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General solution and generalized Hyers-Ulam stability of a two-variable Jensen functional equation

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Abstract

In this paper, we obtain the general solution of the following two-variable Jensen functional equation:

$$2F(\frac{x+u}{2}, \frac{y+v}{2}) = F(x,y) + F(u,v),$$

and investigate the generalized Hyers-Ulam stability.

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Introduction

Hyers in [1] investigated the stability problem of functional equations related to a question of Ulam [2] concerning the stability of group homomorphisms. Aoki in [3] generalized the results of Hyers for additive mappings. Also, Rassias [4] extended the same results to linear mappings by considering an unbounded cauchy difference. Later on, the stability problems of several functional equations have been investigated by a number of authors (see [5-10] and the references cited therein).

A mapping $f : X \to Y$ satisfying the equation

$$2f(\frac{x+y}{2}) = f(x) + f(y); \quad x, y \in X, \tag{1}$$

where X and Y are the space of real numbers and are called the Jensen function. The most general continuous solution of Equation 1 is $f(x) = ax + b$ where a and b are arbitrary constants. The stability of the Jensen equation has been investigated at first by Kominek [12]. Since then, further generalizations have been extensively investigated by a number of mathematicians (see [8,9,11]). In 2006, Bae and Park [15] obtained the generalized Hyers-Ulam stability of a bi-Jensen function. Moreover, the stability problem for the bi-Jensen functional equation was discussed by a number of authors (see [16,17]).

In this paper, we investigate the general solution and generalized Hyers-Ulam stability for a new two-variable Jensen functional equation of the following form:

$$2F(\frac{x+u}{2}, \frac{y+v}{2}) = F(x,y) + F(u,v) \quad x, y, u, v \in X, \tag{2}$$

where F is a mapping from X into Y.

Throughout this paper, we assume that X and Y are normed linear spaces. In the 'General solution of functional Equation 2' section this paper, we investigate the relationship between Equations 1 and 2; also, we obtain the general solution of the two-variable Jensen functional Equation 2. Finally, in 'Generalized Hyers-Ulam stability' section, we investigate the generalized Hyers-Ulam stability of the functional Equation 2.

Methods

General solution of functional Equation 2

In this section, we consider the two-variable Jensen functional Equation 2 and give a representation theorem for the general solution of Equation 2 in terms of Jensen functions, i.e., solutions of Equation 1. It is easy to see that if we let $F : X \times X \to Y$ be a mapping satisfying Equation 2 and $r : X \to Y$ be a mapping given by

$$r(x) = F(x,x),$$

for all $x \in X$, then r satisfies Equation 1.

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Moreover, we can prove that if $a, b \in \mathbb{R}$ and $r : X \to Y$ be a mapping satisfying Equation 1 and $F : X \times X \to Y$ is a mapping given by

$$F(x,y) = ar(x) + br(y),$$

for all $x, y \in X$, then F satisfies Equation 2.

Now, we are ready to investigate the relationship between the Jensen functions (1) and (2).

Theorem 1. $F : X \times X \to Y$ is a mapping that satisfies Equation 2 if and only if there exist two Jensen functions $r_1, r_2 : X \to Y$ such that

$$F(x,y) = r_1(x) + r_2(y) \tag{3}$$

for all $x, y \in X$.

Proof. Assume that F is a solution of Equation 2. If in Equation 2 we put $u = y = 0$, then we see that

$$2F(\frac{x}{2}, \frac{v}{2}) = F(x,0) + F(0,v).$$

It follows that

$$2F(x,v) = F(2x,0) + F(0,2v).$$

Define $r_1(x) = \frac{F(2x,0)}{2}$ and $r_2(y) = \frac{F(0,2y)}{2}$, for all $x, y \in X$, we claim that $r_1(x)$ and $r_2(y)$ are Jensen functions. From the definition of r_1, we have

$$\begin{aligned} 2r_1(\frac{x+y}{2}) &= F(\frac{2x+2y}{2}, \frac{0+0}{2}) \\ &= \frac{1}{2}F(2x,0) + \frac{1}{2}F(2y,0) \\ &= r_1(x) + r_1(y) \ \text{ forall } \ x, y \in X. \end{aligned}$$

It means that r_1 is a Jensen function. Similarly, we can prove that r_2 is a Jensen function.

Conversely, assume that there exist two Jensen functions $r_1, r_2 : X \to Y$ such that $F(x,y) = r_1(x) + r_2(y)$ for all $x, y \in X$. Hence,

$$\begin{aligned} 2F(\frac{x+u}{2}, \frac{y+v}{2}) &= 2r_1(\frac{x+u}{2}) + 2r_2(\frac{y+v}{2}) \\ &= r_1(x) + r_1(u) + r_2(y) + r_2(v) \\ &= r_1(x) + r_2(y) + r_1(u) + r_2(v) \\ &= F(x,y) + F(u,v), \end{aligned}$$

for all $x, u, y, v \in X$. □

Corollary 1. Consider the functional Equation 2 with $X = \mathbb{R}^n$ and $Y = \mathbb{R}$. Then the continuous general solution can be represented by the formula

$$F(x,y) = a.x + b.y + e,$$

where $a, b \in \mathbb{R}^n$ and $e \in \mathbb{R}$.

Proof. Let F be a continuous solution of Equation 2. Theorem 1 implies that there exist two Jensen functions $r_1, r_2 : \mathbb{R}^n \to \mathbb{R}$ such that

$$F(x,y) = r_1(x) + r_2(y)$$

for all $x, y \in \mathbb{R}^n$. Since r_1 and r_2 are continuous solutions of the Jensen functional equation such that $r_1(x) = a.x + c$ and $r_2(x) = b.x + d$ for some $a, b \in \mathbb{R}^n$ and $c, d \in \mathbb{R}$; hence, we have

$$\begin{aligned} F(x,y) &= a.x + c + b.y + d \\ &= a.x + b.y + c + d \\ &:= a.x + b.y + e. \end{aligned}$$

□

Generalized Hyers-Ulam stability

In this section, we investigate the generalized Hyers-Ulam stability for the functional Equation 2.

Theorem 2. Let Y be a Banach space and $F : X^2 \to Y$ is a mapping for which there exists a function $\Phi : X^4 \to Y$ with the condition

$$\lim_{n\to\infty} 3^{-n}\Phi(3^{-n}x, 3^{-n}x, 3^{-n}y, 3^{-n}y) = 0, \tag{4}$$

such that the function inequality

$$\|2F(\frac{x+u}{2}, \frac{y+v}{2}) - F(x,y) - F(u,v)\| \leq \frac{1}{3}\Phi(x,u,y,v) \tag{5}$$

holds for all $x, u, y, v \in X$. Then there exists a unique two-variable Jensen mapping $J : X^2 \to Y$ satisfying the functional Equation 2 and

$$\|F(x,y) - J(x,y)\| \leq \sum_{i=0}^{\infty} 3^{-i-1}\Phi\left(3^{-i-1}x, 3^{-i-1}x, 3^{-i-1}y, 3^{-i-1}y\right), \tag{6}$$

for all $x, y \in X$. The mapping $J(x,y)$ is define by

$$J(x,y) = \lim_{n\to\infty} 3^{-n}F(3^{-n}x, 3^{n}y)$$

for all $x, y \in X$.

Proof. If in inequality (5) we replace x, u, y, v with $2x, 0, 2y, 0$, respectively, then we observe that

$$\|2F(x,y) - F(2x,2y)\| \leq \frac{1}{3}\Phi(2x,0,2y,0). \tag{7}$$

If in inequality (5) we replace (x,u,y,v) with $(3x,x,3y,y)$, respectively, we obtain

$$\|2F(2x,2y) - F(3x,3y) - F(x,y)\| \leq \frac{1}{3}\Phi(3x,x,3y,y). \tag{8}$$

Combining Equations 7 and 8, we see that

$$\|3F(x,y)-F(3x,3y)\| \leq \frac{2}{3}\Phi(2x,0,2y,0)+\frac{1}{3}\Phi(3x,x,3y,y),$$

and so

$$\|F(x,y)-\frac{1}{3}F(3x,3y)\| \leq \frac{2}{9}\Phi(2x,0,2y,0)+\frac{1}{9}\Phi(3x,x,3y,y). \tag{9}$$

Now, replacing (x,y) by $(3x,3y)$ in Equation 9, dividing by 3, and adding with Equation 9, we obtain

$$\begin{aligned}&\|F(x,y)-\frac{1}{9}F(9x,9y)\|\\ &\leq \sum_{i=0}^{1}\frac{2}{3^{i+2}}\Phi(2\times 3^i x,0,2\times 3^i y,0)\\ &\quad+\sum_{i=0}^{1}\frac{1}{3^{i+2}}\Phi(3^{i+1}x,3^i x,3^{i+1}y,3^i y).\end{aligned}$$

By induction on a positive integer n, we see that

$$\begin{aligned}&\|F(x,y)-\frac{1}{3^n}F(3^n x,3^n y)\|\\ &\leq \sum_{i=0}^{n-1}\frac{2}{3^{i+2}}\Phi(2\times 3^i x,0,2\times 3^i y,0)\\ &\quad+\sum_{i=0}^{n-1}\frac{1}{3^{i+2}}\Phi(3^{i+1}x,3^i x,3^{i+1}y,3^i y)\\ &\leq \sum_{i=0}^{\infty}\frac{2}{3^{i+2}}\Phi(2\times 3^i x,0,2\times 3^i y,0)\\ &\quad+\sum_{i=0}^{\infty}\frac{1}{3^{i+2}}\Phi(3^{i+1}x,3^i x,3^{i+1}y,3^i y)\end{aligned} \tag{10}$$

for all $x,y\in X$. In order to prove the convergence of the sequence $\{\frac{1}{3^n}F(3^n x,3^n y)\}$, we replace (x,y) by $(3^p x,3^p y)$, in inequality (10) and dividing by 3^p, we find that for $n>p>0$

$$\begin{aligned}&\|\frac{1}{3^p}F(3^p x,3^p y)-\frac{1}{3^{n+p}}F(3^{n+p}x,3^{n+p}y)\|\\ &\quad=\frac{1}{3^p}\|F(3^p x,3^p y)-\frac{1}{3^n}F(3^{n+p}x,3^{n+p}y)\|\\ &\quad\leq \sum_{i=0}^{\infty}\frac{2}{3^{i+p+2}}\Phi(2\times 3^{i+p}x,0,2\times 3^{i+p}y,0)\\ &\quad\quad+\sum_{i=0}^{\infty}\frac{1}{3^{i+p+2}}\Phi(3^{i+p+1}x,3^{i+p}x,3^{i+p+1}y,3^{i+p}y)\\ &\quad\quad\to 0 \ \ as\ \ p\to\infty.\end{aligned}$$

This means that the sequence $\{\frac{1}{3^n}F(3^n x,3^n y)\}$ is a cauchy sequence. Since Y is a Banach space, the sequence $\{\frac{1}{3^n}F(3^n x,3^n y)\}$ is converged to some $J(x,y)$. Thus, we can well define a mapping $J: X^2\to Y$ by

$$J(x,y)=\lim_{n\to\infty}3^{-n}F(3^{-n}x,3^n y).$$

Next, we will show that J satisfies Equation 2, hence it is a bi-Jensen function. Let $x,y,u,v\in X$, then we have

$$\begin{aligned}&\|2J(\frac{x+u}{2},\frac{y+v}{2})-J(x,y)-J(u,v)\|\\ &\quad=\lim_{n\to\infty}\frac{1}{3^n}\|2F(3^n(\frac{x+u}{2}),3^n(\frac{y+v}{2}))\\ &\quad\quad-F(3^n x,3^n y)-F(3^n u,3^n v)\|\\ &\quad\leq \lim_{n\to\infty}\frac{1}{3^n}\Phi(3^n x,3^n u,3^n y,3^n v)=0.\end{aligned} \tag{11}$$

So J satisfies Equation 2.

Next we claim that T is unique. Let us assume that there exists another bi-Jensen mapping J_1 from X^2 into Y, which satisfies the required inequality. Now, if we use inequality (6), then we obtain

$$\begin{aligned}&\|J(x,y)-J_1(x,y)\|\\ &\quad=3^{-n}\|J(3^{-n}x,3^{-n}y)-J_1(3^{-n}x,3^{-n}y)\|\\ &\quad\leq 3^{-n}\|F(3^{-n}x,3^{-n}y)-J(3^{-n}x,3^{-n}y)\|\\ &\quad\quad+3^{-n}\|J_1(3^{-n}x,3^{-n}y)-F(3^{-n}x,3^{-n}y)\|\\ &\quad\leq 2\sum_{i=0}^{\infty}3^{-i-n-1}\Phi(3^{-i-n-1}x,3^{-i-n-1}x,\\ &\quad\quad 3^{-i-n-1}y,3^{-i-n-1}y),\end{aligned}$$

for all $(x,y)\in X^2$, which tends to become 0 as $n\to\infty$. It follow that $J(x,y)=J_1(x,y)$, for all $(x,y)\in X$. □

Competing interests
Both authors declare that they have no competing interests.

Authors' contributions
Both author contributed equally in the development of this manuscript. Both authors read and approved the final version of this manuscript.

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