(n_\epsilon(\tau), u_\epsilon(\tau))_2|\}\, d\tau. \end{aligned}$$

Then, taking into account (30), (34), and (35) and for η small

$$W_\epsilon(t+1) \leq W_\epsilon(s) + \frac{2\eta}{5}.$$

Therefore, by (47), we obtain

$$W_\epsilon(t+1) \leq \int_t^{t+1} W_\epsilon(s)\, ds + \frac{2\eta}{5} \leq \eta.$$

Consequently, in both cases, (21) and (22),

$$W_\epsilon(t+1) \leq \max\{\eta, W_\epsilon(t)\}.$$

Notice that for every $t \geq 0$, there exists a natural number N such that $N \leq t \leq N+1$, then the last estimate implies that

$$W_\epsilon(t) \leq \max\{\eta, \max_{s\in[N-1,N]} W_\epsilon(s)\}.$$

Hence, applied recursively backwards,

$$W_\epsilon(t) \leq \max\{\eta, \max_{s\in[0,1]} W_\epsilon(s)\}. \tag{48}$$

Since the solution $(u, v) : [0,1] \to H$ is uniformly continuous, W_ϵ has the same property due to (20). Then, (48) implies that $t \mapsto W_\epsilon(t)$ is uniformly continuous for any $t \geq 0$, and the same holds for the solution in H, again in virtue of (20), that is, we have proven (16).

Precompactness

Next, we shall prove (15), that is, the orbit is a precompact subset of H. We start with $\{v(t)\}_{t\geq 0} \subset L^2(\Omega)$.

Notice that because of (16),

$$\begin{aligned} \left\| v(t) - \frac{1}{\epsilon}\int_t^{t+\epsilon} v(\tau)d\tau \right\|_2 &\leq \frac{1}{\epsilon}\int_t^{t+\epsilon} \|v(t) - v(\tau)\|_2\, d\tau \\ &\leq \sup_{\tau\in[t,t+\epsilon]} \|v(t) - v(\tau)\|_2 \\ &\leq \sqrt{2\eta}. \end{aligned} \tag{49}$$

Since $\{u(t)\}_{t\geq 0}$ is bounded in B, then

$$\begin{aligned} \left\| \frac{1}{\epsilon}\left(\int_t^{t+\epsilon} v(\tau)d\tau\right) \right\|_B &\leq \frac{1}{\epsilon}\|u(t+\epsilon) - u(t)\|_B \\ &\leq \frac{2}{\epsilon}\sup_{t\geq 0}\|u(t)\|_B \\ &\leq \frac{2}{\epsilon}\sqrt{2\hat{E}_0}. \end{aligned}$$

Since $B \hookrightarrow L^2(\Omega)$ is compact, $\{\frac{1}{\epsilon}\int_t^{t+\epsilon} v(\tau)d\tau\}_{t\geq 0}$ is precompact or, equivalently, totally bounded in $L^2(\Omega)$. Also, by (49), $\{v(t)\}_{t\geq 0}$ is precompact in $L^2(\Omega)$.

In a similar way, from (16), we estimate

$$\left\| u(t) - \frac{1}{\epsilon}\int_t^{t+\epsilon} u(\tau)d\tau \right\|_B \leq \sup_{\tau\in[t,t+\epsilon]} \|u(t) - u(\tau)\|_B \leq \sqrt{2\eta}. \tag{50}$$

We shall prove that $\{\mathcal{L}\int_t^{t+\epsilon} u(\tau)\, d\tau\}_{t\geq 0}$ is precompact in $B' \equiv$ the dual space of B, where

$$\mathcal{L} \equiv \Delta^2 : B \to B',$$

then precompactness of $\{u(t)\}_{t\geq 0}$ in B follows from (50) since

$$\mathcal{L}^{-1} \equiv (\Delta^2)^{-1} : B' \to B$$

is a linear and continuous operator.

According to the dense and continuous inclusions

$$B \hookrightarrow L^\lambda(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^{\lambda^*}(\Omega) \hookrightarrow B',$$

where $L^{\lambda^*}(\Omega) = (L^\lambda(\Omega))'$, $\lambda^* = \lambda/(\lambda - 1)$, we extend the inner product in $L^2(\Omega)$ to the duality product in $B' \times B$. Now, we integrate Equation 1, and since $\mathcal{L} : B \to B'$ is closed, we get, in the sense of B',

$$\mathcal{L}\int_t^{t+\epsilon} u(\tau)d\tau = v(t) - v(t+\epsilon) + \int_t^{t+\epsilon} M(\|\nabla u(\tau)\|_2^2)\Delta u(\tau)d\tau - \int_t^{t+\epsilon} g(v(\tau))d\tau + \int_t^{t+\epsilon} f(u(\tau))d\tau. \tag{51}$$

By Hölder inequality and since $v(t)$ is bounded in $L^2(\Omega)$,

$$\|v(t) - v(t+\epsilon)\|_{\lambda^*} \leq 2|\Omega|^{\frac{\lambda-2}{2\lambda}} \sup_{t\geq 0} \|(v(t))\|_2 \leq 2|\Omega|^{\frac{\lambda-2}{2\lambda}} C(\hat{E}_0). \tag{52}$$

By Hölder inequality and (39),

$$\left\|\int_t^{t+\epsilon} g(v(\tau))d\tau\right\|_{\lambda^*} \leq \delta \int_t^{t+\epsilon} \|(v(\tau))\|_\lambda^{\lambda-1}d\tau \leq \delta^{\frac{1}{\lambda}}(E_0 - E_\infty)^{\frac{\lambda-1}{\lambda}}. \tag{53}$$

By the boundedness of $u(t)$ in B, Hölder inequality and the injection $B \hookrightarrow L^{2(r-1)}(\Omega)$, we obtain the estimate

$$\begin{aligned}\left\|\int_t^{t+\epsilon} f(u(\tau))d\tau\right\|_{\lambda^*} &\leq |\Omega|^{\frac{\lambda-2}{2\lambda}} \left\|\int_t^{t+\epsilon} f(u(\tau))d\tau\right\|_2 \\ &\leq \mu|\Omega|^{\frac{\lambda-2}{2\lambda}} \int_t^{t+\epsilon} \|(u(\tau))\|_{2(r-1)}^{r-1}d\tau \\ &\leq \mu C(\Omega)\sup_{t\geq 0}\|u(t)\|_B, \\ &\leq \mu C(\Omega)C(\hat{E}_0).\end{aligned} \tag{54}$$

Also,

$$\begin{aligned}&\left\|\int_t^{t+\epsilon} M(\|\nabla u(\tau)\|_2^2)\Delta u(\tau)d\tau\right\|_{\lambda'} \\ &\leq |\Omega|^{\frac{\lambda-2}{2\lambda}} \left\|\int_t^{t+\epsilon} M(\|\nabla u(\tau)\|_2^2)\Delta u(\tau)d\tau\right\|_2 \\ &\leq \epsilon|\Omega|^{\frac{\lambda-2}{2\lambda}} \sup_{t\geq 0}\{M(\|\nabla u(t)\|_2^2)\|u(t)\|_B\} \\ &\leq \epsilon|\Omega|^{\frac{\lambda-2}{2\lambda}} C(\hat{E}_0),\end{aligned} \tag{55}$$

Therefore, (52) to (55) in (51) imply that

$$\left\|\mathcal{L}\int_t^{t+\epsilon} u(\tau)\ d\tau\right\|_{\lambda^*} \leq C, \tag{56}$$

for some constant $C > 0$ and every $t \geq 0$.

$B \hookrightarrow L^\lambda(\Omega)$ is compact by assumption. By Schauder's theorem (see Brézis [25]), $B \hookrightarrow L^\lambda(\Omega)$ is compact if and only if $L^{\lambda^*}(\Omega) \hookrightarrow B'$ is compact. Then, (56) implies that $\{\mathcal{L}\int_t^{t+\epsilon}\ u(\tau)\ d\tau\}_{t\geq 0}$ is precompact in B'. The proof is complete. □

Remark 1. We observe that the main difficulty in the proof of the last theorem is to show precompactness of bounded orbits. This has been accomplished for semi-linear wave equations by Haraux [15]. Here, we extend this technique for a Timoshenko equation with nonlinear damping and a source term. This source term has an amplifying effect instead of a restoring one in case of $r \geq 2(\gamma+1)$ (see [23]). Indeed, in another study [23], we proved that, depending on the initial conditions, every solution of (1) either blows up in a finite time or there exists for all time. In this last case, again depending on the initial conditions, the solution is either unbounded or bounded and tends to the set of equilibria $\mathcal{E}$, as time goes to infinity. On the other hand, Theorem 2 shows that every solution of Equation 1 converges to $\mathcal{E}$, whenever $2 < r < 2(\gamma+1)$, that is, when the source term is dominated, then every solution is bounded and tends to the equilibria set as time goes to infinity. Consequently, we give a complete panorama of the qualitative behavior of the solutions of the nonlinear Timoshenko equation, Equation 1. A dynamic analysis of more realistic rod models (see for instance [1]) requires more effort and research.

Competing interests
The author declare that he have no competing interests

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References

1. Antman, SS: Nonlinear Problems of Elasticity, 2nd ed. Springer, New York (2005)
2. Alves, CO, Cavalcanti, MM: On existence, uniform decay rates and blow up for solutions of the 2-D wave equation with exponential source. Calc. Var. Partial Differential Equations. **34**, 377–411 (2009)
3. Barbu, V, Lasiecka, I, Rammaha, MA: Blow-up of generalized solutions to wave equations with nonlinear degenerate damping and source terms. Indiana Univ. Math. J. **56**, 995–1021 (2007)
4. Cavalcanti, MM, Domingos Cavalcanti, VN, Prates Filho, JS, Soriano, JA: Existence and exponential decay for a Kirchhoff-Carrier model with viscosity. J. Math. Anal. Appl. **226**, 40–60 (1998)
5. Cavalcanti, MM: Existence and uniform decay for the Euler-Bernoulli viscoelastic equation with nonlocal boundary dissipation. Discrete Contin. Dyn. Syst. **8**, 675–695 (2002)
6. Cavalcanti, MM, Domingos Cavalcanti, VN: Existence and asymptotic stability for evolution problems on manifolds with damping and source terms. J. Math. Anal. Appl. **291**, 109–127 (2004)
7. Cavalcanti, MM, Domingos Cavalcanti, VN, Martinez, P: Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term. J. Differential Equations. **203**, 119–158 (2004)
8. Cavalcanti, MM, Domingos Cavalcanti, VN, Soriano, JA: Global existence and asymptotic stability for the nonlinear and generalized damped extensible plate equation. Commun. Contemp. Math. **6**, 705–731 (2004)
9. Cavalcanti, MM, Domingos Cavalcanti, VN, Lasiecka, I: Well-posedness and optimal decay rates for the wave equation with nonlinear boundary damping-source interaction. J. Differential Equations. **236**, 407–459 (2007)
10. Rammaha, MA: The influence of damping and source terms on solutions of nonlinear wave equations. Bol. Soc. Parana. Mat. **25**(3), 77–90 (2007)

11. Rammaha, MA, Sakuntasathien, S: Global existence and blow up of solutions to systems of nonlinear wave equations with degenerate damping and source terms. Nonlinear Anal. **72**, 2658–2683 (2010)
12. Todorova, G, Vitillaro, E: Blow-up for nonlinear dissipative wave equations in $\mathbb{R}^n$. J. Math. Anal. Appl. **303**, 242–257 (2005)
13. Vitillaro, E: Global existence for the wave equation with nonlinear boundary damping and source terms. J. Differential Equations. **186**, 259–298 (2002)
14. Hale, J: Asymptotic behavior of dissipative systems. Mathematical Surveys and Monographs 25, AMS, Providence, Rhode Island (1988)
15. Haraux, A: Semilinear hyperboloic problems in bounded domains. Mathematical Reports 3(1), Harwood Academic Publishers, Gordon and Breach, London (1987)
16. Payne, LE, Sattinger, DH: Saddle points and instability of nonlinear hyperbolic equations. Isr. J. Math. **22**, 273–303 (1975)
17. Georgiev, V, Todorova, G: Existence of a solution of the wave equation with nonlinear damping and source terms. J. Differential Equations. **109**, 295–308 (1994)
18. Ikehata, R: Some remarks on the wave equations with nonlinear damping and source terms. Nonlinear Anal. **27**, 1165–1175 (1996)
19. Esquivel-Avila, JA: The dynamics of a nonlinear wave equation. J. Math. Anal. Appl. **279**, 135–150 (2003)
20. Esquivel-Avila, JA: Qualitative analysis of a nonlinear wave equation. Discrete Contin. Dyn. Syst. **3**, 787–804 (2004)
21. Bainov, D, Minchev, E: Upper estimate of the interval of existence of solutions of a nonlinear Timoshenko equation. Georgian Math. J. **4**, 219–222 (1997)
22. Esquivel-Avila, JA: The dynamics around the ground state of the Timoshenko equation with a source term. In: *Proceedings of Dynamic Systems and Applications, vol. 4*, pp. 198–205. Dynamic Publishers Inc., Atlanta, (2004)
23. Esquivel-Avila, JA: Dynamics analysis of a nonlinear Timoshenko equation. Abstr. Appl. Anal. **2011**, 1–36 (2011)
24. Ball, J: On the asymptotic behaviour of generalized processes, with applications to nonlinear evolution equations. J. Differential Equations. **27**, 224–265 (1978)
25. Brézis, H: Analyse Fonctionnelle, Théorie et Applications. Masson, Paris (1983)

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