Journal of Mathematical Extension Vol. 1, NO. 1, (2006), 21-26

## Limit Points of Trigonometric Sequences

M. Faghih Ahmadi<sup>1</sup> Islamic Azad University - Sepidan Branch

> **K. Hedayatian** Shiraz University

**Abstract:** In this article, we find the set of all limit points of sequences of polynomials with real coefficients, in  $\cos n$ ,  $n = 1, 2, 3, \cdots$  with degree less than or equal to three. Also, when the degree is four, the mentioned set is found in some special cases.

**AMS Subject Classification:** 40-XX. **Key words and Phrases:** Limit points, trigonometric, sequences.

# 1. Introduction

Finding the limit points of a sequences or, at least, finding some topological properties of the limit points of a sequence is one of the remarkable problems in analysis. For instance, in [2], the authors have found some necessary and sufficient conditions for the connectedness of the set of all limit points of a sequence in a metric space. Some other results on the limit points of certain sequences is obtained, for example, in [3] and [4].

<sup>&</sup>lt;sup>1</sup>Research of the first author is supported by a grant from Islamic Azad University

Our claim in this article is to find the set of all limit points of sequence of polynomials with real coefficients, in  $\cos n, n = 1, 2, 3, \cdots$  with degree less than or equal to three. Also, when the degree is four, the mentioned set is found in some cases.

## 2. Main Results

**Theorem 1.** Suppose f is a real valued continuous, periodic function on the real numbers  $\mathbb{R}$  and its period is an irrational number  $\alpha$ . Then the set of all limit points of the sequence  $\{f(n)\}_{-\infty}^{+\infty}$  is the closed interval [m, M] where  $m = Min\{f(x) : x \in \mathbb{R}\}$  and  $M = Max\{f(x) : x \in \mathbb{R}\}$ .

**Proof.** Since f is continuous and periodic, it is uniformly continuous. So for  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for every x, y in  $\mathbb{R}$ , if  $|x - y| < \delta$ then  $|f(x) - f(y)| < \varepsilon$ . But the set  $\mathbb{Z} + \alpha \mathbb{Z} = \{m + \alpha n : m \in \mathbb{Z}, n \in \mathbb{Z}\}$ is a countable dense subset of  $\mathbb{R}$  where  $\mathbb{Z}$  denotes the set of all integers. Therefore, for each  $x \in \mathbb{R}$ , integers m and n can be found so that

$$|m - (n\alpha + x)| < \delta,$$

and consequently,  $|f(m) - f(x)| < \varepsilon$ . Now, considering the fact that  $f(\mathbb{R})$  is a connected subset of  $\mathbb{R}$ , the result follows.  $\Box$ 

We remark that for an irrational number  $\alpha$ ,  $\mathbb{N} + \alpha \mathbb{Z}$  is not dense

in  $\mathbb{R}$  where  $\mathbb{N}$  denotes the natural numbers and so this proof can not be used when replacing  $\{f(n)\}_{-\infty}^{+\infty}$  by  $\{f(n)\}_{n=1}^{\infty}$ . Nevertheless, a direct conclusion of the above theorem runs as follows:

**Theorem 2.** Let the function f satisfy the hypotheses of the preceding theorem. Suppose, furthermore, that f is an even function. Then the set of limit points of the sequence  $\{f(n)\}_{n=1}^{\infty}$  is the range of f.

set of limit points of the sequence  $\{f(n)\}_{n=1}^{\infty}$  is the range of f. In all that follows, for a sequence  $\{p(n)\}_{n=1}^{\infty}$  let  $L_p$  be the set of all limit points of this sequence.

**Theorem 3.** Let  $q(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ , where  $a_3 \neq 0$ , and take  $p(n) = q(\cos n)$ . If  $a_2^2 - 3a_1a_3 < 0$  then  $L_p = [m, M]$  where m and M are, respectively, the minimum and maximum of the set  $\{q(1), q(-1)\}$ . If  $a_2^2 - 3a_1a_3 \ge 0$  then  $L_p = [m, M]$  where m and M are, respectively, the minimum and maximum of the set

$$\{q(1), q(-1), q(\frac{-a_2 + \sqrt{a_2^2 - 3a_1a_3}}{3a_3}), q(\frac{-a_2 - \sqrt{a_2^2 - 3a_1a_3}}{3a_3})\}$$

**Proof.** Consider the function p defined on  $[0, 2\pi]$  by  $p(x) = q(\cos x)$ . Then p(x) is clearly even and periodic, allowing us to use Theorem 2; it remains to find the range of p. If  $a_2^2 - 3a_1a_3 < 0$  then p'(x) = 0 implies that  $x = 0, \pi, 2\pi$ , and so the only values that  $\cos x$  can take are 1 and -1. On the other hand, when  $a_2^2 - 3a_1a_3 \ge 0$ , an easy argument shows that if p'(x) = 0 then  $\cos x$  can be

$$1, -1, \frac{-a_2 + \sqrt{a_2^2 - 3a_1a_3}}{3a_3}, \text{ or } \frac{-a_2 - \sqrt{a_2^2 - 3a_1a_3}}{3a_3}.$$

**Theorem 4.** Let  $q(x) = a_0 + a_1x + a_2x^2$  for  $a_2 \neq 0$ , and let  $p(n) = q(\cos n)$ . Then  $L_p = [m, M]$  where m and M are, respectively, the minimum and maximum of the set

$$\{a_0 + a_1 + a_2, a_0 - a_1 + a_2, a_0 - \frac{a_1^2}{4a_2}\}$$

**Proof.** Considering  $p(x) = a_0 + a_1 \cos x + a_2 \cos^2 x$ ,  $x \in [0, 2\pi]$ ; it is sufficient to find x in the interval  $[0, 2\pi]$  such that p'(x) = 0. Then apply Theorem 2.

Theorem 5. Let

$$q(x) = a_0 + a_1 a_3 x + \frac{a_2 a_3}{2} x^2 + \frac{a_1 a_4}{3} x^3 + \frac{a_2 a_4}{4} x^4,$$

where  $a_2a_4 \neq 0$ ; and for  $n \in \mathbb{N}$ , take  $p(n) = q(\cos n)$ . If  $a_3a_4 \leq 0$ then  $L_p = [m, M]$  where m and M are, respectively, the minimum and

24

maximum of the set

$$\{q(1), q(-1), q(\pm \sqrt{-\frac{a_3}{a_4}}), q(-\frac{a_1}{a_2})\}$$

and if  $a_3a_4 > 0$  then we use the set  $\{q(1), q(-1), q(-\frac{a_1}{a_2})\}$ .

**Proof.** If  $\frac{d}{dx}(q(\cos x)) = 0$  then  $\sin x = 0$  or

$$(a_4\cos^2 x + a_3)(a_2\cos x + a_1) = a_1a_3 + a_2a_3\cos x$$

 $+a_1a_4\cos^2 x + a_2a_4\cos^3 x = 0.$ 

Consequently, if the inequality  $a_3a_4 \leq 0$  holds, we get  $\sin x = 0$  or  $\cos x = \pm \sqrt{-a_3/a_4}$  or  $\cos x = -a_1/a_2$ . Also, whenever  $a_3a_4 > 0$  we get  $\sin x = 0$  or  $\cos x = -a_1/a_2$ . In each case, the result holds from Theorem 2.  $\Box$ 

**Remark 1.** An immediate consequence of Theorem 2, is that the set of limit points of the sequence  $\{\cos n\}_{n=1}^{\infty}$  is [-1,1]. This fact has been proved before, using more complicated techniques. For instance, one can see [1, Problem 3.15, p.14].

**Remark 2.** In Theorems 3, 4 and 5, substituting  $\cos n$  by  $\sin n$ , one can show that the same results hold for the sequence  $\{\sin n\}_{-\infty}^{+\infty}$ .

25

#### LIMIT POINTS OF TRIGONOMETRIC SEQUENCES

#### References

- D. Aliprantis and O. Burkinshaw, Problems in Real Analysis, Academic Press, Inc. 1990.
- [2] M. D. A\V si\'c, and D.D. Admaovi \'C, Limit points of sequences in metric spaces, Amer. Math. Monthly 77 (1970), 613-616.
- [3] H. G. Barone, Limit points of sequences and their transforms by methods of summability, Duke Math. J. 5, (1939), 740-752.
- [4] P. Hellekalek and G. Larcher, On Weyl sums and skew product over irrational rotations, *Theoretical Computer science*, Vol. 65, Issue 2, (1989), 189-196.

#### Masoumeh Faghih Ahmadi

Islamic Azad University - Sepidan Branch Sepidan, Iran E-mail: faghiha@shirazu.ac.ir E-mail: m\_faghih\_a@yahoo.com

### Karim Hedayatian

Department of Mathematics College of Sciences Shiraz University Shiraz 71454, Iran E-mail: hedayatian@susc.ac.ir