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## Irrational Rotation Algebra as

## a Crossed Product

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**Abstract:** In this paper we will consider the crossed product  $C(T) \times_{\alpha} \mathbf{Z}$ , where T is the unit circle,  $\alpha(n) = \alpha_n$  is a rotation through the angle  $-2\pi n\theta$  for  $n \in \mathbf{Z}$ , and  $\theta$  is a fixed irrational number. We will apply some results about patial actions to represent this crossed product as a  $C^*$ -subalgebra of  $B(L^2(T))$ . Also, by a different method form the proof of Davidson, we show that this crossed product is isomorphic to the irrational rotation algebra.

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# 1. Introduction

Let G be a discrete group and  $\theta = (\{\theta_t\}, \{U_t\})_{t \in G}$  be a partial homeomorphism [3] of a locally compact space X. Put  $D_t = C_0(U_t)$  and define  $\alpha_t : D_{t^{-1}} \to D_t$  by

 $\alpha_t(f)(x) := f(\theta_{t^{-1}}(x))$ , for  $f \in D_{t^{-1}}$  and  $x \in U_t$ .

Then  $\alpha = (\{\alpha_t\}, \{D_t\})_{t \in G}$  is a partial action of G on the  $C^*$ -algbera  $C_0(X)$  in the sense of [2] and [6], which is called the partial action of G on  $C_0(X)$  corresponding to  $\theta$  ([4]).

**Definition 1.1.** ([4]) The partial dynamical system  $(C_0(X), G, \alpha)$  is topologically free if for every  $t \in G \setminus \{e\}$ , the set

$$F_t := \{ x \in U_{t^{-1}} : \theta_t(x) = x \}$$

has empty interior.

The concepts of reduced and full crossed products for actions are generalized by McClanahan in [6] to parital actions. It is surprising that in some situations the faithfulness of a representation of the reduced crossed product  $C_0(X) \times_r G$  depends only on that of  $C_0(X)$ . In this relation we bring the following theorem. For the proof see [4, Theorem 2.6].

**Theorem 1.2.** Suppose  $(C_0(X), G, \alpha)$  is topologically free. A representation of the reduced crossed product  $C_0(X) \times_r G$  is faithful if and only if it is faithful on  $C_0(X)$ .

We remark that when G is an amenable group (especially when G is abelian), the reduced and full crossed products are identified with each other, and so in this case, the preceding theorem is valid for the full crossed product, with a similar proof.

## 2. Main Result

Fix an irrational number  $\theta$ . Let  $R_{\theta_n}$  be the rotation through the angle  $2\pi n\theta$ . That is,  $R_{\theta_n}: T \to T$  is defined by  $R_{\theta_n}(z) = ze^{2\pi i n\theta}$  for  $n \in \mathbb{Z}$ . So the map  $R_{\theta}: n \mapsto R_{\theta_n}$  is a partial action (indeed, an action) on  $\mathbb{Z}$ . It is clear that  $R_{\theta_n}$  is a homeomorphism on the compact space T. Thus,  $R_{\theta}$  is a partial homeomorphism.

Now, let  $\alpha$  be the partial action of  $\mathbb{Z}$  on  $C_0(T) = C(T)$  corresponding to  $R_{\theta}$ . So  $\alpha_n : C(T) \to C(T)$  is defined by

$$\alpha_n(f)(z) = f(R_{\theta_{-n}}(z)) = f(ze^{-2\pi i n\theta})$$

for  $f \in C(T)$  and  $z \in T$ . Note that  $\alpha$  is an action in this case.

To identify the crossed product  $C(T) \times_{\alpha} \mathbb{Z}$  ([1], [6]), more explicitly, first we find a faithful representation of  $C(T) \times_{\alpha} \mathbb{Z}$ . Indeed, we represent the crossed product as a  $C^*$ -subalgebra of  $B(L^2(T))$ . Let  $M : C(T) \to$  $B(L^2(T))$  be given by

$$M_f(g) = fg$$

for  $f \in C(T)$  and  $g \in L^2(T)$ . Also, define  $\lambda : \mathbb{Z} \to B(L^2(T))$  by

$$\lambda_n(\xi)(z) = \xi(ze^{-2\pi i n\theta})$$

for  $\xi \in L^2(T)$  and  $z \in T$ .

Note that  $\lambda_n^* = \lambda_{-n}$  and  $\lambda_n = \lambda_1^n$ . It is clear that M is a nondegenerate representation and  $\lambda$  is a unitary representation. Also it can be easily verified that

$$M(\alpha_n(f)) = \lambda_n o M_f o \lambda_n^*$$

for  $f \in C(T)$ . Therefore,  $(M, \lambda, L^2(T))$  is a covariant representation of the  $C^*$ -dynamical system  $(C(T), \mathbf{Z}, \alpha)$ . By the correspondence between the covariant representations of a partial action and the representations of the associated crossed product [6], we conclude that  $M \times \lambda$  is a representation of  $C(T) \times_{\alpha} \mathbf{Z}$ . Since  $\mathbf{Z}$  is an abelian group, we can identify  $C(T) \times_{\alpha} \mathbf{Z}$  with  $C(T) \times_r \mathbf{Z}$ . On the other hand, M is faithful on C(T). So Theorem 1.2 implies that  $M \times \lambda$  is a faithful representation. Note that Theorem 1.2 can be used because for every irrational  $\theta, R_{\theta}$  is topologically free. In fact, for  $n \in \mathbf{Z} \setminus \{0\}, F_n = \{z \in T : ze^{-2\pi i n \theta} = z\} = \emptyset$ because  $\theta$  is an irrational number and so  $e^{2\pi i n \theta} \neq 1$ .

We know that  $(M \times \lambda)(f\delta_n) = M_f\lambda_n = M_f\lambda_1^n$ , where  $f\delta_n$  is a

generator of  $C(T) \times_{\alpha} \mathbf{Z}$ . Since  $\iota(z) = z$ , generates the  $C^*$ -algebra C(T)and 1 generates the group  $\mathbf{Z}$ , we have  $(M \times \lambda)(C(T) \times_{\alpha} \mathbf{Z}) = C^*(M_z, \lambda_1)$ .

We can summarize the above discussions in the following lemma.

**Lemma 2.1.** Assume that  $M_z$  and  $\lambda_1$  are defined as following

$$M_z(g) = zg$$

for  $g \in L^2(T)$ , and

$$\lambda_1(\xi)(z) = \xi(ze^{-2\pi i\theta})$$

for  $\xi \in L^2(T)$  and  $z \in T$ . Then

$$C(T) \times_{\alpha} \mathbf{Z} \simeq C^*(M_z, \lambda_1).$$

**Remark 2.2.** Set  $U = M_z$  and  $V = \lambda_1$ . Then U and V are unitaries

satisfying

$$(*) \qquad \qquad UV = e^{2\pi i \theta} V U.$$

**Lemma 2.3.** Assume that U and V are two unitaries in  $B(L^2(T))$ , sat-

is fying the relation (\*). Let  $\pi$  be the representation of C(T) on  $B(L^2(T))$ taking  $\iota$  to U, where  $\iota(z) = z$  for all z in T. Also let  $\Lambda$  be the representation of  $\mathbf{Z}$  on  $B(L^2(T))$  taking 1 to V. Then  $(\pi, \Lambda)$  is a covariant representation of  $(C(T), \mathbf{Z}, \alpha)$ . **Proof.** It is clear that  $\pi$  is a non-degenerate representation and  $\Lambda$  is a unitary representation. It suffices to show that

$$\Lambda_n \pi(g) \Lambda_n^* = \pi(\alpha_n(g))$$

for all  $g \in C(T)$  and  $n \in \mathbb{Z}$ . Since U is unitary, Proposition 4.1.1 (iii) of [5] implies that  $sp(U) \subset T$ . On the other hand, sp(U) is invariant under the rotation  $R_{\theta}$  through the irrational angle  $2\pi\theta$ , because

$$e^{2\pi i\theta}sp(U) = sp(e^{2\pi i\theta}U) = sp(V^*UV) = sp(UV^*V) = sp(U).$$

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Thus considering the fact that  $\theta$  is irrational, we conclude that sp(U) = T. So we can use the Functional Calculus. For any polynomial  $p(z) = \sum_{k=-N}^{N} a_k z^k$ , one has

$$Vp(U)V^* = \sum_{k=-N}^{N} a_k (VUV^*)^k = \sum_{k=-N}^{N} e^{-2\pi i k\theta} a_k U^k = \alpha_1(p)U.$$

Similarly, we have  $V^*p(V)V = \alpha_{-1}(p)U$ . It is easily verified by induction that

$$V^n p(U) V^{n^*} = \alpha_n(p) U$$
 for all  $n \in \mathbb{Z}$ .

So we have

$$\Lambda_n \pi(p) \Lambda_n^* = V^n p(U) V^{n^*} = \alpha_n(p) U = p(e^{-2\pi i n\theta} U) = \pi(\alpha_n(p)).$$

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Since C(T) is the closure of such these polynomials, the result follows.

In [1] K. R. Davidson has defined the irrational rotation algebra  $\mathcal{A}_{\theta}$  as the following:

**Definition 2.4.** The universal  $C^*$ -algebra  $\mathcal{A}_{\theta}$  satisfying (\*) is called the irrational rotation algebra.

Recall that  $\mathcal{A}_{\theta}$  is universal for the relation (\*) provided that it is generated by two unitaries  $\tilde{U}$  and  $\tilde{V}$  satisfying (\*), and whenever  $\mathcal{A} = C^*(U, V)$  is another  $C^*$ -algebra satisfying (\*), there is a homomorphism of  $\mathcal{A}_{\theta}$  onto  $\mathcal{A}$  which carries  $\tilde{U}$  to U and  $\tilde{V}$  to V.

**Remark 2.5.** Let  $(\mathcal{A}, G, \alpha)$  be a  $C^*$ -dynamical system. Then the crossed product  $\mathcal{A} \times_{\alpha} G$  has the universal property [1]. That is, if  $(\pi, \Lambda)$  is any covariant representation of  $(\mathcal{A}, G, \alpha)$ , then there is a representation of  $\mathcal{A} \times_{\alpha} G$  into  $C^*(\pi(\mathcal{A}), \Lambda(G))$  obtained by setting

$$\sigma(f) = \sum_{t \in G} \pi(A_t) \Lambda_t \quad for \quad f = \sum_{t \in G} A_t \delta_t \in \mathcal{A}G$$

and then extending by continuity. In the unital case, this map is surjective.

**Theorem 2.6.** The crossed product  $C(T) \times_{\alpha} \mathbf{Z}$  can be identified with

the irrational rotation algebra  $\mathcal{A}_{\theta}$ .

**Proof.** By Remark 2.2,  $M_z$  and  $\lambda_1$  are unitaries satisfying (\*). Now, since  $\mathcal{A}_{\theta}$  is simple, by Theorem VI.1.4 of [1],  $C^*(M_z, \lambda_1)$  is isomorphic to  $\mathcal{A}_{\theta}$ . Thus Lemma 2.1 implies that  $C(T) \times_{\alpha} \mathbb{Z} \simeq \mathcal{A}_{\theta}$ .

**Remark 2.7.** There is another proof of the theorem due to Davidson [1] which we bring here.

Suppose that  $\mathcal{A}_{\theta} = C^*(\tilde{U}, \tilde{V})$  such that  $\tilde{U}$  and  $\tilde{V}$  are unitaries satisfying (\*). Then by Lemma 2.3,  $(\pi, \Lambda)$  is a covariant representation of  $(C(T), \mathbf{Z}, \alpha)$ , where  $\pi : \iota \mapsto \tilde{U}$  and  $\Lambda : 1 \mapsto \tilde{V}$ . By Remark 2.5, there is a homomorphism of  $C(T) \times_{\alpha} \mathbf{Z}$  onto  $C^*(\pi(C(T)), \Lambda(\mathbf{Z})) =$  $C^*(\pi(\iota), \Lambda(1)) = C^*(\tilde{U}, \tilde{V}) = \mathcal{A}_{\theta}$ . Conversely, by Remark 2.2,  $M_z$  and  $\lambda_1$  are unitaries satisfying (\*). Therefore, the universal property of  $\mathcal{A}_{\theta}$ implies that there is a homomorphism of  $\mathcal{A}_{\theta}$  onto  $C^*(M_z, \lambda_1)$ , and so by Lemma 2.1, there is a homomorphism of  $\mathcal{A}_{\theta}$  onto  $C(T) \times_{\alpha} \mathbf{Z}$ . Clearly these homomorphisms are inverses.

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