

On CCC - Properties of Almost Regular Closed Lindeloff Space

H. R. Sahebi

Islamic Azad University-Ashtian Branch

Abstract. A topological space is said to be almost regular closed Lindeloff (=ARC-Lindeloff) if every cover by regular closed sets has a countable subfamily whose union is dense. In this paper we investigate properties of ARC-Lindeloff. In addition, several example will be provided to illustrate our results.

AMS Subject Classification: 54A05.

Keywords and Phrases: RC-Lindeloff, ARC-Lindeloff, Isbell space, Luzin space.

1. ARC-Lindeloff Space

Among the various properties of topological spaces a lot of attention has been paid to regular closed sets. The starting point was Thompson's paper on S-closed spaces ([11]). The class of regular closed lindeloff spaces (RC-lindeloff) defined by Jankovic and Konsadilaki([3, 9]). In this paper we consider a class of RC-Lindeloff spaces.

Throughout this paper relative and coarser topology on space (X, τ) is denoted by X^* and X^{**} , respectively. For a subset S of a topological

space (X, τ) the closure of S and the interior of S will be denoted by $Cl_X S$ and $int_X S$.

Definition 1.1. A space X satisfies CCC (=countable chain condition) if every family of pairwise disjoint nonempty open sets in X is at most countable .

Definition 1.2. A filter F on a subset S of X is then a subset of $P(S)$ with the following properties:

- 1) $S \in F$ and $\phi \notin F$.
- 2) If A and $B \in F$, then $A \cap B \in F$.
- 3) If $A \in F$ and $B \subset S$, then $A \subset B$ implies that $B \in F$.

The first three properties imply that a filter has the finite intersection property.

Definition 1.3. Let S be a subset of X , a filter base is a subset B of $P(S)$ with the following properties:

The intersection of any two sets of B contains a set of B is non-empty and the empty set is not in B .

Definition 1.4. A subset S of space X is said to be semi-open if $S \subset Cl_X(int S)$.

Definition 1.5. A subset S of space X is said to be semi-preopen if

$$S \subset Cl_X(int(Cl_X S)).$$

Definition 1.6. A subset S of space X is said to be locally dense if

$$S \subset int(Cl_X S).$$

Example 1.1. Consider the set \mathcal{R} of real numbers with the usual topology and let $S = [0, 1] \cup ((1, 2) \cap \mathcal{Q})$ where \mathcal{Q} stands for the set of rational numbers. Then S is neither semi-open nor pre-open. Let $T = [0, 1] \cap \mathcal{Q}$ then T is semi-preopen .

Remark 1.1. Clearly, every open set is semi-open and locally dense. Also, every semi-open set is semi-preopen, and every locally dense set is semi-per-open.

Definition 1.7. There exists an infinite family F of infinite subsets of \mathbb{N} such that the intersection of any two is finite. Let $D = \{w_{E \in F}\}$ be a new set of distinct points, and define $\Psi = N \cup D$ with the following topology: the points of N are isolate, while a neighborhood of a point w_E is any set containing w_E , Ψ with this topology is called Ψ -space or Isbell space [8].

Definition 1.8. A subset S of space X is called regular open if

$S = int(Cl_X S)$ and $S \subset X$ is called regular closed if $X - S$ is regular open, i.e.

$$S = Cl_X(int S).$$

The families of regular open subsets of space X and regular closed subsets of space X are denoted by $RO(X)$ and $RC(X)$, respectively.

Remark 1.2. *If D is locally dense in a space X , then*

$$RC(D^*) = \{F \cap D : F \in RC(X)\}$$

Lemma 1.1. *If S_i is a family of semi-open sets, then there exists a family O_a of pairwise disjoint open sets such that O_a is a refinement of S_i and the union of O_a is dense in the union of S_i .*

Proof. This is proved by using Zorn's Lemma in the standard way (see ([10, page 39])). \square

Definition 1.9. *A topological space X is called*

- 1) *SC if every regular closed cover has a finite subcover.*
- 2) *Countably SC, if every countable regular closed cover has finite subcover.*
- 3) *RC-Lindelöf if every regular closed cover has a countable subcover.*
- 4) *Almost Lindelöf, if every open cover has a countable subfamily whose union is dense.*
- 5) *ARC-Lindelöf, if every regular closed cover has a countable subfamily whose union is dense.*

6) *Weakly Lindeloff, if every open cover has a countable subfamily such that the closure of whose members cover X .*

2. Properties Of ARC-Lindeloff Space

Theorem 2.1. *For a space X the following are equivalent:*

- 1) *X is RC-Lindeloff.*
- 2) *every semi-open cover of X has a countable subfamily whose union is dense.*
- 3) *Every semi-preopen cover of X has a countable subfamily whose union is dense.*
- 4) *Every regular open filterbase $\{G_i : i \in I\}$ on X satisfying*

$$\text{int} \bigcap \{G_i : i \in J\} \neq \emptyset$$

for each countable subset J of I , has nonempty intersection.

Proof. 3) \implies 2) \implies 1): This is clearly since every semi-open set is semi-preopen and every regular closed set is semi-open.

1) \implies 3): Let $\{A_i : i \in I\}$ be a semi-preopen cover of X . Then each ClA_i is regular closed, therefore there exists a countable subset J of I such that $\bigcup \{ClA_i : i \in J\}$ is dense. One easily checks that $\bigcup \{A_i : i \in J\}$ is also dense.

1) \implies 4): Let $\{F_i : i \in I\}$ be a regular open filterbase satisfying $\text{int} \bigcap \{G_i : i \in J\} \neq \emptyset$ for countable $J \subset I$. Suppose that $\bigcap \{F_i :$

$i \in I\} = \emptyset$. Then $\{X - F_i : i \in I\}$ is a regular closed cover of X . By assumption, there exist a countable subset J of I such that $\bigcup\{X - F_i : i \in J\}$ is dense. Hence $\text{int} \bigcap\{F_i : i \in J\} \neq \emptyset$, a contradiction.

4) \implies 1): Let $\{F_i : i \in I\}$ be a regular closed cover of X and suppose that $\bigcup_{i \in J} F_i$ is not dense for countable subset J of I . Let $K_J = \text{Cl}(\bigcup_{i \in J} F_i)$, then clearly $K_J \in \text{RC}(X)$ and $\{X - K_J : J \subset I, J \text{ is countable}\}$ is a regular open filterbase satisfying the hypothesis of (4). By assumption there exists $x \in X$ with $x \in \bigcap\{X - K_J : J \subset I, J \text{ countable}\}$. Pick $i^* \in I$ with $x \in F_{i^*}$ and let $J = \{i^*\}$. Then $x \in K_J = F_{i^*}$, a contradiction. \square

Remark 2.1. *It is obvious that every SC-space is RC-Lindeloff, however, a countable discrete space is RC-Lindeloff but no SC-space.*

In the next theorem we show relation between CCC and ARC-Lindeloff spaces.

Theorem 2.2. *If X satisfies CCC properties, then X is an ARC-Lindeloff space.*

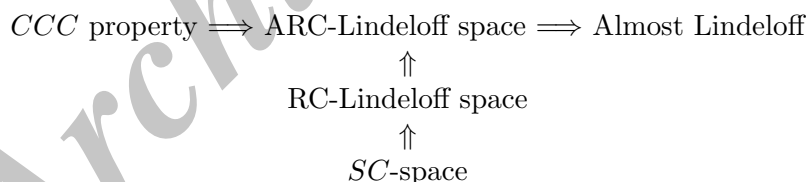
Proof. Let $\{F_i : i \in I\}$ be a regular closed cover of X . By Lemma 1.1, there exists a family $\{G_j : j \in J\}$ of pairwise disjoint nonempty open sets in X such that its union is dense. By assumption, J is at most countable. For each $j \in J$ pick $i_j \in I$ with $G_j \subset F_{i_j}$. Then

$\bigcup \{F_{i_j} : j \in J\}$ is dense in X which proves that X is an ARC -Lindeloff space. \square

Example 2.1. Let X be an uncountable discrete space and βX be its Stone-Cech compactification. Then βX is SC -space ([11]) and thus RC -Lindeloff (then ARC -Lindeloff) but fails to satisfy CCC property.

It is obvious that every SC -closed space is RC -Lindeloff. Note, that a countable discrete space is RC -Lindeloff but not SC -closed. Every RC -Lindeloff space is ARC -Lindeloff and weakly Lindeloff. Moreover, every weakly Lindeloff space is clearly almost Lindeloff, and by Theorem 2.1, every ARC -Lindeloff space is almost Lindeloff.

The following diagram summarizes the observations we have made so far (see [3],[9],[10],[11]).



Definition 2.1. A Hausdorff space X is called Luzin space if we have

- 1) every nowhere dense set in X is countable.
- 2) X has at most countably many isolated point.
- 3) X is uncountable.

Theorem 2.3. Let X be an uncountable first countable T_3 space with

at most countably many isolated point. Then X is RC -Lindeloff space iff X is a Luzin space.

Proof. See [9, page 106].

We give several examples to show that non of the implications in our diagram is reversible.

Example 2.2. Let R be the real line with the usual topology. Then R satisfies CCC and hence is ARC -Lindeloff space. However, R fails to be RC -Lindeloff space.

Example 2.3. The Isbell space Ψ (Definition 1.7) is clearly CCC and so by Theorem 2.2, RC -Lindeloff space. But Ψ fails to be RC -Lindeloff space.

Example 2.4. Let $X = \beta N - \{\sigma\}$, where $\sigma \in \beta N - N$. Then X is separable, it satisfies CCC , and also it is ARC -Lindeloff space. In addition, X is countably SC -space but not SC -space and thus cannot be RC -Lindeloff space.

3. Product ARC -Lindeloff Space

Recall that a topological property (P) is said to be semi-regular provided that a space X satisfies (P) if and only if X^{**} satisfies (P) . The property (P) is called contagious if a space X satisfies (P) whenever a dense

subspace of X has property (P) .

Theorem 3.1. *Let (P) denotes the property ARC-Lindeloff. Then (P) is both semi-regular and contagious.*

Proof. First note that for every space X we have $RC(X) = RC(X^{**})$, and if F is a union of regular closed sets, then $Cl_X F = Cl_{X^{**}} F$. From this it follows immediately that (P) is semi-regular.

Now suppose that D is a dense ARC-Lindeloff subspace X . if $\{F_i : i \in I\}$ denotes a regular closed cover of X then, by Remark 1.2, $\{F_i \cap D : i \in I\} \subset RC(D^*)$ is a cover of D , so there are countably many $F_i \cap D$ whose many F_i is dense in X . Consequently, the union of countably many F_i 's is dense in X and so X is ARC-Lindeloff space.

Corollary 3.1. *Let Y be an ARC-Lindeloff subspace of space X and let $Y \subset Z \subset Cl_X Y$ then space of Z^* is ARC-Lindeloff.*

Theorem 3.2. *If X is ARC-Lindeloff space, then we have*

- 1) *if $Y \in RO(X)$, then Y^* is ARC-Lindeloff space.*
- 2) *if $Y \in RC(X)$, then Y^* is ARC-Lindeloff space.*

Proof. 1) Let $\{G_i : i \in I\} \subset RC(Y^*)$ be a cover of Y . Since Y is locally dense, by Lemma 1.1, for each $i \in I$, we will have $A_i = Y \cap F_i$ where $F_i \in RC(X)$. Since $\{F_i : i \in I\} \cup \{X - Y\}$ is a regular closed cover of X , then there is a countable subset $J \subset I$ such that $X = Cl(\bigcup \{F_i : i \in$

$J\} \cup (X - Y))$. Consequently, $\bigcup\{G_i : i \in J\}$ is dense in Y^* and so Y^* is *ARC*-Lindeloff space.

2) We know that $\text{int}Y \in RO(X)$ and $\text{int}Y$ is dense in Y , it by (1) and Corollary 3.1, it follows that Y^* is *ARC*-Lindeloff space.

Example 3.1. *Let X be an uncountable discrete space and βX be its Stone-Cech compactification. Clearly X is an open and dense subspace of βX and $Y = \{(x, x) : x \in X\}$ is regular open and discrete subspace of $\beta X \times \beta X$. However, the space Y^* is not *ARC*-Lindeloff space.*

This example with Theorem 3.2, also shows that the product of two *ARC*-Lindeloff space need not be *ARC*-Lindeloff.

References

- [1] D. Andrijevic, Semi-preopen sets, *Mat. Vesnik*, 38 (1986), 24-32.
- [2] H. H. Coarson and E. Michael, Metrizable of certain countable unions, *Illinois J. Math.*, 8 (1964), 351-360.
- [3] K. Džurina, N. Ergun, and M. Ganster, Countably S -closed spaces, *Math. Pannonica*, 17 (1) (1996), 97-111.
- [4] R. Engelking, *General topology*, PWN Polish Scientific publishers, 1977.
- [5] Z. Frolik, Remarks concerning the invariance of Bair spaces under mappings, *Czechoslovak Math. J.*, 11 (1961), 381-385.
- [6] M. Ganster, D. S. Jankovic, and I. L. Reilly, On compactness with respect to semi-open sets, *Comment. Math. Univ. Carolinae*, 31 (1) (1990), 37-39.
- [7] M. Ganster, T. Noiri, and I. L. Reilly, Weak and strong forms of θ -irresolute functions, *J. Inst. Math. Univ. Carolina*, 31 (1) (1988), 19-29.

- [8] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, Princeton, 1960.
- [9] D. Jankovic and Ch. Konstadilaki, On covering properties by regular closed sets, *Mathematica Pannonica*, 7 (1) (1996), 97-111.
- [10] N. Levine, Semi-open sets and semi-continuity in topological spaces, *Amer. Math. Monthly*, 70 (1963), 36-41.
- [11] T. Thompson, S -closed spaces, *Proc. Amer. Math. Soc.*, 60 (1976), 335-38.
- [12] R. C. Walker, *The Stone-Cech Compactification*, Springer Verlag, 1974.
- [13] R. G. Woods, Characterizations of some C^* -embedded of βN , *Pacific. J. Math.*, 65 (1976), 573-579.

Hamid Reza Sahebi

Department of Mathematics

Islamic Azad University-Ashtian Branch

Email: hrsaau@Gmail.com