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The polynomial automorphisms of some certain groups

Bahman Askari* and Fariba Fattahi

Abstract

Let $A(G)$ denote the automorphism group of a group G. A polynomial automorphism of G is an automorphism of the form $x \mapsto (v_1^{-1}x^{\varepsilon_1}v_1)\dots(v_m^{-1}x^{\varepsilon_m}v_m)$. We shall write $P(G) = \langle P_0(G)\rangle$ such that $P_0(G)$ is the set of polynomial automorphisms of G. In this paper, we will prove that $P_0(D_8) \cong V_4$ and $P(\mathbb{Q}) = A(\mathbb{Q})$, where $\mathbb{Q}$ is the additive group.

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Introduction

Let G be a group. We shall write $A(G)$ for the automorphism group of G. According to Schweigert [1], we say that an element $f \in A(G)$ is a polynomial automorphism of G if there exist integers $\varepsilon_1, \dots, \varepsilon_m \in \mathbb{Z}$ and elements $u_0, \dots, u_m \in G$ such that

$$f(x) = u_0x^{\varepsilon_1}u_1 \dots u_{m-1}x^{\varepsilon_m}u_m,$$

for all $x \in G$. Since $f(1) = 1$, it is easy to see that $f(x)$ can be expressed as a product of inner automorphisms, that is,

$$f(x) = (v_1^{-1}x^{\varepsilon_1}v_1)\dots(v_m^{-1}x^{\varepsilon_m}v_m).$$

We shall write $P_0(G)$ for the set of polynomial automorphisms of G. Actually, Schweigert defined a polynomial automorphism in the context of finite groups. In particular, in this context, the set $P_0(G)$ is clearly a subgroup of $A(G)$. On the other hand, this is not necessarily the case when G is infinite.

In this paper, we shall consider the subgroup $P(G) = \langle P_0(G)\rangle$ of $A(G)$, generated by all polynomial automorphisms of G. Hence, $P_0(G) = P(G)$, when G is finite. For instance, we will prove that $P_0(D_8) \cong V_4$. Instead, $P(G)$ is distinct from $P_0(G)$ when G is the additive group of a rational number; in this case, we will prove that $P(\mathbb{Q}) = A(\mathbb{Q})$, and the set of polynomial automorphisms forms a monoid with respect to the operation of functional composition, which is isomorphic to the multiplicative monoid $\mathbb{Z} \setminus \{0\}$.

It is easy to verify that $P_0(G)$ is a normal subset of $A(G)$. Thus, $P(G)$ is a normal subgroup of $A(G)$. In addition, we have

$$I(G) \trianglelefteq P(G) \trianglelefteq A(G),$$

where $I(G)$ is the group of inner automorphisms of G.

Preliminaries

If G is abelian, each polynomial automorphism is of the form $x \mapsto x^{\varepsilon}$, and so $P(G)$ is abelian. We show here that if G is a nilpotent group of class $k = 2$, then $P(G)$ is abelian.

Lemma 1. *Let f, g be two functions over a group G, respectively defined by the relations*

$$f(x) = (v_1^{-1}x^{\varepsilon_1}v_1)\dots(v_m^{-1}x^{\varepsilon_m}v_m),$$

$$g(x) = (w_1^{-1}x^{\eta_1}w_1)\dots(w_n^{-1}x^{\eta_n}w_n)$$

(we do not suppose that f and g are automorphisms). Let t be an element of G such that any two conjugates of t commute. Then, we have the relation

$$f(g(t)) = \prod_{i=1}^{m}\prod_{j=1}^{n} t^{\varepsilon_i\eta_j}[\,t^{\varepsilon_i\eta_j}, v_i]\,[\,t^{\varepsilon_i\eta_j}, w_j]\,[\,t^{\varepsilon_i\eta_j}, w_j, v_i]$$

(notice that in this product, the order of the factors is of no consequence).

*Correspondence: bahman_askari2003@yahoo.com
Department of Mathematics, Ghorveh Branch, Islamic Azad University, Ghorveh 66619-83435, Iran

Proof. Using the fact that any two conjugates of t commute, we can write

$$f(g(t)) = \prod_{i=1}^{m} v_i^{-1}(\prod_{j=1}^{m} w_j^{-1}t^{\eta_j}w_j)^{\varepsilon_i}v_i$$

$$= \prod_{i=1}^{m}\prod_{j=1}^{n} v_i^{-1}w_j^{-1}t^{\varepsilon_i\eta_j}w_jv_i$$

$$= \prod_{i=1}^{m}\prod_{j=1}^{n} t^{\varepsilon_i\eta_j}[\, t^{\varepsilon_i\eta_j}, w_j, v_i]\,.$$

We conclude thanks to the relation $[\,x, yz] = [\,x, z]\,[\,x, y]\,[\,x, y, z]$. □

When G is a finite nilpotent group of class ≤ 2, it is proved in [2], that $P(G)$ is abelian. In a nilpotent group G of class ≤ 2, two conjugates of any element $t \in G$ commute. Therefore, as an immediate consequence of Lemma 1, we observe that any two polynomial automorphisms of G commute. Since these automorphisms generate $P(G)$, we obtain:

Proposition 1. *If G is a nilpotent group of class* ≤ 2*, then* $P(G)$ *is abelian.*

Proof. It is enough to show that all generators of $P(G)$ commute. Let G be a nilpotent group of class 1. Then, by [3], we have

$$P_0(G) = \{f \mid f(x) = x^{\varepsilon}, \varepsilon \in \mathbb{Z} \setminus \{0\}\}.$$

Now, we consider $f(x) = x^{\varepsilon}$,$g(x) = x^{\delta}$. We have

$$f(g(x)) = f(x^{\delta}) = x^{\delta\varepsilon},$$

$$g(f(x)) = g(x^{\varepsilon}) = x^{\varepsilon\delta},$$

where $\delta, \varepsilon \in \mathbb{Z} \setminus \{0\}$. Hence, $f(g(x)) = g(f(x))$.

Let G be a nilpotent group of class 2 and let f, g be two elements of $P_0(G)$ such that

$$f(x) = (v_1^{-1}x^{\varepsilon_1}v_1)\ldots(v_m^{-1}x^{\varepsilon_m}v_m),$$

$$g(x) = (w_1^{-1}x^{\eta_1}w_1)\ldots(w_n^{-1}x^{\eta_n}w_n).$$

Then by Lemma 1, for all $t \in G$, we have

$$f(g(t)) = \prod_{i=1}^{m}\prod_{j=1}^{n} t^{\varepsilon_i\eta_j}[\, t^{\varepsilon_i\eta_j}, v_i]\,[\, t^{\varepsilon_i\eta_j}, w_j]\,[\, t^{\varepsilon_i\eta_j}, w_j, v_i]\,,$$

$$g(f(t)) = \prod_{i=1}^{m}\prod_{j=1}^{n} t^{\varepsilon_i\eta_j}[\, t^{\varepsilon_i\eta_j}, w_j]\,[\, t^{\varepsilon_i\eta_j}, v_i]\,[\, t^{\varepsilon_i\eta_j}, v_i, w_j]\,.$$

Since G is a nilpotent group of class 2, $\gamma_3(G) = 1$. So,

$$[\, t^{\varepsilon_i\eta_j}, w_j, v_i] = [\, t^{\varepsilon_i\eta_j}, v_i, w_j] = 1.$$

Therefore, $f(g(t)) = g(f(t))$ and the proof is complete. □

Main results

In this section, we suppose that D_8 is the dihedral group of order 8, V_4 is the Klein 4−group, and $\mathbb{Q}$ is the additive group of a rational number [4,5]. First, in Theorem 1, we will show that $P_0(D_8) \cong V_4$, and then in Theorem 2, we will prove that $P(\mathbb{Q}) = A(\mathbb{Q})$.

Theorem 1. *Let* D_8 *be the dihedral group of order 8, and let* V_4 *be the Klein* 4−*group. Then,* $P_0(D_8) \cong V_4$.

Proof. Since $D_8 = \langle t, s | t^2 = s^4 = 1, (ts)^2 = 1\rangle$, so the eight elements of D_8 are

$$D_8 = \{1, s, s^2, s^3, t, ts, ts^2, ts^3\}.$$

It is straightforward to verify that $Aut(D_8) \cong D_8$. On the other hand, we have $D_8/Z(D_8) \cong Inn(D_8)$. Since $Z(D_8) = \langle s^2\rangle$, we have $|Inn(D_8)| = 4$. The order of each non-trivial element of $D_8/Z(D_8)$ is 2, so $D_8/Z(D_8) \cong V_4$; hence, $Inn(D_8) \cong V_4$. Therefore,

$$V_4 \trianglelefteq P_0(D_8) \trianglelefteq D_8.$$

Since $|D_8 : V_4| = 2$, so $P_0(D_8) \cong D_8$ or V_4. However, D_8 is the nilpotent group of order 2; according to the Proposition 1, $P_0(D_8)$ is abelian group. The result now follows. □

Theorem 2. *Let* $\mathbb{Q}$ *be the additive group of rational numbers. Then, the set of polynomial automorphisms forms a monoid with respect to the operation of functional composition, which is isomorphic to the multiplicative monoid* $\mathbb{Z} \setminus \{0\}$*. Further, we have* $P(\mathbb{Q}) = A(\mathbb{Q}) \cong (\mathbb{Q} \setminus \{0\}, \cdot)$.

Proof. Since $\mathbb{Q}$ is the additive group, so each element of $P_0(\mathbb{Q})$ is of the form $f(x) = kx$ for every $x \in \mathbb{Q}$, where $k \in \mathbb{Z} \setminus \{0\}$. Now, since $\mathbb{Q}$ is a torsion-free group, so $f(x) = kx$ is the element of $P_0(\mathbb{Q})$ for every $k \in \mathbb{Z} \setminus \{0\}$. Hence,

$$P_0(\mathbb{Q}) = \{f : \mathbb{Q} \to \mathbb{Q} | f(x) = kx, k \in \mathbb{Z} \setminus \{0\}\}.$$

It can be easily verified that $P_0(\mathbb{Q})$ has only two commutative elements. We consider the mapping $\varphi : P_0(\mathbb{Q}) \to \mathbb{Z} \setminus \{0\}$ defined like this: for any $f \in P_0(\mathbb{Q})$, $\varphi(f(x)) = k$, where $k \in \mathbb{Z} \setminus \{0\}$ and $x \in \mathbb{Q}$. It is easy to see that $P_0(\mathbb{Q}) \cong (\mathbb{Z} \setminus \{0\}, \cdot)$.

It is straightforward to verify that every element of $End(\mathbb{Q})$, for every $x \in \mathbb{Q}$, is the form $f_t(x) = tx$, where $t = f_t(1)$ is the arbitrary element of $\mathbb{Q}$. The mapping $\psi : End(\mathbb{Q}) \to \mathbb{Q}$ of the form

$$g_t \mapsto g_t(1) \quad (g_t \in End(\mathbb{Q}))$$

is an isomorphism.

Since $A(\mathbb{Q}) \leq End(\mathbb{Q})$ and $A(\mathbb{Q})$ is the group of invertible elements of $End(\mathbb{Q})$, so we have $A(\mathbb{Q}) \cong (\mathbb{Q} \setminus \{0\}, \cdot)$.

Further, we have

$$A(\mathbb{Q}) = \{f : \mathbb{Q} \to \mathbb{Q} | f(x) = tx,\ t \in \mathbb{Q} \setminus \{0\}\}.$$

It is clear that $P(\mathbb{Q}) \leq A(\mathbb{Q})$. Let $f \in A(\mathbb{Q})$. Then, there exist $m, n \in \mathbb{Z} \setminus \{0\}$ such that

$$f(x) = \frac{m}{n}x$$

for every $x \in \mathbb{Q}$.

Let f_1, f_2 be two elements of $P_0(\mathbb{Q})$, respectively defined by the relations

$$f_1(x) = mx, \quad f_2(x) = nx.$$

Then, $f_1 o f_2^{-1}(x) = \frac{m}{n}x = f(x)$. Now, since $f_1 o f_2^{-1}$ is the element of $P(\mathbb{Q})$, so $f(x)$ is the element of $P(\mathbb{Q})$. This complete the proof of Theorem 2. □

Competing interests
Both authors declare that they have no competing interests.

Authors' contributions
BA and FF contributed equally. Both authors read and approved the final manuscript.

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