Journal of Mathematical Extension
Vol. 1, N0. 2, (2007), 105-115

On CCC - Properties of Almost Regular Closed Lindeloff Space

H. R. Sahebi
Islamic Azad University-Ashtian Branch

Abstract. A topological space is said to be almost regular closed Lindeloff (=ARC-Lindeloff) if every cover by regular closed sets has a countable subfamily whose union is dense. In this paper we investigate properties of ARC-Lindeloff. In addition, several example will be provided to illustrate our results.

AMS Subject Classification: 54A05.
Keywords and Phrases: RC-Lindeloff, ARC-Lindeloff, Isbell space, Luzin space.

1. ARC-Lindeloff Space

Among the various properties of topological spaces a lot of attention has been paid to regular closed sets. The starting point was Thompson's paper on S-closed spaces ([11]). The class of regular closed lindeloff spaces (RC-lindeloff) defined by Jankovic and Konsadilaki([3, 9]). In this paper we consider a class of RC-Lindeloff spaces.

Throughout this paper relative and coarser topology on space (X, τ) is denoted by $X^*$and X^{**},respectively. For a subset S of a topological space (X, τ) the closure of S and the interior of S will be denoted by $Cl_X S$ and $int_X S$.

Definition 1.1. *A space X satisfies CCC (=countable chain condition) if every family of pairwise disjoint nonempty open sets in X is at most countable .*

Definition 1.2. *A filter F on a subset S of X is then a subset of $P(S)$ with the following properties:*

1) $S \in F$ and $\phi \notin F$.

2) If A and $B \in F$, then $A \bigcap B \in F$.

3) If $A \in F$ and $B \subset S$, then $A \subset B$ implies that $B \in F$.

The first three properties imply that a filter has the finite intersection property.

Definition 1.3. *Let S be a subset of X, a filter base is a subset B of $P(S)$ with the following properties:*

The intersection of any two sets of B contains a set of B is non-empty and the empty set is not in B.

Definition 1.4. *A subset S of space X is said to be semi-open if* $S \subset Cl_X(intS)$.

Definition 1.5. *A subset S of space X is said to be semi-preopen if*

$S \subset Cl_X(int(Cl_X S))$.

Definition 1.6. *A subset S of space X is said to be locally dense if* $S \subset int(Cl_X S)$.

Example 1.1. *Consider the set $\mathcal{R}$ of real numbers with the usual topology and let $S = [0,1] \bigcup ((1,2) \bigcap Q)$ where Q stands for the set of rational numbers. Then S is neither semi-open nor pre-open. Let $T = [0,1] \bigcap Q$ then T is semi-preopen .*

Remark 1.1. *Clearly, every open set is semi-open and locally dense. Also, every semi-open set is semi-preopen, and every locally dense set is semi-per-open.*

Definition 1.7. *There exists an infinite family F of infinite subsets of $\mathbb{N}$ such that the intersection of any two is finite. Let $D = \{w_{E \in F}\}$ be a new set of distinct points, and define $\Psi = N \bigcup D$ with the following topology: the points of N are isolate, while a neighborhood of a point w_E is any set containing w_E, Ψ with this topology is called Ψ-space or Isbell space [8].*

Definition 1.8. *A subset S of space X is called regular open if* $S = int(Cl_X S)$ *and $S \subset X$ is called regular closed if $X - S$ is regular open, i.e.*

$S = Cl_X(intS)$.

The families of regular open subsets of space X and regular closed subsets of space $X$are denoted by $RO(X)$and $RC(X)$, respectively.

Remark 1.2. *If D is locally dense in a space X, then*

$$RC(D^*) = \{F \bigcap D : \ F \in RC(X)\}$$

Lemma 1.1. *If S_i is a family of semi-open sets, then there exists a family O_a of pairwise disjoint open sets such that O_a is a refinement of S_i and the union of O_a is dense in the union of S_i.*

Proof. This is proved by using Zorn's Lemma in the standard way (see ([10, page 39]). □

Definition 1.9. *A topological space X is called*

1) SC if every regular closed cover has a finite subcover.

2) Countably SC, if every countable regular closed cover has finite subcover.

3) RC-Lindelöf if every regular closed cover has a countable subcover.

4) Almost Lindeloff, if every open cover has a countable subfamily whose union is dense.

5) ARC-Lindeloff, if every regular closed cover has a countable subfamily whose union is dense.

6) Weakly Lindeloff, if every open cover has a countable subfamily such that the closure of whose members cover X.

2. Properties Of ARC-Lindeloff Space

Theorem 2.1. *For a space X the following are equivalent:*
1) X is RC-Lindeloff.
2) every semi-open cover of X has a countable subfamily whose union is dense.
3) Every semi-preopen cover of X has a countable subfamily whose union is dense.
4) Every regular open filterbase $\{G_i : i \in I\}$ on X satisfying

$$int \bigcap \{G_i : i \in J\} \neq \emptyset$$

for each countable subset J of I, has nonempty intersection.

Proof. 3)$\Longrightarrow$ 2)$\Longrightarrow$ 1): This is clearly since every semi-open set is semi-preopen and every regular closed set is semi-open.

1)$\Longrightarrow$ 3): Let $\{A_i \ : i \in I\}$ be a semi-preopen cover of X.Then each ClA_i is regular closed, therefore there exists a countable subset J of I such that $\bigcup\{ClA_i \ : i \in J\}$ is dense. One easily checks that $\bigcup\{A_i \ : i \in J\}$ is also dense.

1)$\Longrightarrow$ 4): Let$\{F_i : \ i \in I\}$ be a regular open filterbase satisfying $int \bigcap \{G_i : \ i \in J\} \neq \emptyset$ for countable $J \subset I$. Suppose that $\bigcap\{F_i \ :$

$i \in I\} = \emptyset$. Then $\{X - F_i \ : i \in I\}$ is a regular closed cover of X. By assumption, there exist a countable subset J of I such that $\bigcup\{X - F_i : \ I \in J\}$ is dense. Hence $int \bigcap\{F_i : \ i \in J\} \neq \emptyset$, a contradiction.

4)$\Longrightarrow$1): Let $\{F_i \ : i \in I\}$ be a regular closed cover of X and suppose that $\bigcup_{i \in J} F_i$ is not dense for countable subset J of I. Let $K_J = Cl(\bigcup_{i \in J} F_i)$, then clearly $K_J \in RC(X)$ and $\{X - K_J : \ J \subset I, J \text{ is countable}\}$ is a regular open filterbase satisfying the hypothesis of (4). By assumption there exists $x \in X$ with $x \in \bigcap\{X - K_J : \ J \subset I, J \text{ countable}\}$. Pick $i^* \in I$ with $x \in F_{i^*}$ and let $J = \{i^*\}$. Then $x \in K_J = F_{i^*}$, a contradiction. □

Remark 2.1. *It is obvious that every SC-space is RC-Lindeloff, however, a countable discrete space is RC-Lindeloff but no SC-space.*

In the next theorem we show relation between CCC and ARC-Lindeloff spaces.

Theorem 2.2. *If X satisfies CCC properties, then X is an ARC-Lindeloff space.*

Proof. Let $\{F_i \ : i \in I\}$ be a regular closed cover of X. By Lemma 1.1, there exists a family $\{G_j \ : j \in J\}$ of pairwise disjoint nonempty open sets in X such that it's union is dense. By assumption, J is at most countable. For each $j \in J$ pick $i_j \in I$ with $G_j \subset F_{i_j}$. Then

$\bigcup\{F_{i_j} : j \in J\}$ is dense in X which proves that X is an ARC-Lindeloff space. □

Example 2.1. *Let X be an uncountable discrete space and βX be its Stone-Cech compactification. Then βX is SC-space ([11]) and thus RC-Lindeloff (then ARC-Lindeloff) but fails to satisfy CCC property.*

It is obvious that every SC-closed space is RC-Lindeloff. Note, that a countable discrete space is RC-Lindeloff but not SC-closed. Every RC-Lindeloff space is ARC-Lindeloff and weakly Lindeloff. Moreover, every weakly Lindeloff space is clearly almost Lindeloff, and by Theorem 2.1, every ARC-Lindeloff space is almost Lindeloff.

The following diagram summarizes the observations we have made so far (see [3],[9],[10],[11]).

$$
\begin{array}{ccccc}
CCC \text{ property} & \Longrightarrow & \text{ARC-Lindeloff space} & \Longrightarrow & \text{Almost Lindeloff} \\
 & & \Uparrow & & \\
 & & \text{RC-Lindeloff space} & & \\
 & & \Uparrow & & \\
 & & SC\text{-space} & &
\end{array}
$$

Definition 2.1. *A Hausdroff space X is called Luzin space if we have*

1) every nowhere dense set in X is countable.

2) X has at most countably many isolated point.

3) X is uncountable.

Theorem 2.3. *Let X be an uncountable first countable T_3 space with* *at most countably many isolated point. Then X is RC-Lindeloff space iff X is a Luzin space.*

Proof. See [9, page 106].

We give several examples to show that non of the implications in our diagram is reversible.

Example 2.2. *Let R be the real line with the usual toplogy. Then R satisfies CCC and hence is ARC-Lindeloff space. However, R fails to be RC-Lindeloff space.*

Example 2.3. *The Isbell space Ψ (Definition 1.7) is clearly CCC and so by Theorem 2.2, RC-Lindeloff space. But Ψ fails to be RC-Lindeloff space.*

Example 2.4. *Let $X = \beta N - \{\sigma\}$, where $\sigma \in \beta N - N$. Then X is separable, it satisfies CCC, and also it is ARC-Lindeloff space. In addition, X is countably SC-space but not SC-space and thus cannot be RC-Lindeloff space.*

3. Product ARC-Lindeloff Space

Recall that a topological property (P) is said to be semi-regular provided that a space X satisfies (P) if and only if X^{**} satisfies (P). The property (P) is called contagious if a space X satisfies (P) whenever a dense subspace of X has property (P).

Theorem 3.1. *Let (P) denotes the property ARC-Lindeloff. Then (P) is both semi-regular and contagious.*

Proof. First note that for every space X we have $RC(X) = RC(X^{**})$, and if F is a union of regular closed sets, then $Cl_X F = Cl_{X^{**}} F$. From this it follows immediately that (P) is semi-regular.

Now suppose that D is a dense ARC-Lindeloff subspace X. if $\{F_i \ : i \in I\}$ denotes a regular closed cover of X then, by Remark 1.2, $\{F_i \bigcap D : \ i \in I\} \subset RC(D^*)$ is a cover of D, so there are countably many $F_i \bigcap D$ whose many F_i is dense in X. Consequently, the union of countably many F_i's is dense in X and so X is ARC-Lindeloff space.

Corollary 3.1. *Let Y be an ARC-Lindeloff subspace of space X and let $Y \subset Z \subset Cl_X Y$ then space of Z^* is ARC-Lindeloff.*

Theorem 3.2. *If X is ARC-Lindeloff space, then we have*

1) if $Y \in RO(X)$, then Y^* is ARC-Lindeloff space.

2) if $Y \in RC(X)$, then Y^* is ARC-Lindeloff space.

Proof. 1) Let $\{G_i \ : i \in I\} \subset RC(Y^*)$ be a cover of Y. Since Y is locally dense, by Lemma 1.1, for each $i \in I$, we will have $A_i = Y \bigcap F_i$ where $F_i \in RC(X)$. Since $\{F_i : \ i \in I\} \bigcup \{X - Y\}$ is a regular closed cover of X, then there is a countable subset $J \subset I$ such that $X = Cl(\bigcup\{F_i : \ i \in$

$J\} \bigcup (X - Y))$. Consequently, $\bigcup \{G_i : i \in J\}$ is dense in Y^* and so Y^* is ARC-Lindeloff space.

2) We know that $intY \in RO(X)$ and $intY$ is dense in Y, it by (1) and Corollary 3.1, it follows that Y^* is ARC-Lindeoff space.

Example 3.1. *Let X be an uncountable discrete space and βX be its Stone-Cech compactification. Clearly X is an open and dense subspace of βX and $Y = \{(x, x) : x \in X\}$ is regular open and discrete subspace of $\beta X \times \beta X$. However, the space Y^* is not ARC-Lindeloff space.*

This example with Theorem 3.2, also shows that the product of two ARC-Lindeloff space need not be ARC-Lindeloff.

References

[1] D. Andrijevic, Semi-preopen sets, *Mat. Vesnik*, 38 (1986), 24-32.

[2] H. H. Coarson and E. Michael, Metrizability of certain countable unions, *Illinois J. Math.*, 8 (1964), 351-360.

[3] K. Dlaska, N.Ergun, and M. Ganester, Countably S-closed spaces, *Math.Pannonica*, 17 (1) (1996), 97-111.

[4] R. Engelking, *General topology*, PWN Polish Scientific publishers, 1977.

[5] Z. Frolik, Remarks concerning the invariance of Bair spaces under mappings, *Czecholovak Math. J.*, 11 (1961), 381-385.

[6] M. Ganester, D. S. Jankovic, and I . L. Reily, On compactness with respect to semi-open sets, *Comment. Math. Univ. Carolinae*, 31 (1) (1990), 37-39.

[7] M. Ganester, T. Noiri, and I . L. Reily, Weak and strong forms of θ-irreslute functions, *J. Inst. Math. Univ. Carolina*, 31 (1) (1988), 19-29.

[8] L. Gillman and M. Jerison, *Rings of Continous Function*, Van Nostrand, Princeton, 1960.

[9] D. Jankovic and Ch. Konstadilaki, On covering properties by regular closed sets, *Mathematica Pannonica*, 7 (1) (1996), 97-111.

[10] N. Levine, Semi-open sets and semi-continuty in topological spaces, *Amer. Math. Monthly*, 70 (1963), 36-41.

[11] T. Thompson, S-closed spaces, *Proc. Amer. Math. Soc.*, 60 (1976), 335-38.

[12] R. C. Walker, *The Stone-Cech Compactification*, Springer Verlag, 1974.

[13] R. G. Woods, Characterizations of some C^*-embedded of βN, *Pacific. J. Math.*, 65 (1976), 573-579.

Hamid Reza Sahebi
Department of Mathematics
Islamic Azad University-Ashtian Branch
Email: hrsaau@Gmail.com