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On the Weighted Hardy Spaces

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Abstract: Let $\left\{\beta(n)\right\}_{n=0}^{\infty}$ be a sequence of positive numbers and $1 < p < \infty$. We consider the weighted Hardy space $H^p(\beta)$. We investigate the relation between the generating function and the functional of point evaluations. Also, under a sufficient condition we determine the structure of all non-zero multiplicative linear functionals on $H^p(\beta)$.

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1. Introduction

First in the following, we generalize the definitions coming in [3]. Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$ and $1 < p < \infty$. We consider the space of sequences $f = \left\{\hat{f}(n)\right\}_{n=0}^{\infty}$ such that

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ shall be used whether or not the series converges for any value of z. These are called formal power series. Let $H^p(\beta)$ denotes the space of such formal power series. It is usually called as weighted Hardy spaces. These are reflexive Banach spaces with the norm $\|.\|_\beta$ and the dual of $H^p(\beta)$ is $H^q(\beta^{\frac{p}{q}})$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $\beta^{\frac{p}{q}} = \{\beta(n)^{\frac{p}{q}}\}_n$ ([4]). Also if

$$g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n \in H^q(\beta^{\frac{p}{q}}),$$

then

$$\|g\|^q = \sum_{n=0}^{\infty} |\hat{g}(n)|^q \beta(n)^p.$$

The Hardy, Bergman and Dirichlet spaces can be viewed in this way when $p = 2$ and respectively $\beta(n) = 1, \beta(n) = (n+1)^{-1/2}$ and $\beta(n) = (n+1)^{1/2}$. It is convenient and helpful to introduce the notation $\langle f, g \rangle$ to stand for $g(f)$ where $f \in H^p(\beta)$ and $g \in H^p(\beta)^*$. Note that

$$\langle f, g \rangle = \sum_{n=0}^{\infty} \hat{f}(n) \overline{\hat{g}(n)} \beta(n)^p.$$

Let $\hat{f}_k(n) = \delta_k(n)$. So $f_k(z) = z^k$ and then $\{f_k\}_k$ is a basis such that $\|f_k\| = \beta(k)$. Clearly M_z, the multiplication operator by z on $H^p(\beta)$ shifts the basis $\{f_k\}_k$.

Remember that a complex number λ is said to be a bounded point evaluation on $H^p(\beta)$ if the functional of point evaluation at λ, e_λ, is bounded. The functional of evaluation of the j-th derivative at λ is dentoed by $e_\lambda^{(j)}$. These spaces are also studied in [1, 2, 4, 5, 6, 7, 8].

If Ω is a bounded domain in the complex domain $\mathbb{C}$, then by $H(\Omega)$ we mean the set of analytic functions on Ω. We will denote the open unit disc by U.

2. Main Results

We will investigate the relation between the generating function and the functional of point evaluations on $H^p(\beta)$. Also we will determine the structure of all non-zero linear functionals on $H^p(\beta)$ that are multiplicative. This extends some results of [1] into Banach spaces of formal power series. The differential of functionals of point evaluations are also considered.

Definition 1. *The generating function for the weighted Hardy space $H^p(\beta)$ is the function*

$$g(z) = \sum_{n=0}^{\infty} \frac{z^n}{\beta(n)^q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Lemma 2. *If g is the generating function for a weighted Hardy space contained in $H(U)$, then $g \in H(U)$.*

Proof. Let g be the generating function for the weighted Hardy space $H^p(\beta)$ where $H^p(\beta) \subset H(U)$. Define

$$\hat{f}(n) = \begin{cases} 0 & n = 0 \\ \dfrac{1}{n\beta(n)} & n \neq 0 \end{cases}$$

and let $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$. Now we have

$$\sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p = \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty.$$

So $f \in H^p(\beta)$ and by assumption, it is analytic in the open unit disk U. Thus the radius of convergence of its power series, R, is at least 1. Thus

$$\frac{1}{R} = \limsup_{n \longrightarrow \infty} |\hat{f}(n)|^{\frac{1}{n}} = \limsup_{n \longrightarrow \infty} \left(\frac{1}{n\beta(n)}\right)^{\frac{1}{n}} \leqslant 1,$$

and so

$$\limsup_{n} \left(\frac{1}{\beta(n)^q}\right)^{\frac{1}{n}} = \limsup_{n} \left[\left(\frac{1}{n\beta(n)}\right)^{\frac{1}{n}}\right]^q \leqslant 1,$$

which implies that $g \in H(U)$. This completes the proof. $\square$

The next theorem shows the principle role of g: it generates the reproducing kernels for $H^p(\beta)$.

Lemma 3. *Let $H^p(\beta)$ be a weighted Hardy space contained in $H(U)$. For each point λ in U, the functional of evaluation at λ, e_λ, is a bounded linear functional and $\|e_\lambda\|^q = g(|\lambda|^q)$.*

Proof. For $|\lambda| < 1$, the analyticity of g on U implies that e_λ is in $H^q(\beta^{\frac{p}{q}})$. Indeed,

$$\|e_\lambda\|^q = \|\sum_{n=0}^{\infty} \frac{\bar{\lambda}^n}{\beta(n)^p} z^n\|^q = \sum_{n=0}^{\infty} \frac{|\lambda|^{nq}}{\beta(n)^q} = g(|\lambda|^q) < \infty.$$

This completes the proof. □

Theorem 4. *If g, the generating function for $H^p(\beta)$, satisfies $g(1) = \infty$, then each non-zero bounded linear functional H on $H^p(\beta)$ such that $\langle fh, H\rangle = \langle f, H\rangle\langle h, H\rangle$ whenever f, h and fh are in $H^p(\beta)$ is in the form of $H = e_\lambda$ for some point λ in U.*

Proof. Suppose $H \in H^p(\beta)^*$ such that H is *non*-zero and satisfies

$$\langle fh, H\rangle = \langle f, H\rangle\langle h, H\rangle$$

whenever f, h and fh are in $H^p(\beta)$. Since the polynomials are dense in $H^p(\beta)$, this holds when f and h are polynomials.

Also, we note that

$$\langle f, H\rangle = \langle f_0 f, H\rangle = \langle f_0, H\rangle\langle f, H\rangle$$

for all f in $H^p(\beta)$, hence $\langle f_0, H\rangle = 1$. Letting $\lambda = \langle f_1, H\rangle$, it follows that

$$\langle f_2, H\rangle = \langle f_1 f_1, H\rangle = \langle f_1, H\rangle^2 = \lambda^2.$$

By induction, $\langle f_n, H\rangle = \lambda^n$ for all $n \in \mathbb{N}$. Remember that if

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^p(\beta)$$

and

$$G(z) = \sum_{n=0}^{\infty} \hat{G}(n) z^n \in H^q(\beta^{\frac{p}{q}}),$$

then

$$\langle f, G\rangle = \sum \hat{f}(n)\overline{\hat{G}(n)}\beta(n)^p.$$

Now if $H(z) = \sum_{n=0}^{\infty} \hat{H}(n) z^n$, we have

$$1 = \langle f_0, H\rangle = \overline{\hat{H}(0)}\beta(0)^p = \overline{\hat{H}(0)},$$

$$\lambda = \langle f_1, H\rangle = \overline{\hat{H}(1)}\beta(1)^p,$$

$$\vdots$$

$$\lambda^n = \langle f_n, H\rangle = \overline{\hat{H}(n)}\beta(n)^p.$$

So if $|\lambda| < 1$, then

$$H(z) = 1 + \sum_{n=1}^{\infty} \frac{\overline{\lambda^n}}{\beta(n)^p} z^n = e_\lambda(z),$$

and so H is the linear functional of evaluation at λ. If $|\lambda| \geqslant 1$, then we get

$$\|H\|^q = \sum_{n=0}^{\infty} \frac{|\lambda|^{nq}}{\beta(n)^q} \geqslant \sum_{n=0}^{\infty} \frac{1}{\beta(n)^q} = g(1) = \infty,$$

which contradicts the boundedness of the linear functional determined by H. This completes the proof. □

As we saw, for λ in U,

$$e_\lambda(z) = \sum_{n=0}^{\infty} \frac{1}{\beta(n)^p} (\bar{\lambda})^n z^n \in H^q(\beta^{p/q}).$$

This is not true in general if λ is on the unit circle.

Lemma 5. *Let $j \in \mathbb{N} \cup \{0\}$. If $\sum_{n=j}^{\infty} \frac{n^{qj}}{\beta(n)^q} < \infty$, then $e_\lambda^{(j)} \in H^q(\beta^{p/q})$ and $\langle f, e_\lambda^{(j)} \rangle = f^{(j)}(\lambda)$ for all λ in $\bar{U}$ and f in $H^p(\beta)$.*

Proof. For λ in $\bar{U}$ we have

$$e_\lambda^{(j)}(z) = \sum_{n=j}^{\infty} \frac{n(n-1)\cdots(n-j+1)}{\beta(n)^p} (\bar{\lambda})^{n-j} z^n,$$

and so

$$\begin{aligned} \|e_\lambda^{(j)}(z)\|^q &= \sum_{n=j}^{\infty} (\frac{n(n-1)\cdots(n-j+1)}{\beta(n)^p})^q |\lambda|^{(n-j)q} \beta(n)^p \\ &\leqslant \sum_{n=j}^{\infty} \frac{(n(n-1)\cdots(n-j+1))^q}{\beta(n)^q} \end{aligned}$$

$$\leqslant \sum_{n=j}^{\infty} \frac{n^{qj}}{\beta(n)^q} < \infty.$$

Thus $e_\lambda^{(j)} \in H^q(\beta^{p/q})$. Now if

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \in H^p(\beta),$$

then by a Theorem in [4], for all λ in $\bar{U}$ we have:

$$\begin{aligned} \langle f, e_\lambda^{(j)} \rangle &= \sum_n \hat{f}(n) \overline{(\widehat{e_\lambda^{(j)}}(n))} \beta(n)^p \\ &= \sum_{n=j}^{\infty} \hat{f}(n) \frac{n(n-1)\cdots(n-j+1)}{\beta(n)^p} \lambda^{n-j} \beta(n)^p \\ &= \sum_{n=j}^{\infty} n(n-1)\cdots(n-j+1) \hat{f}(n) \lambda^{n-j} = f^{(j)}(\lambda). \end{aligned}$$

This completes the proof. □

Note that if $\sum_{n=j}^{\infty} \frac{n^{qj}}{\beta(n)^q} < \infty$, then $H^p(\beta)$ is very small and for every function in $H^p(\beta)$, the j-th derivative exists and is continuous on the unit circle.

Corollary 6. *If* $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^q} < \infty$, *then the generating function belongs to* $H^p(\beta)$.

Proof. Let g be the generating function. Thus $g(z) = \sum_n \frac{1}{\beta(n)^q} z^n$ and

$$\|g\|^q = \sum_n |\hat{g}(n)|^p \beta(n)^p = \sum_n \frac{1}{\beta(n)^q} < \infty.$$

So $g \in H^p(\beta)$. $\square$

Corollary 7. *If* $\sum_n \frac{1}{\beta(n)^q} < \infty$ *and* $\liminf_n \beta(n)^{1/n} = 1$, *then the generating function is in* $H(U)$.

Proof. Clearly we can see that $H^p(\beta) \subset H(U)$ and so by the Lemma 2, the proof is complete. $\square$

Theorem 8. *In the weighted Hardy space* $H^p(\beta)$ *for which* $g(1) = \infty$ *for all integer* $j \geqslant 0$, *the normalized functional of point evaluations,* $\frac{e_{w_n}^{(j)}}{\|e_{w_n}^{(j)}\|}$, *tends to zero weakly as* $w_n \longrightarrow \xi \in \partial U$.

Proof. For $j = 0$, the norm of the functional of point evaluations are given by the generating function g and indeed $\|e_{w_n}\|^q = g(|w_n|^q)$. Since

$$\lim_n \ g(|w_n|^q) = g(1) = \sum_n \frac{1}{\beta(n)^q} = \infty,$$

it follows that $\|e_{w_n}\|$ tends to infinity and for every polynomial p,

$$\lim_m \ |\langle p, \frac{e_{w_n}}{\|e_{w_n}\|}\rangle| = \lim_n \ \frac{|p(w_n)|}{\|e_{w_n}\|} = 0.$$

But the polynomials are dense in $H^p(\beta)$, thus $\frac{e_{w_n}}{\|e_{w_n}\|} \longrightarrow 0$ weakly as $n \longrightarrow \infty$. If $j > 0$, then

$$\lim_n \ \|e_{w_m}^{(j)}\|^q \ = \ \lim_m \ \sum_{n=j}^{\infty} (\frac{n(n-1)\cdots(n-j+1)}{\beta(n)})^q |w_m|^{q(n-j)}$$

$$= \sum_{n=j}^{\infty} \left(\frac{n(n-1)\cdots(n-j+1)}{\beta(n)}\right)^q$$
$$\geqslant \sum_{n=j}^{\infty} \frac{1}{\beta(n)^q} = \infty.$$

Thus $\|e_{w_m}^{(j)}\|$ tends to infinity and for every polynomial p,

$$\lim_m \left|\langle p, \frac{e_{w_m}^{(j)}}{\|e_{w_m}^{(j)}\|}\rangle\right| = \lim_m \frac{|p^{(j)}(w_m)|}{\|e_{w_m}^{(j)}\|} = 0.$$

Since the polynomials are dense in $H^p(\beta)$, the proof is complete.□

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