

P-Dense Submodules

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Abstract. Let M and P be right R -modules. A submodule K of an R -module M is called P -dense if for each $m \in M$, $(K : m)$ is a P -faithful right ideal of R . P_R is nonsingular if and only if, for each R -module M , every essential submodule of M is a P -dense submodule. For any R -module M , we obtain P -rational extension of M and equivalent condition in order that M is equal with its P -rational extension is found. An R -module P is called right Kasch if every simple R -module can be embedded in P . Finally, we give some equivalent conditions for an R -module P to be right Kasch.

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1. Introduction

Throughout this paper, all rings are associative with non-zero identity and modules are unitary right module. For any R -module M , $E(M)$ is injective hull of M . Let \mathcal{A} be a set of right ideals of R and K be a submodule of an R -module M . K is called \mathcal{A} -submodule of M , if for each $m \in M$, $(K : m) \in \mathcal{A}$. K denoted by $K \subseteq_{\mathcal{A}} M$. Let P be a right

R -module and I be a right ideal of R . I is called P -faithful, if

$$\text{ann}_P(I) = \{p \in P : pI = 0\} = 0.$$

Let \mathcal{A} be the set of all P -faithful right ideals of R and K be a submodule of M . If $K \subseteq_{\mathcal{A}} M$, then K is called a P -dense submodule. Dense submodules have been investigated by several authors some of their recent work are cited in the reference. If K is a P -dense submodule of M , then we denote it by $K \subseteq_P M$. In Section 2, we study properties of P -dense submodules of an R -module M . Equivalent conditions are given for a submodule to be P -dense are found 2.4. and show that P is a nonsingular R -module if and only if for each right R -module M , its essential submodules are P -dense submodules. In Section 3, we study modules are called right Kasch R -modules.

2. P-Dense Submodules

Lemma 2.1. *Let M and P be right R -modules and K be a submodule of M . The following are equivalent.*

- (i) K is a P -dense submodule of M .
- (ii) For each $m \in M$ and non-zero element $p \in P$, there exists $r \in R$ such that $pr \neq 0$ and $mr \in K$.

proof. (i) \Rightarrow (ii) Let $m \in M$ and $0 \neq p \in P$. By (i), $(K : m)$ is P -faithful, then $p(K : m) \neq 0$.

(ii) \Rightarrow (i) Let $m \in M$ and $p \in P$ such that $p(K : m) = 0$. If $p \neq 0$, by (ii), there exists $r \in R$ such that $pr \neq 0$ and $mr \in K$. Thus $p(K : m) \neq 0$, a contradiction. \square

Example 2.2. If K is a dense submodule of a right module M over a ring R , then K is M -dense.

Example 2.3. Every essential submodule of a Z -module is Z -dense.

Theorem 2.4. Let M and P be R -modules and K be a submodule of M . The following are equivalent.

(i) $K \subseteq_P M$.

(ii) $\text{Hom}_R(M/K, E(P)) = 0$.

(iii) For any submodule Q of M such that $K \subseteq Q \subseteq M$, $\text{Hom}_R(Q/K, P) = 0$.

Proof. (i) \Rightarrow (ii) Let $f \in \text{Hom}_R(M, E(P))$ be such that $f(K) = 0$. If $f \neq 0$, then $f(m) \neq 0$ for some non-zero element $m \in M$. Since $P \subseteq_{\text{ess}} E(P)$, there exists $r \in R$ such that $0 \neq f(mr) = f(m)r \in P$. Since $K \subseteq_P M$, there exists $s \in R$ such that $f(mr)s = f(mrs) \neq 0$ and $mrs \in K$, a contradiction.

(ii) \Rightarrow (iii) Assume that, for some Q as in (iii), there exists a non-zero R -homomorphism $g \in \text{Hom}_R(Q, P)$ such that $g(K) = 0$. Since $E(P)$ is an injective R -module, then g can be extended to $\bar{g} \in \text{Hom}_R(M, E(P))$.

Since $\bar{g}(K) = g(K) = 0$, by (ii), $\bar{g} = 0$, a contradiction.

(iii) \Rightarrow (i) Suppose that $p(K : m) = 0$ for some $m \in M$ and $p \in P \setminus \{0\}$.

We define $f : K + mR \rightarrow P$ by

$$f(k + mr) = pr \quad (k \in K, r \in R).$$

This map is well-defined, for, if $k + mr = \acute{k} + m\acute{r}$, then $(k - \acute{k}) = m(\acute{r} - r) \in K$. Hence $p(\acute{r} - r) = 0$. Clearly, f is an R -homomorphism vanishing on K . So by (iii), $0 = f(m) = p$, a contradiction. \square

Proposition 2.5. *Let M and P be R -modules and K and L be submodules of M . Then*

- (i) *If $K \subseteq_P M$, $L \subseteq_P M$, then $K \cap L \subseteq_P M$.*
- (ii) *Let $L \subseteq K \subseteq M$. Then $L \subseteq_P M$ if and only if $L \subseteq_P K$ and $K \subseteq_P M$.*

Proof. (i) Let $m \in M$ and $p \in P \setminus \{0\}$. Since $K \subseteq_P M$, there exists $r \in R$ such that $pr \neq 0$ and $mr \in K$. Since $L \subseteq_P M$, there exists $s \in R$ such that $prs \neq 0$ and $mrs \in L$. Thus $prs \neq 0$ and $mrs \in K \cap L$.

(ii) It is sufficient to prove the "if" part. Assume that $L \subseteq_P K$ and $K \subseteq_P M$. Let $m \in M$ and $p \in P \setminus \{0\}$. There exists $r \in R$ such that $pr \neq 0$ and $mr \in K$ since $K \subseteq_P M$. Since $L \subseteq_P K$, there exists $s \in R$ such that $prs \neq 0$ and $mrs \in L$. \square

Let M be a right module over a ring R . An element $m \in M$ is said

to be singular element of M if the right ideal $\text{ann}_r(m)$ is essential in R_R . The set of all singular elements of M is denoted by $Z(M)$. The right R -module M is called nonsingular module, if $Z(M) = 0$.

Theorem 2.6. *For any R -module P , the following are equivalent.*

- (i) P is a nonsingular R -module.
- (ii) Every essential submodule of any R -module M is a P -dense submodule.

Proof. (i) \Rightarrow (ii) Let M be a right R -module and $K \subseteq_{\text{ess}} M$. Let $m \in M$ and $p \in P \setminus \{0\}$. Since K is an essential submodule of M , then $(K : m) \subseteq_{\text{ess}} R$. If $p(K : m) = 0$, then $(K : m) \subseteq \text{ann}_r(p) \subseteq R$. Since $(K : m) \subseteq_{\text{ess}} R$, then $\text{ann}_r(p) \subseteq_{\text{ess}} R$ and hence $p \in Z(P)$, a contradiction.

(ii) \Rightarrow (i) Let $p \in Z(P)$. Then $\text{ann}_r(p) \subseteq_{\text{ess}} R$. By (ii), $\text{ann}_r(p) \subseteq_P R$. If $p \neq 0$, then there exists $r \in R$ such that $pr \neq 0$ and $1.r \in \text{ann}_r(p)$, a contradiction. \square

Definition 2.7. *For two R -modules M and P , we define*

$$\tilde{E}_P(M) = \{x \in E(M) : \forall f \in \text{Hom}_R(E(M), E(P)), f(M) = 0 \Rightarrow f(x) = 0\}.$$

Lemma 2.8. *Let P and M be R -modules and N be any submodule of $E(M)$ containing M . Then $M \subseteq_P N$ if and only if $N \subseteq \tilde{E}_P(M)$.*

Proof. For the "if" part, it suffices to show that $\text{Hom}_R(N/M, E(P)) =$

0. Assume that, $f \in \text{Hom}_R(N, E(P))$ such that $f(M) = 0$. Since $E(P)$ is an injective R -module, then f can be extended to $\bar{f} \in \text{Hom}_R(E(M), E(P))$. Since $\bar{f}(M) = f(M) = 0$ and $N \subseteq \tilde{E}_P(M)$, then $f(N) = \bar{f}(N) = 0$. Hence $f = 0$. For the "only if" part, assume that $M \subseteq_P N$ and consider $f \in \text{Hom}_R(E(M), E(P))$ such that $f(M) = 0$. If $f(N) \neq 0$, then there exists $n \in N \setminus \{0\}$ such that $f(n) \in E(P) \setminus \{0\}$. Since $P \subseteq_{ess} E(P)$, there exists $r \in R$ such that $f(n)r = f(nr) \in P \setminus \{0\}$. For $nr \in N$ and $f(nr) \in P \setminus \{0\}$, $M \subseteq_P N$ implies that $f(nr).s = f(nrs) \in P \setminus \{0\}$ and $nrs \in M$, for some $s \in R$. It is a contradiction, since $f(M) = 0$. \square

Proposition 2.9. *For two R -modules M and P , we have*

$$\tilde{E}_P(M) = \{m \in E(M) : \forall x \in E(P) \setminus \{0\}, x(M : m) \neq 0\}.$$

Proof. Let $m \in \tilde{E}_P(M)$ and $x \in E(P) \setminus \{0\}$. Since P is an essential submodule of $E(P)$, there exists $r \in R$ such that $xr \in P \setminus \{0\}$. By Lemma 2.8. $M \subseteq_P \tilde{E}_P(M)$ and hence there exists $s \in R$ such that $xrs \neq 0$ and $mrs \in M$, hence $x(M : m) \neq 0$. Conversely, assume $m \in \text{RHS}$ and $f \in \text{Hom}_R(E(M), E(P)) \neq 0$ such that $f(M) = 0$. If $f(m) \neq 0$, then by hypothesis, $f(m)(M : m) \neq 0$. Thus there exists $r \in R$ such that $f(m)r = f(mr) \neq 0$ and $mr \in M$. It is a contradiction, since $f(M) = 0$. \square

Definition 2.10. *An R -module M_R is called rationally P - complete if*

$$\tilde{E}_P(M) = M.$$

Theorem 2.11. *For any R -modules M and P , the following are equivalent.*

- (i) M is rationally P -complete.
- (ii) For any R -modules $A \subseteq B$ such that $\text{Hom}_R(B/A, E(P)) = 0$, any R -homomorphism $f : A \rightarrow M$ can be extended to B .

Proof. (i) \Rightarrow (ii). Let $A \subseteq B$ be R -modules such that $\text{Hom}_R(B/A, E(P)) = 0$, and let $f \in \text{om}_R(A, M)$. Since $E(M)$ is an injective R -module, We can extended f to $g : B \rightarrow E(M)$. We claim that

$$M \subseteq_P M + g(B).$$

Once we have proved this, then lemma 2.8. and (i) imply that $g(B) \subseteq M$ and we are done. By theorem 2.4. it suffices to prove that

$$\text{Hom}_R((M + g(B))/M, E(P)) = 0.$$

Suppose $h \in \text{Hom}_R((M + g(B))/M, E(P))$. Define R -homomorphism $\acute{h} \in \text{Hom}_R(B/A, (M + g(B))/M)$ by $\acute{h}(b + A) = g(b) + M$ ($b \in B$). Since $\text{Hom}_R(B/A, E(P)) = 0$, $h\acute{h} = 0$. Hence for each $m \in M$ and $b \in B$,

$$h((m + g(b)) + M) = h(g(b) + M) = h\acute{h}(b + A) = 0.$$

Therefore $h = 0$. (ii) \Rightarrow (i) By theorem 2.4. $M \subseteq_P \tilde{E}_P(M)$, and so $\text{Hom}_R(\tilde{E}_P$

$(M)/M, E(P)) = 0$. By (ii), the identity map $id : M \rightarrow M$ can be extended to an R -module $g : \tilde{E}_P(M) \rightarrow M$ such that $gi = id$, where $i : M \rightarrow \tilde{E}_P(M)$ is an inclusion map. Therefore $\tilde{E}_P(M) = \ker g \oplus \text{Im } i$. Since $\ker g \cap M = 0$ and M is an essential submodule of $E(M)$, then $\ker g = 0$. Hence

$$\tilde{E}_P(M) = \text{Im } i = M. \quad \square$$

3. Right Kasch R -modules

If S is a simple right module and P is a right module over a ring R , it is of interest to know whether S can be embedded in P . Consideration of this issue leads to the notion of right Kasch R -modules.

Definition 3.1. *Let P be a right module over a ring R . We say that P is right Kasch R -module if every simple right R -module S can be embedded in P .*

Theorem 3.2. *Let P be a right module over a ring R . For any maximal right ideal m of R , the following are equivalent.*

- (i) R/m embeds to P_R .
- (ii) $m = \text{ann}_r(x)$ for some $x \in P$.
- (iii) m is not a P -faithful right ideal.
- (iv) $m = \text{ann}_r(\text{ann}_P(m))$.
- (v) m is not P -dense in R_R .

Proof. (i) \Rightarrow (ii) Let $f \in \text{Hom}_R(R/m, P)$ be a monomorphism and $0 \neq x = f(1 + m)$. For each $r \in m$,

$$xr = f(1 + m)r = f(r + m) = f(m) = 0.$$

Therefore $r \in \text{ann}_r(x)$. Hence $m \subseteq \text{ann}_r(x)$. Since $\text{ann}_r(x) \neq R$, then $m = \text{ann}_r(x)$.

(ii) \Rightarrow (iii) Let $m = \text{ann}_r(x)$ for some non-zero element $x \in P_R$. Then $xm = 0$, therefore m is not a P -faithful right ideal of R .

(iii) \Rightarrow (iv) We know that $\text{ann}_P(m).m = 0$, then $m \subseteq \text{ann}_r(\text{ann}_P(m))$. If $\text{ann}_r(\text{ann}_P(m)) = R$, then $\text{ann}_P(m) = 0$. Hence m is a P -faithful, a contradiction. Therefore $\text{ann}_r(\text{ann}_P(m))$ is a proper right ideal of R and hence

$$m = \text{ann}_r(\text{ann}_P(m)).$$

(iv) \Rightarrow (v) Since $\text{ann}_P(m) \neq 0$, there exists $0 \neq p \in \text{ann}_P(m)$.

For $1 \in R$, $(m : 1) = m$, and hence

$$pm = p(m : 1) = 0.$$

Therefore m is not P -dense in R_R , by Lemma 2.1.

(v) \Rightarrow (i) By Theorem 2.4. since m is not P -dense in R_R , then $\text{Hom}_R(R/m, E(P)) \neq 0$. Let f be a non-zero element of $\text{Hom}_R(R/m, E(P))$. Since R/m is a simple R -module, f is a monomorphism and hence $\text{Im } f \simeq R/m$. Since P is an essential submodule of $E(P)$, then $\text{Im } f \cap$

$P \neq 0$. Since $Im f$ is a simple R -module, then $Im f \cap P = Im f$. Therefore $Im f \subseteq P$. This show that $f : R/m \rightarrow P$ is a monomorphism. \square

Corollary 3.3. *For any right R -module P , the following are equivalent.*

- (i) P is a right Kasch module.
- (ii) Any maximal right ideal in R has the form $ann_r(x)$ for some $x \in P$.
- (iii) The set of P -faithful right ideals of R does not contain any maximal right ideal of R .
- (iv) For any maximal right ideal m of R , $ann_r(ann_P(m)) = m$.
- (v) The only P -dense right ideal in R is R itself.
- (vi) For any proper right ideal I of R , $ann_P(I) \neq 0$.

Proof. (i) \Rightarrow (ii) Let m be a maximal right ideal of R , then the simple R -module R/m embeds in P . By Theorem 3.1, $m = ann_r(x)$ for some $x \in P$.

(ii) \Rightarrow (iii) \Rightarrow (iv) By Theorem 3.1.

(iv) \Rightarrow (v) Let J be a proper P -dense right ideal of R_R . There exists a maximal right ideal m containing J . By Proposition 2.5. m is a P -dense in R_R . By Theorem 3.1. $m \neq ann_r(ann_P(m))$, a contradiction.

(v) \Rightarrow (i) For each maximal right ideal m of R , m is not P -dense in R . By Theorem 3.2. R/m embeds in P . Which implies that P is a right Kasch R -module.

(ii) \Rightarrow (vi) Let I be a proper right ideal of R . There exists a maximal

right ideal m contains I . By (ii), $m = \text{ann}_r(x)$ for some $0 \neq x \in P$. Then $I \subseteq m = \text{ann}_r(x)$, and so $xI = 0$. It implies that $\text{ann}_P(I) \neq 0$.
(vi) \Rightarrow (iii) For any maximal right ideal m of R , $\text{ann}_P(m) \neq 0$. Then m is not P -faithful. \square

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