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# A note on abelian strongly $k$ -Engel $\pi$ -regular rings

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## Abstract

Let  $R$  be an associative ring with identity and let  $k \geq 1$  be a fixed integer. An element  $(x, y) \in R \times R$  is said to be left (right)  $k$ -Engel  $\pi$ -regular if there exists a positive integer  $n$  and an element  $z \in R$  such that  $[x, y]_k^n = z[x, y]_k^{n+1}$  ( $[x, y]_k^n = [x, y]_k^{n+1}z$ ). If every element of  $R \times R$  is left (right)  $k$ -Engel  $\pi$ -regular, then  $R$  is said to be left (right)  $k$ -Engel  $\pi$ -regular. An element  $(x, y) \in R \times R$  is strongly  $k$ -Engel  $\pi$ -regular if it is both left and right  $k$ -Engel  $\pi$ -regular. The ring  $R$  is strongly  $k$ -Engel  $\pi$ -regular if every element of  $R \times R$  is strongly  $k$ -Engel  $\pi$ -regular. In this paper, we investigate properties of abelian strongly  $k$ -Engel  $\pi$ -regular ring.

**Keywords:** Strongly  $\pi$ -regular,  $k$ -Engel, Abelian ring

**MSC:** 16E50, 16D70, 16U99

## Introduction

Let  $R$  be an associative ring with identity. An element  $x \in R$  is said to be right  $\pi$ -regular if there exists a positive integer  $n$  and an element  $y \in R$  such that  $x^n = x^{n+1}y$ . If every element of  $R$  is right  $\pi$ -regular, then  $R$  is said to be right  $\pi$ -regular. By [1], this definition is left-right symmetric. An element of  $R$  is strongly  $\pi$ -regular if it is both left and right  $\pi$ -regular.  $R$  is strongly  $\pi$ -regular if every element of  $R$  is strongly  $\pi$ -regular. In [2], it was shown that if an element  $x$  in the ring  $R$  is strongly  $\pi$ -regular, then there exists a positive integer  $n$  and an element  $y \in R$  such that  $x^n = x^{n+1}y$  and  $xy = yx$ . In the case where  $n = 1$ , the element  $x$  is said to be strongly regular.

If  $(x_i)_{i \in \mathbb{N}}$  is a sequence of elements of  $R$  and  $k$  is a positive integer, we define  $[x_1, \dots, x_{k+1}]$  inductively as follows:

$$[x_1, x_2] = x_1x_2 - x_2x_1,$$
$$[x_1, \dots, x_k, x_{k+1}] = [ [x_1, \dots, x_k], x_{k+1} ].$$

If  $x_1 = x$  and  $x_2 = \dots = x_{k+1} = y$ , the notation  $[x, y]_k$  is used to denote  $[x_1, \dots, x_{k+1}]$ , and  $[x, y]_k$  is called a  $k$ -

Engel element. For  $k = 1$ ,  $[x, y]_k = [x, y]_1$  is usually just denoted by  $[x, y]$ . An element  $(x, y) \in R \times R$  is said to be left (right)  $k$ -Engel  $\pi$ -regular if there exists a positive integer  $n$  and an element  $z \in R$  such that  $[x, y]_k^n = z[x, y]_k^{n+1}$  ( $[x, y]_k^n = [x, y]_k^{n+1}z$ ). If every element of  $R \times R$  is left (right)  $k$ -Engel  $\pi$ -regular, then  $R$  is said to be left (right)  $k$ -Engel  $\pi$ -regular. An element  $(x, y) \in R \times R$  is strongly  $k$ -Engel  $\pi$ -regular if it is both left and right  $k$ -Engel  $\pi$ -regular. The ring  $R$  is strongly  $k$ -Engel  $\pi$ -regular if every element of  $R \times R$  is strongly  $k$ -Engel  $\pi$ -regular. Clearly, if  $(x, y)$  is strongly  $k$ -Engel  $\pi$ -regular, then  $[x, y]_k$  is strongly  $\pi$ -regular. Therefore, there exists a positive integer  $n$  and an element  $z \in R$  such that  $[x, y]_k^n = [x, y]_k^{n+1}z$  and  $[x, y]_kz = z[x, y]_k$  (by [2]).

Division rings are examples of strongly  $k$ -Engel  $\pi$ -regular rings. Other examples include full matrix rings over division rings and triangular matrix rings over fields. It is clear that rings which satisfy the  $k$ -Engel condition are strongly  $k$ -Engel  $\pi$ -regular. In [3], we studied the conditions for strongly  $k$ -Engel  $\pi$ -regular rings to be commutative (hence, has  $k$ -Engel condition). Now in this paper, we investigate the properties of abelian strongly  $k$ -Engel  $\pi$ -regular rings and obtain some characterisations of these rings. All rings in this paper are assumed to have

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identity. A ring is said to be abelian if all of its idempotents are central. For a ring  $R$ , we use the notation  $N(R)$  and  $Id(R)$  to denote the set of all nilpotent elements of  $R$  and the set of all idempotents of  $R$ , respectively.

### Main results

**Proposition 2.1.** *Let  $R$  be an abelian strongly  $k$ -Engel  $\pi$ -regular ring. Suppose that  $N(R)$  is an ideal of  $R$ . Then for each  $x, y \in R$ ,  $[x, y]_k + N(R)$  is strongly regular (hence regular).*

*Proof.* Let  $x, y \in R$ . Then there exist  $z \in R$  and a positive integer  $n$  such that  $[x, y]_k^n = [x, y]_k^{n+1} z$  and  $[x, y]_k z = z[x, y]_k$ . Thus,  $e = [x, y]_k^n z^n \in Id(R)$ , and hence,  $1 - e \in Id(R)$ . Then since  $[x, y]_k^n = [x, y]_k^{2n} z^n = [x, y]_k^n e$ , it follows that  $(1 - e)[x, y]_k^n = 0$ , and hence,  $(1 - e)[x, y]_k \in N(R)$ . Therefore,

$$\begin{aligned} [x, y]_k + N(R) &= e[x, y]_k + N(R) \\ &= [x, y]_k^{n+1} z^n + N(R) \\ &= ([x, y]_k + N(R))^2 ([x, y]_k^{n-1} z^n + N(R)). \end{aligned}$$

It follows that  $[x, y]_k + N(R)$  is strongly regular (hence regular).

The following lemma is well known and can be found for example in p. 72 of [4].  $\square$

**Lemma 2.2.** *Let  $R$  be a ring and  $I$  a nil ideal of  $R$ . Then idempotents of  $R/I$  can be lifted to  $R$ .*

**Proposition 2.3.** *Let  $R$  be an abelian ring. If  $N(R)$  is an ideal of  $R$  and for each  $x, y \in R$ ,  $[x, y]_k + N(R)$  is regular, then  $R$  is strongly  $k$ -Engel  $\pi$ -regular.*

*Proof.* Let  $x, y \in R$ . Since  $[x, y]_k + N(R)$  is regular, there exist some  $z \in R$  such that  $[x, y]_k z[x, y]_k + N(R) = [x, y]_k + N(R)$ . Clearly,  $(z[x, y]_k)^2 + N(R) = z[x, y]_k + N(R)$ . By Lemma 2.2, there is an idempotent  $e \in R$  such that  $e + N(R) = z[x, y]_k + N(R)$ , that is,  $e - z[x, y]_k \in N(R)$ . Thus, there exists an integer  $m \geq 1$  such that  $(e - z[x, y]_k)^m = 0$ . Since  $e$  is central,  $e = t[x, y]_k$  for some  $t \in R$ .

Now  $[x, y]_k + N(R) = ([x, y]_k z[x, y]_k) + N(R) = [x, y]_k e + N(R)$  gives us  $[x, y]_k - [x, y]_k e \in N(R)$ . Hence, there exist some integer  $n \geq 1$  with  $0 = ([x, y]_k - [x, y]_k e)^n = [x, y]_k^n - [x, y]_k^n e$ . Therefore,  $[x, y]_k^n = [x, y]_k^n e = e[x, y]_k^n = t[x, y]_k^{n+1}$ . Thus,  $R$  is strongly  $k$ -Engel  $\pi$ -regular.

By Propositions 2.1 and 2.3, we readily have the following:  $\square$

**Theorem 2.4.** *Let  $R$  be an abelian ring such that  $N(R)$  is an ideal of  $R$ . Then  $R$  is strongly  $k$ -Engel  $\pi$ -regular if and only if for each  $x, y \in R$ ,  $[x, y]_k + N(R)$  is regular.*

**Proposition 2.5.** *Let  $R$  be an abelian strongly  $k$ -Engel  $\pi$ -regular ring and let  $P$  be a prime ideal of  $R$ . Then for each  $x, y \in R$ ,  $[x, y]_k + P$  is nilpotent or a unit.*

*Proof.* Let  $x, y \in R$ . Since  $R$  is strongly  $k$ -Engel  $\pi$ -regular, by the proof of Theorem 2.1 in [3], we may write  $[x, y]_k = fu = uf$  for some near idempotent  $f$  and some unit  $u \in R$ . By near idempotent we mean that there exists a positive integer  $n$  such that  $e = f^n$  is an idempotent. Then  $[x, y]_k^n = eu^n = u^n e$ . Since  $(1 - e)Re = \{0\} \subseteq P$  and  $P$  is a prime ideal, it follows that  $e \in P$  or  $1 - e \in P$ . If  $e \in P$ , then  $[x, y]_k^n = eu^n \in P$ ; hence,  $[x, y]_k + P$  is nilpotent. If  $1 - e \in P$ , then  $[x, y]_k^n + P = eu^n + P = (e + P)(u^n + P) = u^n + P$  is a unit in  $R/P$ . It follows that  $[x, y]_k + P$  is a unit in  $R/P$ .  $\square$

**Proposition 2.6.** *Let  $R$  be a strongly  $k$ -Engel  $\pi$ -regular ring and  $I$  an ideal of  $R$ . Then  $I$  is strongly  $k$ -Engel  $\pi$ -regular as a ring.*

*Proof.* Let  $x, y \in I$ . Since  $R$  is strongly  $k$ -Engel  $\pi$ -regular, then there exist  $z \in R$  and a positive integer  $n$  such that  $[x, y]_k^n = [x, y]_k^{n+1} z$  and  $[x, y]_k z = z[x, y]_k$ . If  $n = 1$ , let  $t = [x, y]_k z^2$ . Then  $t \in I$ ,  $[x, y]_k t = t[x, y]_k$  and  $[x, y]_k^2 t = [x, y]_k^2 [x, y]_k z^2 = [x, y]_k ([x, y]_k^2 z) = [x, y]_k^2 z = [x, y]_k t$ . If  $n \geq 2$ , let  $t = [x, y]_k^{n-1} z^n \in I$ . Then  $[x, y]_k t = t[x, y]_k$ . Therefore,  $[x, y]_k^n = [x, y]_k^{n+1} z = \dots = [x, y]_k^{n+1} ([x, y]_k^{n-1} z^n) = [x, y]_k^{n+1} t$ . Thus,  $I$  is strongly  $k$ -Engel  $\pi$ -regular.  $\square$

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

SS conceived of the idea for the study. AYMC participated in the investigation. Both authors read and approved the final manuscript.

#### Authors' information

AYMC and SS are lecturers in their respective institutions.

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