Kumam et al. Mathematical Sciences 2013, 7:28
http://www.iaumath.com/content/7/1/28

Mathematical Sciences
a SpringerOpen Journal

ORIGINAL RESEARCH Open Access

Viscosity approximation to a common solution of variational inequality problems and fixed point problems for Lipschitzian semigroup in Banach spaces

Poom Kumam[1], Somyot Plubtieng[2] and Phayap Katchang[3*]

Abstract

We introduce a new iterative algorithm based on a viscosity approximation method for finding the common solution of variational inequality problems for an inverse strongly accretive operator and the solution of fixed point problems for Lipschitzian semigroup mappings in Banach spaces. In controlling suitable conditions, strong convergence theorems are proven. Our results extend and improve the recent results of some authors in the literature in this field.

Keywords: Strong convergence; Fixed point; Variational inequality; Lipschitzian semigroup; Banach spaces; Accretive operator; Viscosity approximation

MSC 2000: 47H10; Secondary 47H09, 43A07, 47H20, 47J20

Introduction

Let C be a nonempty closed convex subset of a real Banach space E and E^* be a dual space of E with norm $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ pairing between E and E^*. A self mapping $f : C \to C$ is a contraction on C if there exists a constant $\alpha \in (0, 1)$ and $x, y \in C$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|.$$

Recall that a mapping $A : C \to C$ is said to be as follows:

1. *Lipschitzian* with Lipschitz constant $L > 0$ if
$$\|Ax - Ay\| \leq L\|x - y\|, \quad \forall x, y \in C.$$
2. *Nonexpansive* if
$$\|Ax - Ay\| \leq \|x - y\|, \quad \forall x, y \in C.$$
3. *Asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of positive numbers, satisfying the property $\lim_{n\to\infty} k_n = 1$ and
$$\|A^n x - A^n y\| \leq k_n \|x - y\|, \quad \forall x, y \in C.$$

Clearly, every nonexpansive mapping A is asymptotically nonexpansive with sequence $\{1\}$. Also, every asymptotically nonexpansive mapping is uniformly L-Lipschitzian with $L = \sup_{n\in\mathbb{N}} k_n$.

An operator $A : C \to E$ is said to be as follows:

1. *Accretive* if there exists $j(x - y) \in J(x - y)$ such that
$$\langle Ax - Ay, j(x - y)\rangle \geq 0, \quad \forall x, y \in C.$$
2. β*-Strongly accretive* if there exists a constant $\beta > 0$ such that
$$\langle Ax - Ay, j(x - y)\rangle \geq \beta\|x - y\|^2, \quad \forall x, y \in C.$$
3. β*-Inverse strongly accretive* if, for any $\beta > 0$,
$$\langle Ax - Ay, j(x - y)\rangle \geq \beta\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Evidently, the definition of the inverse strongly accretive operator is based on the inverse strongly monotone operator.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be *strictly convex* if, for any $x, y \in U$, $x \neq y$ implies $\|\frac{x+y}{2}\| \leq 1$. A Banach space E is said to be *uniformly convex* if, for any $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that, for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive

*Correspondence: p.katchang@hotmail.com
[3]Department of Mathematics and Statistics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna Tak, Tak 63000, Thailand
Full list of author information is available at the end of the article

and strictly convex. A Banach space E is said to be *smooth* if $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x, y \in U$. It is also said to be *uniformly smooth* if the limit is attained uniformly for $x, y \in U$. The *modulus of smoothness* of E is defined by the following:

$$\rho(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : x, y \in E, \|x\| = 1, \|y\| = \tau\},$$

where $\rho : [0,\infty) \to [0,\infty)$ is a function. It is known that E is uniformly smooth if and only if $\lim_{\tau\to 0} \frac{\rho(\tau)}{\tau} = 0$. Let q be a fixed real number with $1 < q \leq 2$. A Banach space E is said to be *q-uniformly smooth* if there exists a constant $c > 0$ such that $\rho(\tau) \leq c\tau^q$ for all $\tau > 0$. We note that E is a uniformly smooth Banach space if and only if J_q is single-valued and uniformly continuous on any bounded subset of E. Typical examples of both uniformly convex and uniformly smooth Banach spaces are L^p, where $p > 1$. More precisely, L^p is $\min\{p, 2\}$-uniformly smooth for every $p > 1$. Note also that no Banach space is q-uniformly smooth for $q > 2$; see the work of Xu [1] for more details.

For $q > 1$, the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by the following:

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}$$

for all $x \in E$. In particular, if $q = 2$, the mapping J_2 is called the *normalized duality mapping* and usually write $J_2 = J$. Further, we have the following properties of the generalized duality mapping J_q:

1. $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \in E$ with $x \neq 0$.
2. $J_q(tx) = t^{q-1}J_q(x)$ for all $x \in E$ and $t \in [0,\infty)$.
3. $J_q(-x) = -J_q(x)$ for all $x \in E$.

Let D be a subset of C and $Q : C \to D$, then Q is said to be *sunny* if

$$Q(Qx + t(x - Qx)) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A subset D of C is said to be a *sunny nonexpansive retraction* of C if there exists a sunny nonexpansive retraction Q of C onto D. A mapping $Q : C \to C$ is called a *retraction* if $Q^2 = Q$. If a mapping $Q : C \to C$ is a retraction, then $Qz = z$ for all z is in the range of Q.

The following result describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Proposition 1. *Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Q : E \to C$ be a retraction and let J be the normalized duality mapping on E. Then, the following are equivalent [2]:*

1. *Q is sunny and nonexpansive.*
2. *$\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \forall x, y \in E$.*
3. *$\langle x - Qx, J(y - Qx)\rangle \leq 0, \forall x \in E, y \in C$.*

Proposition 2. *Let C be a nonempty closed convex subset of a uniformly convex and uniformly smooth Banach space E and let T be a nonexpansive mapping of C into itself with $F(T) \neq \emptyset$. Then, the set $F(T)$ is a sunny nonexpansive retract of C [3].*

$\{T(s) : s \in \mathbb{R}^+\}$ is called a *strongly continuous semigroup of Lipschitzian mappings* from C into itself if it satisfies the following conditions:

(i) For each $s > 0$, there exists a function $k(\cdot) : (0,\infty) \to (0,\infty)$ such that
$$\|T(s)x - T(s)y\| \leq k(s)\|x - y\|, \quad \forall x, y \in C.$$
(ii) $T(0)x = x$ for each $x \in C$.
(iii) $T(s_1 + s_2)x = T(s_1)T(s_2)x$ for any $s_1, s_2 \in \mathbb{R}^+$ and $x \in C$.
(iv) For each $x \in C$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into C is continuous.

If $k(s) = L$ for all $s > 0$ in (i), then $\{T(s) : s \in \mathbb{R}^+\}$ is called a *strongly continuous semigroup of uniformly L-Lipschitzian mappings*. If $k(s) = 1$ for all $s > 0$ in (i), then $\{T(s) : s \in \mathbb{R}^+\}$ is called a *strongly continuous semigroup of nonexpansive mappings* (see the work of Sahu and O'Regan [4]). For a semigroup S, we can define a partial preordering $\prec$ on S by $a \prec b$ if and only if $aS \supset bS$. If S is a *left reversible semigroup* (i.e., $aS \cap bS \neq \emptyset$ for $a, b \in S$), then it is a directed set. (Indeed, for every $a, b \in S$, applying $aS \cap bS \neq \emptyset$, there exist $a', b' \in S$ with $aa' = bb'$; by taking $c = aa' = bb'$, we have $cS \subseteq aS \cap bS$, and then $a \prec c$ and $b \prec c$.) If a semigroup S is left amenable, then S is left reversible [5]. Let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of a left reversible semigroup S as Lipschitzian mappings on C with Lipschitz constants $\{k(s) : s \in S\}$. We shall say that $\mathcal{S}$ is an *asymptotically nonexpansive semigroup* on C if there holds the uniform Lipschitzian condition $\lim_s k(s) \leq 1$ on the Lipschitz constants. (Note that a left reversible semigroup is a directed set.) It is worth mentioning that there is a notion of asymptotically nonexpansive defined, depending on left ideals in a semigroup in the works of Holmes and Lau [6] and Holmes [7].

In 2008, Saeidi [8] introduced the following viscosity iterative scheme:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)x_n, \forall n \geq 1, \tag{1}$$

for a representation of S as Lipschitzian mappings on a compact convex subset C of a smooth Banach space E with respect to a left regular sequence $\{\mu_n\}$ of means defined on an appropriate invariant subspace of $l^\infty(S)$. For some related results, we refer the readers to the works of Kirk [9] and Takahashi [10]. In 2011, Katchang and Kumam

[11] extended the result of Saeidi [8] and introduced a new iterative algorithm with a viscosity iteration method for approximating a common fixed point of Lipschitzian semigroups in Banach spaces as the following:

$$\begin{cases} y_n = \delta_n x_n + (1-\delta_n)T(\mu_n)x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n, \forall n \geq 1. \end{cases} \tag{2}$$

Recently, Aoyama et al. [12] first considered the following generalized variational inequality problem in a smooth Banach space. Let A be an accretive operator of C into E. Find a point $x \in C$ such that

$$\langle Ax, j(y-x)\rangle \geq 0 \tag{3}$$

for all $y \in C$. The set of solutions of (3) is denoted by $VI(C,A)$. This problem is connected with the fixed point problem for nonlinear mappings, the problem of finding a zero point of an accretive operator and so on. For the problem of finding a zero point of an accretive operator by the proximal point algorithm, see Kamimura and Takahashi [13,14] . In order to find a solution of the variational inequality (3), Aoyama et al. [12] proved the strong convergence theorem in the framework of Banach spaces which is generalized by Iiduka et al. [15] from Hilbert spaces. In 2011, Yao and Maruster [16] proved some strong convergence theorems for finding a solution of variational inequality problem (3) in Banach spaces. They defined a sequence $\{x_n\}$ iteratively by the arbitrary given $x_0 \in C$ and

$$x_{n+1} = \beta_n x_n + (1-\beta_n)Q_C[\,(1-\alpha_n)(x_n - \lambda A x_n)]\,, \forall n \geq 0, \tag{4}$$

where Q_C is a sunny nonexpansive retraction from a uniformly convex and 2-uniformly smooth Banach space X onto a nonempty closed convex subset C of X, and A is an α-inverse strongly accretive operator of C into X (in the framework of variational inequality problems, also see Katchang and Kumam [17]).

Here, motivated and inspired by the ideas of Aoyama et al. [12], Saeidi [8], and Yao and Maruster [16], we introduce the new iterative algorithm (7) and prove strong convergence theorems for finding the common solution of variational inequality problems (3) involving an inverse strongly accretive operator and the solution of fixed point problems involving a Lipschitzian semigroup mapping in Banach spaces using a viscosity approximation method. Moreover, its applications are also studied.

Preliminaries

Let S be a semigroup. We denote by $l^\infty(S)$ the Banach space of all bounded real-valued functions on S with the supremum norm. For each $s \in S$, we define l_s and r_s on $l^\infty(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for each $t \in S$ and $f \in l^\infty(S)$. Let X be a subspace of $l^\infty(S)$ containing 1, and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(f(t))$, instead of $\mu(f)$ for $\mu \in X^*$ and $f \in X$. Let X be left invariant (with respect to the (resp.) right invariant), i.e., $l_s(X) \subset X$ (resp. $r_s(X) \subset X$), for each $s \in S$. A mean μ on X is said to be left invariant (resp. right invariant) if $\mu(l_s f) = \mu(f)$ (resp. $\mu(r_s f) = \mu(f)$) for each $s \in S$ and $f \in X$. X is said to be left (resp. right) *amenable* if X has a left (resp. right) invariant mean. X is amenable if X is both left and right amenable. A net $\{\mu_\alpha\}$ of means on X is said to be *strongly left regular* if

$$\lim_\alpha \|l_s^* \mu_\alpha - \mu_\alpha\| = 0$$

for each $s \in S$, where l_s^* is the adjoint operator of l_s. Let C be a nonempty closed and convex subset of E. Throughout this paper, S will always denote a semigroup with an identity e. S is called left reversible if any two right ideals in S have nonvoid intersection, i.e., $aS \cap bS \neq \emptyset$ for $a, b \in S$. In this case, we can define a partial ordering $\prec$ on S by $a \prec b$ if and only if $aS \supset bS$. It is easy to see $t \prec ts$, $(\forall t, s \in S)$. Furthermore, if $t \prec s$, then $pt \prec ps$ for all $p \in S$. If a semigroup S is left amenable, then S is left reversible. However, the converse is not true.

$\mathcal{S} = \{T(s) : s \in S\}$ is called a representation of S as Lipschitzian mappings on C if for each $s \in S$, the mapping $T(s)$ is Lipschitzian mapping on C with Lipschitz constant $k(s)$ and $T(st) = T(s)T(t)$ for $s, t \in S$. We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, and we denote by C_a the set of almost periodic elements in C, i.e., all $x \in C$ such that $\{T(s)x : s \in S\}$ is relatively compact in the norm topology of reflexive Banach space E. We will call a subspace X of $l^\infty(S)$, $\mathcal{S}$-stable if the functions $s \mapsto \langle T(s)x, x^*\rangle$ and $s \mapsto \|T(s)x - y\|$ on S are in X for all $x, y \in C$ and $x^* \in E^*$. We know that if μ is a mean on X and if for each $x^* \in E^*$, the function $s \mapsto \langle T(s)x, x^*\rangle$ is contained in X and C is weakly compact, then there exists a unique point x_0 of E such that

$$\mu_s\langle T(s)x, x^*\rangle = \langle x_0, x^*\rangle$$

for each $x^* \in E^*$. We denote such a point x_0 by $T(\mu)x$. Note that $T(\mu)z = z$ for each $z \in F(\mathcal{S})$; see related works [18-20].

We need the following lemmas to prove our main results.

Lemma 1. *Let S be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty weakly compact convex subset C of a Banach space E into C, with the uniform Lipschitzian condition $\lim_s k(s) \leq 1$ on the Lipschitz constants of the mappings. Let X be a left invariant $\mathcal{S}-$ stable subspace of $l^\infty(S)$ containing 1, and μ be a left invariant mean on X. Then, $F(\mathcal{S}) = F(T(\mu)) \cap C_a$ [21].*

Corollary 1. *Let $\{\mu_n\}$ be an asymptotically left invariant sequence of the means on X. If $z \in C_a$ and $\liminf_{n\to\infty} \|T(\mu_n)z - z\| = 0$, then z is a common fixed point for $\mathcal{S}$ [8].*

Lemma 2. *Let S be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty weakly compact convex subset C of a Banach space E into C, with the uniform Lipschitzian condition $\lim_s k(s) \le 1$ on the Lipschitz constants of the mappings. Let X be a left invariant subspace of $l^\infty(S)$ containing 1 such that the mappings $s \mapsto \langle T(s)x, x^*\rangle$ be in X for all $x \in X$ and $x^* \in E^*$, and $\{\mu_n\}$ be a strongly left regular sequence of means on X [8]. Then,*

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T(\mu_n)x - T(\mu_n)y\| - \|x - y\|) \le 0.$$

Remark 1. Taking in Lemma 2,

$$c_n = \sup_{x,y\in C} (\|T(\mu_n)x - T(\mu_n)y\| - \|x - y\|), \forall n, \qquad (5)$$

we obtain $\limsup_{n\to\infty} c_n \le 0$. Moreover,

$$\|T(\mu_n)x - T(\mu_n)y\| \le \|x - y\| + c_n, \forall x, y \in C. \qquad (6)$$

Corollary 2. *Let S be a left reversible semigroup and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty compact convex subset C of a Banach space E into C, with the uniform Lipschitzian condition $\lim_s k(s) \le 1$. Let X be a left invariant $\mathcal{S} -$ stable subspace of $l^\infty(S)$ containing 1, and μ be a left invariant mean on X. Then, $T(\mu)$ is nonexpansive and $F(\mathcal{S}) \ne \emptyset$. Moreover, if E is smooth, then $F(\mathcal{S})$ is a sunny nonexpansive retract of C, and the sunny nonexpansive retraction of C onto $F(\mathcal{S})$ is unique [8].*

Lemma 3. *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C and let A be an accretive operator of C into X. Then, for all $\lambda > 0$, the set $VI(C, A)$ is coincident with the set of fixed points of $Q_C(I - \lambda A)$ [12], that is,*

$$VI(C, A) = F(Q_C(I - \lambda A)).$$

Lemma 4. *Let C be a nonempty bounded closed convex subset of a uniformly convex Banach space X and T be the nonexpansive mapping of C into itself. If $\{x_n\}$ is a sequence of C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then x is a fixed point of T [22].*

Lemma 5. *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$, for all integers $n \ge 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$, then $\lim_{n\to\infty} \|z_n - x_n\| = 0$ [23].*

Lemma 6. *Assume that $\{x_n\}$ is a sequence of nonnegative real numbers such that*

$$x_{n+1} \le (1 - a_n)x_n + b_n, \ n \ge 0,$$

where $\{a_n\}$ is a sequence in $(0, 1)$ and $\{b_n\}$ is a sequence in $\mathbb{R}$ such that [24]

(1) $\sum_{n=1}^{\infty} a_n = \infty$,
(2) $\limsup_{n\to\infty} \frac{b_n}{a_n} \le 0$ or $\sum_{n=1}^{\infty} |b_n| < \infty$.

Then, $\lim_{n\to\infty} x_n = 0$.

Lemma 7. *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space E with the best smooth constant K. Let the mapping $A : C \to E$ be β-inverse strongly accretive (see Lemma 3.1 of [25], see also Lemma 2.8 of [12]. Then, we have*

$$\|(I-\lambda A)x-(I-\lambda A)y\|^2 \le \|x-y\|^2+2\lambda(\lambda K^2-\beta)\|Ax-Ay\|^2.$$

If $\beta \ge \lambda K^2$, then $I - \lambda A$ is nonexpansive.

Lemma 8. *Let X be a real Banach space and let J be the duality mapping. Then, for any given $x, y \in X$ and $j(x+y) \in J(x + y)$, there holds the following inequality [9,10]:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, j(x + y)\rangle.$$

Lemma 9. *Let E be a real 2-uniformly smooth Banach space with the best smooth constant K. Then, the following inequality holds [1]:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, Jx\rangle + 2\|Ky\|^2, \quad \forall x, y \in E.$$

Lemma 10. *Let E be a uniformly convex Banach space and $B_r(0) := \{x \in E : \|x\| \le r\}$ be a closed ball of E. Then, there exists a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x+\mu y+\gamma z\|^2 \le \lambda\|x\|^2+\mu\|y\|^2+\gamma\|z\|^2-\lambda\mu g(\|x-y\|)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda+\mu+\gamma = 1$ [26].

Lemma 11. *Let $r > 0$ and let E be a uniformly convex Banach space. Then, there exists a continuous, strictly increasing, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x+(1-\lambda)y\|^2 \le \lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in B_r := \{z \in E : \|z\| \le r\}$ and $0 \le \lambda \le 1$ [1].

We note that for a given sequence $\{x_n\} \subset C$, let $\omega_w(\{x_n\}) := \{x : \exists x_{n_j} \rightharpoonup x\}$ denote the weak ω-limit set of $\{x_n\}$.

Main result

In this section, we prove a strong convergence theorem in Banach spaces.

Theorem 1. *Let C be a nonempty compact convex subset of a uniformly convex and 2-uniformly smooth Banach space E with weakly sequentially continuous duality mapping and the best smooth constant K, S be a left reversible semigroup, and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from C into itself, with the uniform Lipschitzian condition $\lim_s k(s) \leq 1$. Let X be a left invariant $\mathcal{S}$-stable subspace of $l^{\infty}(S)$ containing 1, $\{\mu_n\}$ be a strongly left regular sequence of means on X such that $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ and $\{c_n\}$ be the sequence defined by (5). Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$, Q_C be a sunny nonexpansive retraction from E onto C, and $A : C \to E$ be a β-inverse strongly accretive with $\beta \geq \lambda K^2$ such that λ be a positive real number. Suppose $\mathcal{F} = F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$ and the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. The following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.
(ii) $\lim_{n\to\infty} \delta_n = 0$.
(iii) $\limsup_{n\to\infty} \frac{c_n}{\alpha_n} \leq 0$, *(note that, by Remark 1,* $\limsup_{n\to\infty} c_n \leq 0$*).*
(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

If for the arbitrary given $x_1 \in C$, the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = \delta_n x_n + (1-\delta_n) Q_C(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n) y_n, \forall n \geq 1, \end{cases} \tag{7}$$

then $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$, which is the unique solution of the variational inequality

$$\langle (f - I)x^*, J(z - x^*) \rangle \leq 0, \forall z \in \mathcal{F}.$$

Equivalently, we have $x^* = Q_{\mathcal{F}} f(x^*)$, where Q is the unique sunny nonexpansive retraction of C onto $\mathcal{F}$.

Proof. First, we claim that for any sequence $\{v_{n+1}\} \in C$, $\|T(\mu_{n+1})v_{n+1} - T(\mu_n)v_{n+1}\| \to$ as $n \to \infty$. Taking $D = \sup\{\|p^*\| : p^* \in E^*\}$, we have the following:

$$\begin{aligned} &\|T(\mu_{n+1})v_{n+1} - T(\mu_n)v_{n+1}\| \\ &= \sup_{\|p^*\|=1} \left| \langle T(\mu_{n+1})v_{n+1} - T(\mu_n)v_{n+1}, p^* \rangle \right| \\ &= \sup_{\|p^*\|=1} \left| (\mu_{n+1})_s \langle T(s)v_{n+1}, p^* \rangle - (\mu_n)_s \langle T(s)v_{n+1}, p^* \rangle \right| \\ &\leq \|\mu_{n+1} - \mu_n\| \sup_{s\in S} \|T(s)v_{n+1}\| \\ &\leq \|\mu_{n+1} - \mu_n\| D. \end{aligned}$$

Since $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$, therefore

$$\lim_{n\to\infty} \|T(\mu_{n+1})v_{n+1} - T(\mu_n)v_{n+1}\| = 0. \tag{8}$$

Next, we show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, and by Lemma 2, we observe that

$$\begin{aligned} \|y_{n+1} - y_n\| &= \|\delta_{n+1}x_{n+1} + (1-\delta_{n+1})Q_C(x_{n+1} - \lambda A x_{n+1}) \\ &\quad - \delta_n x_n - (1-\delta_n)Q_C(x_n - \lambda A x_n)\| \\ &= \|\delta_{n+1}x_{n+1} - \delta_{n+1}x_n + \delta_{n+1}x_n \\ &\quad + (1-\delta_{n+1})Q_C(x_{n+1} - \lambda A x_{n+1}) \\ &\quad - (1-\delta_{n+1})Q_C(x_n - \lambda A x_n) \\ &\quad + (1-\delta_{n+1})Q_C(x_n - \lambda A x_n) \\ &\quad - \delta_n x_n - (1-\delta_n)Q_C(x_n - \lambda A x_n)\| \\ &= \Big\| \delta_{n+1}(x_{n+1} - x_n) + (\delta_{n+1} - \delta_n)x_n + (1-\delta_{n+1}) \\ &\quad \times [\, Q_C(x_{n+1} - \lambda A x_{n+1}) - Q_C(x_n - \lambda A x_n)] \\ &\quad + (\delta_n - \delta_{n+1})Q_C(x_n - \lambda A x_n) \Big\| \\ &\leq \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|x_n\| \\ &\quad + (1-\delta_{n+1})\|Q_C(x_{n+1} - \lambda A x_{n+1}) \\ &\quad - Q_C(x_n - \lambda A x_n)\| \\ &\quad + |\delta_n - \delta_{n+1}|\|Q_C(x_n - \lambda A x_n)\| \\ &\leq \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\|x_n\| \\ &\quad + (1-\delta_{n+1})\|x_{n+1} - x_n\| \\ &\quad + |\delta_n - \delta_{n+1}|\|Q_C(x_n - \lambda A x_n)\| \\ &= \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n| \\ &\quad \times [\, \|x_n\| + \|Q_C(x_n - \lambda A x_n)\|\,]. \end{aligned}$$

Setting $x_{n+1} = (1-\beta_n)z_n + \beta_n x_n$, we see that $z_n = \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$. Then, we compute

$$\begin{aligned} \|z_{n+1} - z_n\| &= \left\| \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1-\beta_n} \right\| \\ &= \Big\| \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}T(\mu_{n+1})y_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\alpha_n f(x_n) + \gamma_n T(\mu_n)y_n}{1-\beta_n} \Big\| \\ &= \Big\| \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}T(\mu_{n+1})y_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}} + \frac{\alpha_{n+1}f(x_n)}{1-\beta_{n+1}} \\ &\quad - \frac{\gamma_{n+1}T(\mu_n)y_{n+1}}{1-\beta_{n+1}} + \frac{\gamma_{n+1}T(\mu_n)y_{n+1}}{1-\beta_{n+1}} \\ &\quad - \frac{\gamma_n T(\mu_n)y_{n+1}}{1-\beta_n} + \frac{\gamma_n T(\mu_n)y_{n+1}}{1-\beta_n} \\ &\quad - \frac{\alpha_n f(x_n) + \gamma_n T(\mu_n)y_n}{1-\beta_n} \Big\| \end{aligned}$$

$$
\begin{aligned}
&= \Big\| \frac{\alpha_{n+1}}{1-\beta_{n+1}} \Big[f(x_{n+1}) - f(x_n) \Big] \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \Big[T(\mu_{n+1})y_{n+1} - T(\mu_n)y_{n+1} \Big] \\
&\quad + \Big[\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \Big] f(x_n) \\
&\quad + \Big[\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \Big] T(\mu_n)y_{n+1} \\
&\quad + \frac{\gamma_n}{1-\beta_n} \Big[T(\mu_n)y_{n+1} - T(\mu_n)y_n \Big] \Big\| \\
&\le \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}} \|x_{n+1} - x_n\| \\
&\quad + \frac{\gamma_{n+1}}{1-\beta_{n+1}} \|T(\mu_{n+1})y_{n+1} - T(\mu_n)y_{n+1}\| \\
&\quad + \Big| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \Big| \|f(x_n)\| \\
&\quad + \Big| \frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \Big| \|T(\mu_n)y_{n+1}\| \\
&\quad + \frac{\gamma_n}{1-\beta_n} \|T(\mu_n)y_{n+1} - T(\mu_n)y_n\| \\
&\le \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}} \|x_{n+1} - x_n\| \\
&\quad + \|T(\mu_{n+1})y_{n+1} - T(\mu_n)y_{n+1}\| \\
&\quad + \Big| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \Big| \\
&\quad \times \Big[\|f(x_n)\| + \|T(\mu_n)y_{n+1}\| \Big] \\
&\quad + \|y_{n+1} - y_n\| + c_n \\
&\le \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}} \|x_{n+1} - x_n\| + \|T(\mu_{n+1})y_{n+1} \\
&\quad - T(\mu_n)y_{n+1}\| + \Big| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \Big| \\
&\quad \times \Big[\|f(x_n)\| + \|T(\mu_n)y_{n+1}\| \Big] \\
&\quad + \|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n| \\
&\quad \times \Big[\|x_n\| + \|Q_C(x_n - \lambda A x_n)\| \Big] + c_n.
\end{aligned}
$$

Since C is bounded and $\limsup_{n\to\infty} \beta_n < 1$, we have big constants $M_1 > 0$ and $M_2 > 0$. Therefore, we observe that

$$
\begin{aligned}
&\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\
&\quad \le \frac{\alpha\alpha_{n+1}}{1-\beta_{n+1}} \|x_{n+1} - x_n\| + \|T(\mu_{n+1})y_{n+1} - T(\mu_n)y_{n+1}\| \\
&\quad\quad + \Big| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \Big| M_1 + |\delta_{n+1} - \delta_n| M_2 + c_n.
\end{aligned}
$$

It follows from (i), (ii), (iv), (8), and Lemma 2 that

$$
\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.
$$

Applying Lemma 5, we obtain $\lim_{n\to\infty} \|z_n - x_n\| = 0$. We also have $\|x_{n+1} - x_n\| = (1 - \beta_n)\|z_n - x_n\|$, therefore, we get

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{9}
$$

On the other hand, let $p \in \mathcal{F}$, and we have the following:

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)y_n - p\|^2 \\
&\le \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 \\
&\quad + \gamma_n \|T(\mu_n)y_n - p\|^2 \\
&\le \alpha_n \|f(x_n) - p\|^2 + \beta_n \|x_n - p\|^2 \\
&\quad + \gamma_n \Big[\|y_n - p\| + c_n \Big]^2 \\
&= \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n - \gamma_n)\|x_n - p\|^2 \\
&\quad + \gamma_n \Big[\|y_n - p\|^2 + 2c_n \|y_n - p\| + c_n^2 \Big] \\
&= \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 \\
&\quad - \gamma_n \Big[\|x_n - p\|^2 - \|y_n - p\|^2 \Big] \\
&\quad + \gamma_n c_n \Big[2\|y_n - p\| + c_n \Big] \\
&= \alpha_n \|f(x_n) - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 \\
&\quad - \gamma_n \Big[\|x_n - p\| - \|y_n - p\| \Big] \\
&\quad \times \Big[\|x_n - p\| + \|y_n - p\| \Big] s \\
&\quad + \gamma_n c_n \Big[2\|y_n - p\| + c_n \Big] \\
&\le \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 \\
&\quad - \gamma_n \|x_n - y_n\|^2 + \gamma_n c_n \Big[2\|y_n - p\| + c_n \Big].
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\gamma_n \|x_n - y_n\|^2 &\le \alpha_n \|f(x_n) - p\|^2 + \|x_n - p\|^2 \\
&\quad - \|x_{n+1} - p\|^2 + \gamma_n c_n \Big[2\|y_n - p\| + c_n \Big] \\
&\le \alpha_n \|f(x_n) - p\|^2 + \|x_n - x_{n+1}\| \\
&\quad \times \Big[\|x_n - p\| + \|x_{n+1} - p\| \Big] \\
&\quad + \gamma_n c_n \Big[2\|y_n - p\| + c_n \Big].
\end{aligned}
$$

From conditions (i) and (iv) and by (5) and (9), we conclude that

$$
\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{10}
$$

We note that

$$
\begin{aligned}
x_{n+1} - y_n &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)y_n - y_n \\
&= \alpha_n \Big[f(x_n) - y_n \Big] + \beta_n (x_n - y_n) \\
&\quad + \gamma_n \Big[T(\mu_n)y_n - y_n \Big].
\end{aligned}
$$

Thus, we have the following:

$$\begin{aligned}\gamma_n\|y_n - T(\mu_n)y_n\| &\leq \alpha_n\|f(x_n) - y_n\| + \beta_n\|x_n - y_n\| \\ &\quad + \|y_n - x_{n+1}\| \\ &\leq \alpha_n\|f(x_n) - y_n\| + \beta_n\|x_n - y_n\| \\ &\quad + \|y_n - x_n\| + \|x_n - x_{n+1}\| \\ &= \alpha_n\|f(x_n)-y_n\|+(1+\beta_n)\|x_n-y_n\| \\ &\quad + \|x_n - x_{n+1}\|.\end{aligned}$$

By (i), (iv), (9), and (10), we obtain the following:

$$\lim_{n\to\infty} \|y_n - T(\mu_n)y_n\| = 0. \tag{11}$$

We consider

$$\begin{aligned}\|x_n - y_n\| &= \left\|x_n - \left[\delta_n x_n + (1-\delta_n)Q_C(x_n - \lambda Ax_n)\right]\right\| \\ &= \Big\|\delta_n(x_n - x_n) + (1-\delta_n) \\ &\quad \times \left[x_n - Q_C(x_n - \lambda Ax_n)\right]\Big\| \\ &= (1-\delta_n)\|x_n - Q_C(x_n - \lambda Ax_n)\|.\end{aligned}$$

By (ii) and (10), we have the following:

$$\lim_{n\to\infty} \|x_n - Q_C(x_n - \lambda Ax_n)\| = 0. \tag{12}$$

Let $\omega\big(\{x_n\}\big)$ be the ω-limit set of $\{x_n\}$. Next, we show that $\omega\big(\{x_n\}\big)$ is a subset of $\mathcal{F} = F(\mathcal{S})\cap VI(C,A)$. Let $z \in \omega\big(\{x_n\}\big)$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that converges strongly to z. Since $\|x_n - y_n\| \to 0$, we obtain $y_{n_k} \to z$. From (11), Lemma 2 and Remark 1, we obtain the following:

$$\begin{aligned}&\limsup_{k\to\infty}\|z - T(\mu_{n_k})z\| \\ &\leq \limsup_{k\to\infty} \left[\|z - y_{n_k}\| + \|y_{n_k} - T(\mu_{n_k})y_{n_k}\|\right. \\ &\qquad \left.+\|T(\mu_{n_k})y_{n_k} - T(\mu_{n_k})z\|\right] \\ &\leq \limsup_{k\to\infty}\left[\|y_{n_k} - T(\mu_{n_k})y_{n_k}\|\right. \\ &\qquad \left.+2\|y_{n_k} - z\| + c_{n_k}\right] \leq 0.\end{aligned}$$

Moreover, we have the following:

$$\liminf_{k\to\infty} \|z - T(\mu_{n_k})z\| \leq \limsup_{k\to\infty} \|z - T(\mu_{n_k})z\| \leq 0.$$

Thus, applying Corollary 1, we get $z \in F(\mathcal{S})$. Next, we show $z \in VI(C,A)$. From (12) and by Lemmas 3 and 4, we have $z \in F(Q_C(I - \lambda A)) = VI(C,A)$. Therefore $z \in \mathcal{F}$.

Next, we show that $\limsup_{n\to\infty}\langle (f-I)x^*, J(x_n - x^*)\rangle \leq 0$, where $x^* = Q_{\mathcal{F}}f(x^*)$. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (f-I)x^*, J(x_n - x^*)\rangle = \lim_{k\to\infty} \langle (f-I)x^*, J(x_{n_k} - x^*)\rangle. \tag{13}$$

Now, from (13), Proposition 1 (iii) and the weakly sequential continuity of the duality mapping J, we have the following:

$$\begin{aligned}&\limsup_{n\to\infty}\langle (f-I)x^*, J(x_n - x^*)\rangle \\ &\quad = \lim_{k\to\infty} \langle (f-I)x^*, J(x_{n_k} - x^*)\rangle \\ &\quad = \langle (f-I)x^*, J(z - x^*)\rangle \leq 0.\end{aligned} \tag{14}$$

From (9), it follows that

$$\limsup_{n\to\infty}\langle (f-I)x^*, J(x_{n+1} - x^*)\rangle \leq 0. \tag{15}$$

Finally, we show that the sequence $\{x_n\}$ converges strongly to $x^* = Q_{\mathcal{F}}f(x^*)$. From Lemma 7 and since Q_C is a nonexpansive, we have the following:

$$\begin{aligned}\|y_n - x^*\| &= \|\delta_n x_n + (1-\delta_n)Q_C(x_n - \lambda Ax_n) - x^*\| \\ &\leq \delta_n\|x_n - x^*\| + (1-\delta_n)\|Q_C(x_n - \lambda Ax_n) \\ &\quad - Q_C(x^* - \lambda Ax^*)\| \\ &\leq \delta_n\|x_n - x^*\| + (1-\delta_n)\|(x_n - \lambda Ax_n) \\ &\quad - (x^* - \lambda Ax^*)\| \\ &\leq \delta_n\|x_n - x^*\| + (1-\delta_n)\|x_n - x^*\| \\ &= \|x_n - x^*\|.\end{aligned} \tag{16}$$

Using Lemmas 8 and 11 and (16), we have the following:

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\alpha_n f(x_n)+\beta_n x_n+\gamma_n T(\mu_n)y_n-x^*\|^2\\
&= \big\|\big[\gamma_n(T(\mu_n)y_n-x^*)+\beta_n(x_n-x^*)\big]\\
&\quad +\alpha_n\big[f(x_n)-x^*\big]\big\|^2\\
&\le \big\|\gamma_n\big[T(\mu_n)y_n-x^*\big]+\beta_n(x_n-x^*)\big\|^2\\
&\quad +2\alpha_n\langle f(x_n)-x^*,J(x_{n+1}-x^*)\rangle\\
&\le (1-\beta_n)\Big\|\frac{\gamma_n}{1-\beta_n}\big[T(\mu_n)y_n-x^*\big]\Big\|^2\\
&\quad +\beta_n\|x_n-x^*\|^2\\
&\quad +2\alpha_n\langle f(x_n)-f(x^*),J(x_{n+1}-x^*)\rangle\\
&\quad +2\alpha_n\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle\\
&= \frac{\gamma_n^2}{1-\beta_n}\big\|T(\mu_n)y_n-x^*\big\|^2+\beta_n\|x_n-x^*\|^2\\
&\quad +2\alpha_n\langle f(x_n)-f(x^*),J(x_{n+1}-x^*)\rangle\\
&\quad +2\alpha_n\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle\\
&\le \frac{\gamma_n^2}{1-\beta_n}\|y_n-x^*\|^2\\
&\quad +\frac{\gamma_n^2 c_n}{1-\beta_n}+\beta_n\|x_n-x^*\|^2\\
&\quad +2\alpha\alpha_n\|x_n-x^*\|\|x_{n+1}-x^*\|\\
&\quad +2\alpha_n\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle\\
&\le \frac{\gamma_n^2}{1-\beta_n}\|x_n-x^*\|^2\\
&\quad +\frac{\gamma_n^2 c_n}{1-\beta_n}+\beta_n\|x_n-x^*\|^2\\
&\quad +2\alpha\alpha_n\|x_n-x^*\|\|x_{n+1}-x^*\|\\
&\quad +2\alpha_n\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle\\
&\le \frac{\gamma_n^2}{1-\beta_n}\|x_n-x^*\|^2\\
&\quad +\frac{\gamma_n^2 c_n}{1-\beta_n}+\beta_n\|x_n-x^*\|^2\\
&\quad +\alpha\alpha_n\big[\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2\big]\\
&\quad +2\alpha_n\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle\\
&= \Big[\frac{\gamma_n^2}{1-\beta_n}+\beta_n+\alpha\alpha_n\Big]\|x_n-x^*\|^2\\
&\quad +\frac{\gamma_n^2 c_n}{1-\beta_n}\\
&\quad +\alpha\alpha_n\|x_{n+1}-x^*\|^2\\
&\quad +2\alpha_n\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle\\
&= \Big[\frac{\big((1-\beta_n)-\alpha_n\big)^2}{1-\beta_n}+\beta_n+\alpha\alpha_n\Big]\\
&\quad \|x_n-x^*\|^2+\frac{\gamma_n^2 c_n}{1-\beta_n}+\alpha\alpha_n\|x_{n+1}-x^*\|^2\\
&\quad +2\alpha_n\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle\\
&= \Big[\frac{(1-\beta_n)^2-2(1-\beta_n)\alpha_n+\alpha_n^2}{1-\beta_n}+\beta_n+\alpha\alpha_n\Big]\\
&\quad \|x_n-x^*\|^2+\frac{\gamma_n^2 c_n}{1-\beta_n}\\
&\quad +\alpha\alpha_n\|x_{n+1}-x^*\|^2+2\alpha_n\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle\\
&= \Big[1-\beta_n-2\alpha_n+\frac{\alpha_n^2}{1-\beta_n}+\beta_n+\alpha\alpha_n\Big]\|x_n-x^*\|^2\\
&\quad +\frac{\gamma_n^2 c_n}{1-\beta_n}+\alpha\alpha_n\|x_{n+1}-x^*\|^2\\
&\quad +2\alpha_n\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle\\
&= \Big[(1-\alpha\alpha_n)+(2\alpha\alpha_n-2\alpha_n)+\frac{\alpha_n^2}{1-\beta_n}\Big]\|x_n-x^*\|^2\\
&\quad +\frac{\gamma_n^2 c_n}{1-\beta_n}+\alpha\alpha_n\|x_{n+1}-x^*\|^2\\
&\quad +2\alpha_n\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \Big[1-\frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n}\Big]\|x_n-x^*\|^2\\
&\quad +\frac{\alpha_n^2}{(1-\alpha\alpha_n)(1-\beta_n)}\|x_n-x^*\|^2\\
&\quad +\frac{\alpha_n\gamma_n^2}{(1-\alpha\alpha_n)(1-\beta_n)}\Big(\frac{c_n}{\alpha_n}\Big)\\
&\quad +\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle\\
&\le \Big[1-\frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n}\Big]\|x_n-x^*\|^2+\frac{\alpha_n}{1-\alpha\alpha_n}\\
&\quad \times\Big[\frac{\alpha_n}{1-\beta_n}\|x_n-x^*\|^2+\frac{\gamma_n^2}{1-\beta_n}\Big(\frac{c_n}{\alpha_n}\Big)\\
&\quad +2\langle f(x^*)-x^*,J(x_{n+1}-x^*)\rangle\Big]\\
&= (1-a_n)\|x_n-x^*\|^2+b_n,
\end{aligned}
$$

where

$$
a_n=\frac{2\alpha_n(1-\alpha)}{1-\alpha\alpha_n}
$$

and

$$
b_n=\frac{\alpha_n}{1-\alpha\alpha_n}\Big[\frac{\alpha_n}{1-\beta_n}\|x_n-x^*\|^2+\frac{\gamma_n^2}{1-\beta_n}\Big(\frac{c_n}{\alpha_n}\Big)+2\big\langle f(x^*)-x^*,J(x_{n+1}-x^*)\big\rangle\Big].
$$

Now, from (i), (iii), (iv), and (15) and Lemma 6, we get $\|x_n-x^*\|\to 0$ as $n\to\infty$. This completes the proof. □

Example 1. *We can choose for an example of the control condition α_n in Theorem 1 as follows:*

$$
\alpha_n=\begin{cases}\frac{1}{n+1}+\sqrt{c_n} & \text{if } c_n\ge 0,\\ \frac{1}{n+1} & \text{if } c_n<0.\end{cases} \tag{17}
$$

Corollary 3. *Let C be a nonempty compact convex subset of a uniformly convex and 2-uniformly smooth Banach space E with weakly sequentially continuous duality mapping and the best smooth constant K, S be a left reversible semigroup and* $\mathcal{S} = \{T(s) : s \in S\}$ *be a representation of S as Lipschitzian mappings from C into itself, with the uniform Lipschitzian condition* $\lim_s k(s) \leq 1$. *Let X be a left invariant* $\mathcal{S}$*-stable subspace of* $l^\infty(S)$ *containing 1,* $\{\mu_n\}$ *be a strongly left regular sequence of means on X such that* $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ *and* $\{c_n\}$ *be the sequence defined by (5). Let f be a contraction of C into itself with coefficient* $\alpha \in (0,1)$, Q_C *be a sunny nonexpansive retraction from E onto C and* $A : C \to E$ *be a* β*-inverse-strongly accretive with* $\beta \geq \lambda K^2$ *such that* λ *be a positive real number. Suppose* $\mathcal{F} = F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$ *and the sequences* $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *in* $(0,1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. *The sequence* $\{x_n\}$ *is generated by* $x_1 \in C$ *and*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n) Q_C(x_n - \lambda A x_n), \forall n \geq 1. \quad (18)$$

If the sequence $\{x_n\}$ *satisfy the conditions (i) to (iv) in Theorem 1 then* $\{x_n\}$ *converges strongly to* $x^* \in \mathcal{F}$.

Proof. Taking $\delta_n = 0$ in Theorem 1, we can conclude the desired conclusion easily. This completes the proof. □

Remark 2. Our result extends and improves the results of Saeidi [8], Katchang and Kumam [11], and Yao and Maruster [16].

Applications

Application to the other form of semigroups

Theorem 2. *Let C be a nonempty compact convex subset of a uniformly convex and 2-uniformly smooth Banach space E with weakly sequentially continuous duality mapping and the best smooth constant K, and let* $\mathcal{S} = \{T(t) : t \in \mathbb{R}^+\}$ *be a strongly continuous semigroup of Lipschitzian mappings from C into itself, with the uniform Lipschitzian condition* $\lim_s k(s) \leq 1$ *and* $\{t_n\}$ *be increasing sequence in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = 1$. *Let f be a contraction of C into itself with coefficient* $\alpha \in (0,1)$, Q_C *be a sunny nonexpansive retraction from E onto C and* $A : C \to E$ *be a* β*-inverse strongly accretive with* $\beta \geq \lambda K^2$ *such that* λ *be a positive real number.*

Suppose $\mathcal{F} = F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$ *and the sequences* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, *and* $\{\delta_n\}$ *in* $(0,1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$, *the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.
(ii) $\lim_{n\to\infty} \delta_n = 0$.
(iii) $\limsup_{n\to\infty} \frac{c_n}{\alpha_n} \leq 0$, *where,* $c_n = \sup_{x,y\in C}\{\|\frac{1}{t_n}\int_0^{t_n} T(s)xds - \frac{1}{t_n}\int_0^{t_n} T(s)yds\| - \|x-y\|\}$.
(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

If for arbitrary given $x_1 \in C$, *the sequence* $\{x_n\}$ *is generated by*

$$\begin{cases} y_n = \delta_n x_n + (1-\delta_n) Q_C(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \frac{1}{t_n}\int_0^{t_n} T(s) y_n ds, \forall n \geq 1, \end{cases} \quad (19)$$

then $\{x_n\}$ *converges strongly to* $x^* \in \mathcal{F}$, *which is the unique solution of the following variational inequality:*

$$\langle (f-I)x^*, J(z-x^*)\rangle \leq 0, \forall z \in \mathcal{F}.$$

Equivalently, we have $x^* = Q_{\mathcal{F}} f(x^*)$, *where Q is the unique sunny nonexpansive retraction of C onto* $\mathcal{F}$.

Proof. For $n \geq 1$, define $\mu_n(g) = \frac{1}{t_n}\int_0^{t_n} g(t)dt$ for each $g \in C(\mathbb{R}^+)$, where $C(\mathbb{R}^+)$ is the space of all real-valued bounded continuous functions on $\mathbb{R}^+$ with the supremum norm. Then, $\{\mu_n\}$ is a strongly regular sequence of means and $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ (see [27]). Furthermore, for each $x \in C$, we have $T(\mu_n)x = \frac{1}{t_n}\int_0^{t_n} T(s)xds$. Therefore, we apply Theorem 1 to conclude the result. □

Application to the strongly accretive and Lipschitz continuous operators

Now, we prove a strong convergence theorem for strongly accretive operators.

Theorem 3. *Let C be a nonempty compact convex subset of a uniformly convex and 2-uniformly smooth Banach space E with weakly sequentially continuous duality mapping and the best smooth constant K, S be a left reversible semigroup, and* $\mathcal{S} = \{T(s) : s \in S\}$ *be a representation of S as Lipschitzian mappings from C into itself, with the uniform Lipschitzian condition* $\lim_s k(s) \leq 1$. *Let X be a left invariant* $\mathcal{S}$*-stable subspace of* $l^\infty(S)$ *containing 1,* $\{\mu_n\}$ *be a strongly left regular sequence of means on X such that* $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$, *and* $\{c_n\}$ *be the sequence defined by (5). Let f be a contraction of C into itself with coefficient* $\alpha \in (0,1)$, Q_C *be a sunny nonexpansive retraction from E onto C, and A be an* β*-strongly accretive and L-Lipschitz continuous operator of C into E with* $\beta \geq \lambda K^2 L^2$ *such that* λ *be a positive real number. Suppose* $\mathcal{F} = F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$ *and the sequences* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ *and* $\{\delta_n\}$ *in* $(0,1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. *If the sequence* $\{x_n\}$ *is generated by* $x_1 \in C$ *and (7) such that they satisfy conditions (i) to (iv), then* $\{x_n\}$ *converges strongly to* $x^* \in \mathcal{F}$, *which is the unique solution of the following variational inequality:*

$$\langle (f-I)x^*, J(z-x^*)\rangle \leq 0, \forall z \in \mathcal{F}.$$

Equivalently, we have $x^* = Q_{\mathcal{F}} f(x^*)$, *where Q is the unique sunny nonexpansive retraction of C onto* $\mathcal{F}$.

Proof. Since A is a β-strongly accretive and L-Lipschitz continuous operator of C into E, we have the following:

$$\langle Ax-Ay, j(x-y)\rangle \geq \beta\|x-y\|^2 \geq \frac{\beta}{L^2}\|Ax-Ay\|^2, \forall x,y \in C.$$

Therefore, A is $\frac{\beta}{L^2}$-inverse strongly accretive. Using Theorem 1, we can obtain that $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Application to Hilbert spaces

Let C be a closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a mapping. The classical variational inequality problem is to find $x \in C$ such that

$$\langle Ax, y-x\rangle \geq 0 \tag{20}$$

for all $y \in C$.

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x-y\| \quad \text{for all} y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies the following:

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \tag{21}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{22}$$

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{23}$$

for all $x \in H, y \in C$.

It is well known in Hilbert spaces that the smooth constant $K = \frac{\sqrt{2}}{2}$ and $J = I$ (identity mapping). From Theorem 1, we can obtain the following result immediately.

Theorem 4. *Let C be a nonempty compact convex subset of a real Hilbert space H, S be a left reversible semigroup, and $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as Lipschitzian mappings from C into itself, with the uniform Lipschitzian condition $\lim_s k(s) \leq 1$. Let X be a left invariant $\mathcal{S}$-stable subspace of $l^\infty(S)$ containing 1, $\{\mu_n\}$ be a strongly left regular sequence of means on X such that $\lim_{n\to\infty}\|\mu_{n+1}-\mu_n\| = 0$ and $\{c_n\}$ be the sequence defined by (5). Let f be a contraction of C into itself with coefficient $\alpha \in (0,1)$, P_C be a nonexpansive mapping of H onto C, and $A : C \to H$ be a β-inverse strongly monotone with $\lambda \in (0, 2\beta)$. Suppose $\mathcal{F} = F(\mathcal{S}) \cap VI(C,A) \neq \emptyset$ and the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$, the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$.
(ii) $\lim_{n\to\infty}\delta_n = 0$.
(iii) $\limsup_{n\to\infty}\frac{c_n}{\alpha_n} \leq 0$, *(note that, by Remark 1,* $\limsup_{n\to\infty} c_n \leq 0$*).*
(iv) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

If for the arbitrary given $x_1 \in C$ the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = \delta_n x_n + (1-\delta_n)P_C(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n) y_n, \forall n \geq 1, \end{cases} \tag{24}$$

then $\{x_n\}$ converges strongly to $x^* = P_{\mathcal{F}} f(x^*) \in \mathcal{F}$, which is the unique solution of the following variational inequality:

$$\langle (f-I)x^*, z - x^*\rangle \leq 0, \forall z \in \mathcal{F}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
PK, SP, and PK contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Acknowledgements
This research was partially supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission. Phayap Katchang was supported by the Commission on Higher Education, the Thailand Research Fund, and the Rajamangala University of Technology Lanna Tak (grant MRG5580233).

Author details
[1]Department of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), Bang Mod, Bangkok 10140, Thailand. [2]Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand. [3]Department of Mathematics and Statistics, Faculty of Science and Agricultural Technology, Rajamangala University of Technology Lanna Tak, Tak 63000, Thailand.

Received: 25 March 2013 Accepted: 17 May 2013
Published: 26 May 2013

References
1. Xu, HK: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**, 1127–1138 (1991)
2. Reich, S: Asymptotic behavior of contractions in Banach spaces. J. Math. Anal. Appl. **44**, 57–70 (1973)
3. Kitahara, S, Takahashi, W: Image recovery by convex combinations of sunny nonexpansive retractions. Method Nonlinear Anal. **2**, 333–342 (1993)
4. Sahu, C, O'Regan, D: Convergence theorems for semigroup-type families of non-self mappings. Rend. Circ. Mat. Palermo. **57**, 305–329 (2008)
5. Holmes, RD, Lau, AT: Non-expansive actions of topological semigroups and fixed points. J. London Math. Soc. **5**, 330–336 (1972)
6. Holmes, RD, Lau, AT: Asymptotically non-expansive actions of topological semigroups and fixed points. Bull. London Math. Soc. **3**, 343–347 (1971)
7. Holmes, RD, Narayanaswamy, PP: On asymptotically nonexpansive semigroups of mappings. Canad. Math. Bull. **13**, 209–214 (1970)
8. Saeidi, S: Approximating common fixed points of Lipschitzian semigroup in smooth Banach spaces. Fixed Point Theory Appl. **2008**, 363257 (2008)
9. Kirk, B, Sims, W (eds): Handbook of Metric Fixed Point Theory. Kluwer Academic Publishers, Dordrecht (2001)
10. Takahashi, W: Nonlinear Functional Analysis: Fixed Point Theory and its Applications. Yokohama Publishers, Yokohama (2000)
11. Katchang, P, Kumam, P: A composite explicit iterative process with a viscosity method for Lipschitzian semigroup in a smooth Banach space. Bull. Iranian Math. Soc. **37**, 143–159 (2011)
12. Aoyama, K, Iiduka, H, Takahashi, W: Weak convergence of an iterative sequence for accretive operators in Banach spaces. Fixed Point Theory Appl. **2006**, 35390 (2006)

13. Kamimura, S, Takahashi, W: Approximating solutions of maximal monotone operators in Hilbert space. J. Approximation Theory. **106**, 226–240 (2000)
14. Kamimura, S, Takahashi, W: Weak and strong convergence of solutions to accretive operator inclusions and applications. Set-Valued Anal. **8**, 361–374 (2000)
15. Iiduka, H, Takahashi, W, Toyoda, M: Approximation of solutions of variational inequalities for monotone mappings. Panamerican Math. J. **14**, 49–61 (2004)
16. Yao, Y, Maruster, S: Strong convergence of an iterative algorithm for variational inequalities in Banach spaces. Math. Comput. Modell. **54**, 325–329 (2011)
17. Katchang, P, Kumam, P: An iterative algorithm for finding a common solution of fixed points and a general system of variational inequalities for two inverse strongly accretive operators. Positivity. **15**, 281–295 (2011)
18. Hirano, N, Kido, K, Takahashi, W: Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces. Nonlinear Anal. **12**, 1269–1281 (1988)
19. Saeidi, S: Existence of ergodic retractions for semigroups in Banach spaces. Nonlinear Anal. **69**, 3417–3422 (2008)
20. Takahashi, W: A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space. Proc. Amer. Math. Soc. **81**, 253–256 (1981)
21. Saeidi, S: Strong convergence of Browder's type iterations for left amenable semigroups of Lipschitzian mappings in Banach spaces. Fixed Point Theory Appl. **5**, 93–103 (2009)
22. Browder, FE: Nonlinear operators and nonlinear equations of evolution in Banach spaces, In: Nonlinear Functional Analysis, Proceedings of Symposia in Pure Mathematics Chicago April 1968, 18, part 2, pp.1–308. American Mathematical Society, Providence, (1976)
23. Suzuki, T: Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals. J. Math. Anal. Appl. **305**, 227–239 (2005)
24. Xu, HK: Viscosity approximation methods for nonexpansive mappings. J. Math. Anal. Appl. **298**, 279–291 (2004)
25. Yao, Y, Noor, MA, Noor, KI, Liou, Y-C, Yaqoob, H: Modified extragradient methods for a system of variational inequalities in Banach spaces. Acta Applicandae Math. **110**, 1211–1224 (2010)
26. Cho, YJ, Zhou, HY, Guo, G: Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings. Compt. Math. Appl. **47**, 707–717 (2004)
27. Atsushiba, S, Takahashi, W: Strong convergence of Mann's-type iterations for nonexpansive semigroups in general Banach spaces. Nonlinear Anal. **61**, 881–899 (2005)

doi:10.1186/2251-7456-7-28
Cite this article as: Kumam *et al.*: Viscosity approximation to a common solution of variational inequality problems and fixed point problems for Lipschitzian semigroup in Banach spaces. *Mathematical Sciences* 2013 **7**:28.