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An Application of Linear Algebra over Lattices

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Abstract. In this paper, first we consider L^n as a semimodule over a complete bounded distributive lattice L. Then we define the basic concepts of module theory for L^n. After that, we proved many similar theorems in linear algebra for the space L^n. An application of linear algebra over lattices for solving linear systems, was given.

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1. Introduction

Fuzzy linear systems of equations and inequalities over a bounded chain have been studied by many authors [6], [8], [7]. To extend this concept to L-fuzzy linear systems over a bounded distributive lattice L, we need some basic definitions of linear algebra over lattices such as linearly independent subset, a subsemimodule generated by a set and so on. For more details see [3], [2]. By defining subsemimodule generated by a set, we can find a theoretical necessary and sufficient condition for consistency of the linear system of equations $A * X = b$ over a bounded distributive lattice.

Definition 1.1. *Let* $(H, *)$ *be a commutative semigroup (monoid) with a reflexive and transitive order* $\leqslant$ *on it.* $(H, *, \leqslant)$ *is called an ordered commutative semigroup (monoid) if*

$$a \leqslant b \Longrightarrow a * c \leqslant b * c \quad \forall a, b, c \in H.$$

Definition 1.2. *Let* $(H, *)$ *be a commutative group (resp. semigroup, monoid) with a partial order* $\leqslant$*.* $(H, *, \leqslant)$ *is called a lattice-ordered commutative group (resp. semigroup, monoid), if*

$$a \leqslant b \Longrightarrow a * c \leqslant b * c, \quad \forall a, b, c \in H.$$

For simplicity, we call it l-group (resp. l-semigroup, l-monoid).

Example 1.3. Every lattice $(L, \leqslant)$ is a l-semigroup, by letting $* = \wedge$. Clearly a bounded lattice is a l-monoid in this way.

Definition 1.4. *Let* $Mat_{n\times m}(L)$ *be the set of all* $n\times m$ *matrices over the lattice* $(L, \leqslant)$*. Define a partial order relation on* $Mat_{n\times m}(L)$ *as follows:*
$X \leqslant Y \Leftrightarrow x_{ij} \leqslant y_{ij}; \quad for\ all \quad i = 1, 2, ..., n \quad and \quad j = 1, 2, ..., \ m,$

where $X, Y \in Mat_{n\times m}(L)$*. One can see that* $(Mat_{n\times m}(L), \leqslant)$ *is a lattice where its supremum and infimum are defined componentwise on* $Mat_{n\times m}(L)$ *induced by the supremum and infimum of lattice* L*, respectively.*

Definition 1.5 ([10]). *Let $(R, \oplus)$ be a commutative monoid with neutral element 0 and $(R, \otimes)$ be a monoid with neutral element 1 where $0 \neq 1$. Then, $(R, \oplus, \otimes)$ is called a semiring with unity 1 and zero 0, if for all $a, b, c \in R$, the following conditions hold:*

(a) $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$,

(b) $(b \oplus c) \otimes a = (b \otimes a) \oplus (c \otimes a)$,

(c) $0 = a \otimes 0 = 0 \otimes a$.

Example 1.6. Let L be a bounded distributive lattice. Then, $(L, \vee, \wedge)$ and $(L, \wedge, \vee)$ are semirings.

Definition 1.7 ([10]). *$(R, \oplus, \otimes, \leqslant)$ is called an ordered semiring if*

(a) *$(R, \oplus, \otimes)$ is a semiring,*

(b) *$(R, \oplus, \leqslant)$ is an ordered commutative monoid,*

(c) *for all $a, b, c, d \in R$,*

(i) $a \leqslant b$ and $c \geqslant 0 \Longrightarrow a \otimes c \leqslant b \otimes c$ and $c \otimes a \leqslant c \otimes b$,

(ii) $a \leqslant b$ and $d \leqslant 0 \Longrightarrow a \otimes d \geqslant b \otimes d$ and $d \otimes a \geqslant d \otimes b$.

Definition 1.8 ([10]). *Let $(H, *, \leqslant)$ be a commutative ordered monoid with neutral element e and let $(R, \oplus, \otimes)$ be a semiring with unity 1 and zero 0.*

Moreover, suppose that. $: R \times H \longrightarrow H$ is a scalar multiplication such that for all $\alpha, \beta \in R$ and for all $a, b \in H$:

(a) $(\alpha \otimes \beta).a = \alpha.(\beta.a)$,

(b) $(\alpha \oplus \beta).a = (\alpha.a) \oplus (\beta.a)$,

(c) $\alpha.(a * b) = (\alpha.a) * (\alpha.b)$,

(d) $0.a = e$,

(e) $1.a = a$,

then, $(R, \oplus, \otimes, H, *, .)$ *is called an ordered semimodule over* R.

Remark 1.9. *Let* L *be a bounded distributive lattice.*

Then, $(L, \vee, \wedge, L, \vee, \wedge)$ *and* $(L, \wedge, \vee, L, \wedge, \vee)$ *are semimodules over* $(L, \vee, \wedge)$ *and* $(L, \wedge, \vee)$*, respectively.*

Upward and downward sets, as important notions in optimization (see [4], [5]), are used in [9] as in the following definition.

Definition 1.10. *Let* $(L, \leqslant)$ *be a lattice.*

(i) A subset $U \subseteq L$ *is called upward set if* $(a \in U, x \geqslant a) \Longrightarrow x \in U$.

(ii) A subset $D \subseteq L$ *is called downward set if* $(a \in D, x \leqslant a) \Longrightarrow x \in D$.

Example 1.11. Let $(L, \leqslant)$ be a lattice and $a \in L$. Then $\{x \in L | x \geqslant a\}$ is an upward set and $\{x \in L | x \leqslant a\}$ is a downward set.

We can easily prove the following proposition.

Proposition 1.12. *Let* $(L, \leqslant)$ *be a lattice and* $M_i \subseteq L$ *for* $i \in I$*. Then* $\bigcup_{i \in I} M_i$ *is an upward (resp. downward) set if each* M_i*;* $i \in I$ *is upward (resp. downward) set.*

2. Basis For Semimodules

In this section we need to extend some basic definition of linear algebra to concepts of lattices. In this case suppose L is a complete distributive lattice and consider L^n as $Mat_{n\times 1}(L)$, the set of all $n \times 1$ matrices over L. By Definition 1.4., L^n is a lattice. Clearly L^n is a distributive complete lattice if L is so. For every bounded distributive lattice L, $(L, \vee, \wedge)$ is a semiring by Example 1.6. and hence $(L^n, \wedge, \leqslant)$ is a lattice-ordered commutative monoid, by Example 1.3. So we can construct a semimodule as follows.

Theorem 2.1. *Let L be a distributive complete lattice. Then $(L^n, \vee, \leqslant)$ is a semimodule over $(L, \vee, \wedge)$.*

Proof. Let L be a bounded distributive lattice. Then $(L^n, \vee, \leqslant)$ is a semimodule over $(L, \vee, \wedge)$ with scalar multiplication $\bar{\wedge}$ defined by $\bar{\wedge} : L \times L^n \longrightarrow L^n$ such that

$$\alpha \bar{\wedge} \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{pmatrix} = \begin{pmatrix} \alpha \wedge a_1 \\ \alpha \wedge a_2 \\ . \\ . \\ . \\ \alpha \wedge a_n \end{pmatrix},$$

which for simplification, we write it as $\wedge$.

In this way $(L^n, \vee, \leqslant)$ satisfies all conditions of Definition 1.8. Note that the identity element of $(L^n, \vee)$ is a column matrix which all of its entry are equal to 0. $\square$

Definition 2.2. *Let $(H, *, \leqslant)$ be a semimodule over semiring $(R, \oplus, \otimes)$ and K be a subset of H such that $(K, *, \leqslant)$ is a monoid. Then $(K, *, \leqslant)$ is called a subsemimodule of $(H, *, \leqslant)$ if it is a semimodule over $(R, \oplus, \otimes)$ and it is denoted by $K \leqslant_m H$.*

The following theorem can be proved easily.

Theorem 2.3. *Let $(H, *, \leqslant)$ be a semimodule over semiring $(R, \oplus, \otimes)$ and K be a subset of H. Then $K \leqslant_m H$ if and only if*

(i) $e \in K$

(ii) $x * y \in K \quad for\ all \quad x, y \in K,$

(iii) $a.x \in K \quad for\ all \quad a \in R, \quad and \quad x \in K.$

Corollary 2.4. *Let L be a distributive complete lattice and K be a sublattice of L which contains 0. Then $(K^n, \vee, \leqslant)$ is a semimodule over $(L, \vee, \wedge)$ if and only if for every elements $x \in L$ and $y \in K$, we have $x \wedge y \in K$.*

Example 2.5. Let $L = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ and $x \leqslant y$ if x divides y. Consider the sublattice $K = \{1, 2, 3, 6\}$. Then, L and K satisfy on Corollary 2.4. Hence $(K^n, \vee, \leqslant)$ is a semimodule over $(L, \vee, \wedge)$.

Definition 2.6. *Let $(H, *, \leqslant)$ be a semimodule over $(R, \oplus, \otimes)$ and X be* *a subset of H.*

(i) *The subsemimodule hull of (or subsemimodule generated by) X is the intersection of all subsemimodules of H which contains X and denoted by $< X >$. Hence*

$$< X > \quad = \bigcap_{X \subseteq K \leqslant H} K.$$

In the other words, $< X >$ is the smallest subsemimodule of H which contains X.

(ii) *The upward hull of (or upward set generated by) X is defined as the intersection of all upward subsets of H which contains X and is denoted by $< X^* >$. So, $< X^* > \;\; = \;\; \bigcap\{K : X \subseteq K$ and K is an upward subset of $H\}$. In the other words, $< X^* >$ is the smallest upward subset of H which contains X.*

(iii) *The downward hull of (or downward set generated by) X is defined as the intersection of all downward subsets of H which contains X and is denoted by $< X_* >$. So, $< X_* > \;\; = \;\; \bigcap\{K : X \subseteq K$ and K is a downward subset of $H\}$. In the other words, $< X_* >$ is the smallest downward subset of H which contains X.*

Lemma 2.7. *Let $(H, *, \leqslant)$ be a semimodule over $(R, \oplus, \otimes)$ and $x \in H$. Then,*

(i) $< \{x\}^* >= \{a \in H : a \geqslant x\}$, *and*

(ii) $< \{x\}_* >= \{a \in H : a \leqslant x\}$.

Definition 2.8. *Let* $(H, *, \leqslant)$ *be a semimodule over semiring* $(R, \oplus, \otimes)$ *with scalar multiplication "." and* X *be a subset of* H. *By a linear combination of elements* $x_1, ..., x_m \in X$, *we mean* $(a_1.x_1) * ... * (a_m.x_m)$ *where* $a_1, ..., a_m \in R$ *and* m *is a positive integer.*

Theorem 2.9. *Let* $(H, *, \leqslant)$ *be a semimodule over* $(R, \oplus, \otimes)$ *and* X *be a subset of* H.

(i) *Consider* $M = \{(a_1.x_1) * \ldots * (a_m.x_m) | x_1, \ldots, x_m \in X, a_1, \ldots, a_m \in R$ *and* m *is a positive integer* $\}$; *as the set of all finite linear combinations of elements of* X. *Then,* $< X >= M$.

(ii) $< X^* >= \bigcup_{x \in X} < \{x\}^* >$.

(iii) $< X_* >= \bigcup_{x \in X} < \{x\}_* >$.

Proof. The proofs of (i)-(iii) follow from Lemma 2.7. Definition 2.8. and Proposition 1.12. □

Example 2.10. Let $L = [0, 10]$; the bounded chain of real numbers between 0 and 10. Consider semimodule $(L^2, \vee, \wedge)$ over $(L, \vee, \wedge)$, where $\leqslant$ is usual partial order on L. For $X_1 = \{(2, 3)^T, (5, 1)^T\}$ the subsemimodule generated by X_1 is shown in Fig. 1.

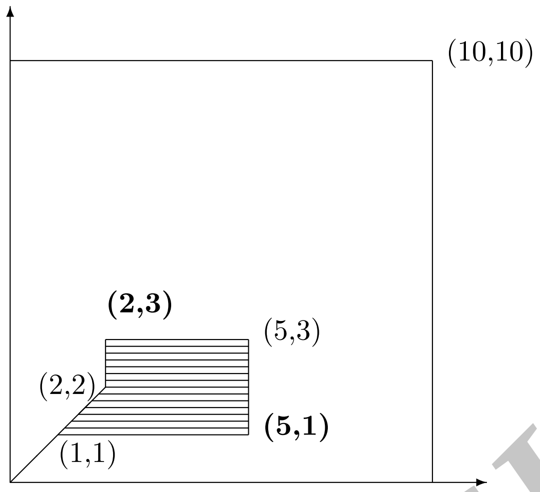

Fig. 1. Subsemimodule hull of X_1

The upward hull of X_1 is shown in Fig. 2.

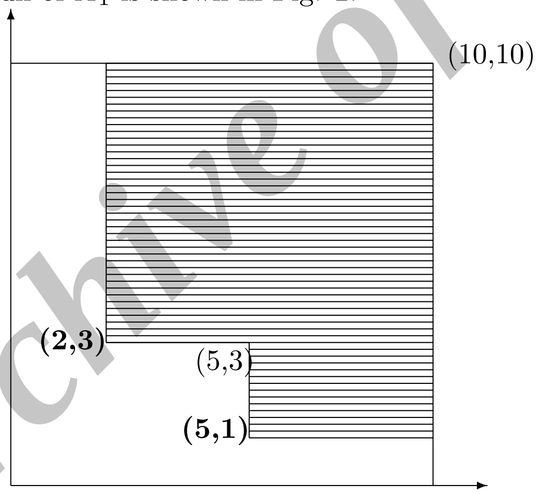

Fig. 2. Upward hull of X_1

The downward hull of X_1 is shown in Fig. 3.

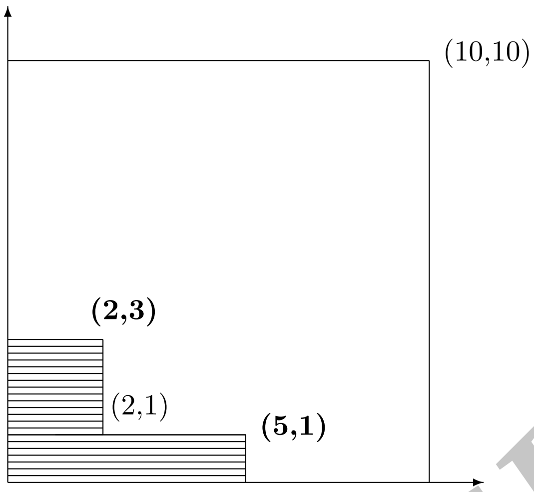

Fig. 3. Downward hull of X_1

Now consider $X_2 = \{(2,4)^T, (5,9)^T\}$. The subsemimodule hull of X_2 is shown in Fig. 4.

Fig. 4. Subsemimodule hull of X_2

The subsemimodule $< X_3 >$, where $X_3 = \{(3,1)^T, (5,2)^T, (2,4)^T\}$, is as follows:

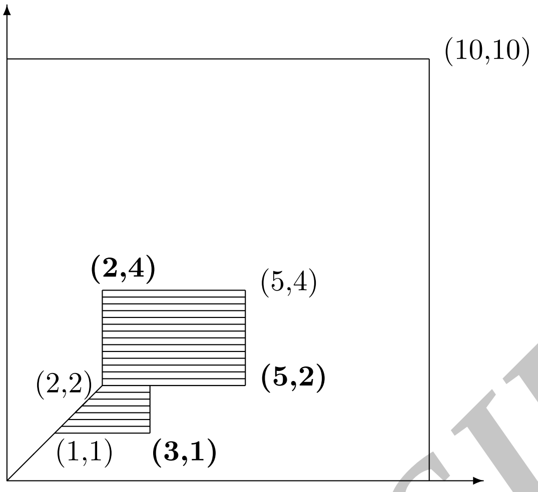

Fig. 5. Subsemimodule hull of X_3

Definition 2.11. *Let* $(H, *, \leqslant)$ *be a semimodule over* $(R, \oplus, \otimes)$ *with zero 0. A subset* X *of* H *is called linearly independent if for all finite subset* $\{x_1, \ldots, x_m\} \subseteq X$*, and elements* $a_1, \ldots, a_m \in R$*;* $(a_1.x_1) * \ldots * (a_m.x_m) = e$ *imply* $a_1 = \ldots = a_m = 0$.
If the subset X *is not linearly independent, it is called linearly dependent.*

Example 2.12. Let $L = \{1, 2, 3, 6\}$ and $x \leqslant y$ means that x divides y. Clearly $(L, \vee, \leqslant)$ is a semimodule over $(L, \vee, \wedge)$ with zero 1. Since $2 \wedge 3 = 1$, the set $\{3\}$ is not linearly independent.

Remark 2.13. *By the previous example, it is not true that if* $x \neq 0$ *then* $\{x\}$ *is linearly independent. But if* L *is a chain, then for every* *non-zero element x, the set $\{x\}$ is linearly independent.*

Definition 2.14. *Let $(H, *, \leqslant)$ be a semimodule over $(R, \oplus, \otimes)$. A linearly independent subset B of H is called a basis for H over R, if $< B >= H$.*

Example 2.15. Let L be as in Example 2.5. (see Fig. 6).

In this lattice the following subsets of L are linearly independent:

$K_1 = \{6\}, \quad K_2 = \{6, 12\}, \quad K_3 = \{12, 18\}$

$K_4 = \{6, 12, 36\}, \quad K_5 = \{6, 12, 18, 36\}$

But the following subsets are linearly dependent:

$K_6 = \{9\}, \quad K_7 = \{2, 3\}, \quad K_8 = \{4, 9\}, \ K_9 = \{6, 9\}$

Some sublattices generated by above subsets of L are as follows:

$< K_9 >= \{1, 2, 3, 6, 9, 18\}, \quad < K_3 >=< K_4 >=< K_5 >=< K_8 >= L,$

$< K_6 >= \{1, 3, 9\}$

Clearly K_3, K_4 and K_5 are bases of L. Also

$< (K_9)_* >= \{1, 2, 3, 6, 9\}, < K_9^* >= \{6, 9, 12, 18, 36\}$

$< (K_5)_* >= L, < K_5^* >= K_5.$

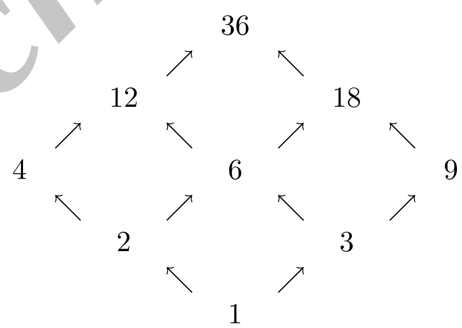

Fig. 6. The relationship between elements of L

Remark 2.16. (i) *Note that although* $< K_8 >= L$, *but* K_8 *contains no linearly independent subset.*
(ii) *For the basis* K_3 *we have* $6 = (6 \wedge 12) \vee (6 \wedge 18) = (2 \wedge 12) \vee (3 \wedge 18) = (3 \wedge 12) \vee (2 \wedge 18)$. *Therefore, representation of any elements of* L *in terms of a linear combination of elements of a basis is not unique.*

Example 2.17. Suppose $(L, \leqslant)$ be a bounded distributive lattice. Clearly, $\{1\}$ is a basis for $(L, \wedge, \leqslant)$ over $(L, \wedge, \vee)$. Note that in semimodule $(L^2, \wedge, \leqslant)$, the set $\{(1,1)^T\}$ is linearly independent but $< \{(1,1)^T\} > \neq L^2$.

3. Consistency of $\mathbf{A} * \mathbf{X} = \mathbf{b}$.

In this section we consider semimodule $(H, *, \leqslant)$ over semiring $(R, \oplus, \otimes)$. By a linear system of equations $A * X = b$ over R we mean the following equations:

$$\begin{cases} (a_{11}.x_1) * (a_{12}.x_2) * \ldots * (a_{1n}.x_n) = b_1 \\ (a_{21}.x_1) * (a_{22}.x_2) * \ldots * (a_{2n}.x_n) = b_2 \\ \quad\vdots \\ (a_{m1}.x_1) * (a_{m2}.x_2) * \ldots * (a_{mn}.x_n) = b_m \end{cases} \qquad (*)$$

where $a_{ij} \in R$ and $x_i, b_j \in H$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

Theorem 3.1. *Let L be a bounded distributive lattice. Consider $(L^n, \vee, \leqslant)$ as a semimodule over semiring $(L, \vee, \wedge)$ with scalar multiplication "$\wedge$". Let A, X and b are $m \times n$, $n \times 1$ and $m \times 1$ matrices over L, respectively. The linear system $A \vee X = b$ has a solution if and only if b belongs to the subsemimodule generated by columns of A.*

Proof. If we show the columns of A by $A_1, A_2, ..., A_n$; then the linear system $A \vee X = b$ can shown by

$$(x_1 \wedge A_1) \vee (x_2 \wedge A_2) \vee ... \vee (x_n \wedge A_n) = b$$

and clearly the linear system has a solution if and only if
$b \in < \{A_1, \dots, A_n\} >$ by Theorem 2.9. □

Example 3.2. Let L, K_9 and K_8 be as in Example 2.15. consider the linear equation

$$(6 \wedge x_1) \vee (9 \wedge x_2) = 3 \tag{1}$$

Then the set of all solutions of (1) is
$\{(1,3)^T, (1,6)^T, (1,12)^T, (3,1)^T, (3,3)^T, (3,6)^T, (3,12)^T, (3,2)^T, (3,4)^T, (9,1)^T, (9,3)^T, (9,6)^T, (9,12)^T, (9,2)^T, (9,4)^T\}$.
Linear equation (1) has solution since $3 \in < K_9 >$; the subsemimodule generated by $\{6, 9\}$. But if we change right hand side of (1) to 12 we have:

$$(6 \wedge x_1) \vee (9 \wedge x_2) = 12 \tag{2}$$

Clearly (2) doesn't have any solution since $12 \notin K_9$. Now consider

$$(4 \wedge x_1) \vee (9 \wedge x_2) = b \tag{3}$$

Since $< \{4, 9\} >=< K_8 >= L$, so (3) has solution for all $b \in L$.

Remark 3.3. *Note that Theorem 3.1. gives a theoretical necessary and sufficient condition for consistency of (*). A computational necessary and sufficient condition for consistency of (*) over a bounded chain was given in [6]. Finding such a condition(s) over a bounded distributive lattice is still an open problem.*

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