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Stochastic Differential Equations and Markov Processes in the Modeling of Electrical Circuits

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Abstract. Stochastic differential equations(SDEs), arise from physical systems that possess inherent noise and certainty. We derive a SDE for electrical circuits. In this paper, we will explore the close relationship between the SDE and autoregressive(AR) model. We will solve SDE related to RC circuit with using of AR(1) model (Markov process) and however with Euler-Maruyama(EM) method. Then, we will compare this solutions. Numerical simulations in MATLAB are obtained.

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1. Introduction

One of the most important subjects in mathematical science is convenient mathematical model fitting for prediction of events that will occur in future. Deterministic and stochastic differential equations(SDEs), are fundamental for the modeling in science and engineering.
A SDE is a differential equation in which one or more of the terms is a stochastic process, thus resulting in a solution which is itself a stochastic process.

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In physical science, SDEs are usually written as Langevin equation[15]. These consist of an ordinary differential equation containing a deterministic part and an additional random white noise term.
One of the more important applications of SDE, is in the modeling electrical networks. A RC network, is an electric circuit composed of resistors and capacitors driven by a voltage or current source. The 1st order RC circuit composed of one resistor and one capacitor in series, is the simplest example of an RC circuit ([19]).
The deterministic model of the circuit is replaced by a stochastic model by adding a noise term in both the potential source and the resistance. The random behavior of a RC circuit can be viewed as a stochastic process. SDEs are powerful mathematical tools to analyze such processes. Based on the theory of SDEs ([12,15]) randomly disturbed electric circuits can be modeled in the time domain.
With adding a noise term to the right side of a deterministic equation in the Kirchoff circuit laws, we introduce a SDE ([14]). Two typs of noises can exist in an electrical circuit; external noise and internal noise. The effects of both types of noises in electrical circuits has been studied by many researchers ([5]).
For example, Kampowsky et al (1992), described classification and numerical simulation of electrical circuits with white noise ([10]). Additionally, Penski (1999), was presented a new numerical method for SDEs with white noise and its application in circuit simulation ([16]). Recently, Rawat (2008), showed an application of the Ito stochastic calculus to the problem of modeling a series RC Circuit with white noise and colored noise, including numerical solution ([17]).
Many SDEs cannot be solved explicitly. For this reason, it is convenient to develop numerical methods that provide approximated simulations of these equations.
Numerical solution of SDEs has been studied by many researchers (see, for example, [2,3,11,12,13,18], and the references therein). In this paper, we focus on numerically solution of stochastic model RC circuit with using of the first-order Autoregressive model.
The organization of this paper is as follows: In Section 2, stochastic calculus theory is reviewed. Section 3 will introduced SDEs for RC circuit.

Numerical solutions results will be presented in Section 4. A first-order AR model, as a discrimination of the SDE, is discussed in Section 5. The paper ends with a conclusion in Section 6.

2. Stochastic Calculus

In many physical applications, there are many processes in which the random variables depend on space and/or time and this introductory material will be the subject of this section.
A stochastic process $X(t)$, is a family of random variables $X(t,\omega)$ of two variables $t \in T$, $\omega \in \Omega$ on a common probability space (Ω, F, P) which assumes real values and is $P-$measurable as a function of ω for each fixed t. The parameter t is interpreted as time, with T being a time interval. $X(t,.)$ represents a random variable on the above probability space Ω, while $X(.,\omega)$ is called a sample path or trajectory of the stochastic process, ([6]).

The best-known stochastic process to which stochastic calculus is applied is the Wiener process or Brownian motion. A Wiener process is a time continuous process with the property $W(t) \sim N(0,t)$ $(0 \leqslant t \leqslant T)$, usually it is differentiable almost nowhere.

For a generalized stochastic process, derivatives of any order can be defined. Suppose that $W(t)$ is a generalized version of a wiener process which is used to model the motion of stock prices. An example of a generalized stochastic processes is white noise. White noise $\xi(t)$ is defined as $\xi(t) = \frac{dW(t)}{dt}$ ([15]). Stochastic calculus is a branch of mathematics that operates on stochastic processes. It allows a consistent theory of integration to be defined for integrals of stochastic processes with respect to stochastic processes. The main part of stochastic calculus are the Ito calculus and Stratonovich. Ito calculus, extends the methods of calculus to stochastic processes such as Brownian motion.
We go back to the definition of an integral:

$$\int_0^t f(t)dt = \lim_{n\to\infty} \sum_{j=1}^{n} f(\tau_j)(t_{j+1} - t_j)$$

where τ_j, is in the interval $[t_j, t_{j+1}]$. More generally has Riemann-Stieltjes integral:

$$\int_0^t f(t)dg(t) = \lim_{n\to\infty} \sum_{j=1}^{n} f(\tau_j)(g(t_{j+1}) - g(t_j)),$$

for a smooth measure $g(t)$, limit converges to a unique value regardless of where τ_j, taken in interval $[t_j, t_{j+1}]$. The Ito and Stratonovich calculus follows the same rules as for the regular Riemann-Stieltjes calculus. If our choose is lower end point, of the partition $[t_n, t_{n+1}]$, we have the case Ito integral, but if we choose midpoint $\frac{t_n+t_{n+1}}{2}$, we have Stratonovich case (where the symbol o, is employed).

$$(I)\int_0^t W(s)dW(s) = \frac{1}{2}[W^2(t) - t],$$

while

$$(S)\int_0^t W(s)o\ dW(s) = \frac{1}{2}[W^2(t)].$$

A SDE is given by

$$X'(t) = f(t, X(t)) + g(t, X(t))\xi(t),\ \ X(0) = x_0, (0 \leqslant t), \tag{1}$$

where f is the deterministic part, $g\xi(t)$ is the stochastic part, and $\xi(t)$ denotes a generalized stochastic process ([12]).
If we replace $\xi(t)dt$ by $dW(t)$ in equation (1), an Ito SDE can be rewritten as [9,15]:

$$dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), \tag{2}$$

where $f(t, X(t))$ and $g(t, X(t))$ are drift and diffusion term, respectively, and $X(t)$ is a solution which we try to find based on the experimental data ([3]). We can represent the SDE in the integral form

$$X(t) = X(0) + \int_0^T f(s, X(s))ds + \int_0^T g(s, X(s))dW(s) \tag{3}$$

where the first integral in (3) is an ordinary Riemann integral, and the second integral in (3) is the Ito stochastic integral. Let $0 \leqslant t_1 \leqslant ... \leqslant$

$t_n = T$, be a partition of the interval $[0, T]$ and $\lambda_n = max(t_i - t_{i-1})$. The Ito integral $\int_0^T g(s, X(s))dW(s)$ is defined as the limit in the quadratic mean:

$$\lim_{\lambda_n \to 0} \sum_{i=1}^{n} (g(t_{i-1}, X(t_{i-1}))(W(t_i) - W(t_{i-1}))). \tag{4}$$

If the integrand g is jointly measurable and $\int_0^T E(|g(s, X(s))|^2)ds < \infty$, then the limit in (4) exists ([15]).

3. Modeling Electrical Circuit

Any electrical circuit consists of resistor (R), capacitor (C) and inductor (L). These circuit elements can be combined to form an electrical circuits in four distinct ways: the RC, RL, LC and RLC circuits. Then a resistor-capacitor circuit(RC), or RC network, is an electric circuit composed of resistor and capacitors driven by a voltage or current source. We consider model stochastic RC circuit by Rawat and Parthasarathy ([17]). Let $Q(t)$ be the charge on the capacitor and $V(t)$ be the potential source applied to the input of a RC circuit. Using Kirchoff's second law,

$$V(t) = I(t)R + \frac{Q(t)}{C}, \tag{5}$$

and since $I(t) = \frac{dQ(t)}{dt} = Q'(t)$, the following equation holds [9]:

$$Q'(t) + (RC)^{-1}Q(t) = R^{-1}V(t). \tag{6}$$

If $V(t)$ is a piecewise continuous function, the solution of the first order linear differential equation (6) is:

$$Q(t) = Q(0)\exp(\frac{-t}{RC}) + \frac{1}{R}\int_0^t \exp(\frac{-(t-s)}{RC})V(s)ds, \tag{7}$$

where $Q(0)$ is the initial charge stored in the capacitor. The resistance and potential source may not be deterministic but of the form:

$$R^* = R + \text{"}noise\text{"} = R + \alpha W(t), \tag{8}$$

$$V^*(t) = V(t) + "noise" = V(t) + \beta N_1(t), \tag{9}$$

where $W(t)$ is a zero mean, exponentially correlated stationary process and $N_1(t)$ is a white noise process of mean Zero and variance one, and α, β, are non negative constant, known as the intensity of noise. Substituting (8) and (9) in (6) we get:

$$dQ(t) = -\frac{Q(t)}{C(R + \alpha W(t))}dt + \frac{V(t)}{(R + \alpha W(t))}dt + \frac{\beta dN_2(t)}{(R + \alpha W(t))} \tag{10}$$

where, $dN_2(t) = N_1(t)dt$ and $N_2(t)$ is the Brownian motion process, independent of $N_1(t)$. Form (10) is a model stochastic RC circuit [17].

4. Numerical Solution of SDE with EM Method

When a differential equation model for some physical phenomenon is formulated preferably the exact solution can be obtained. However, even for ODEs, this is generally not possible and numerical methods must be used.
The Euler-Maruyama(EM) numerical method is used for the simulation of $X(t)$. There are similar relationships the numerical methods for ordinary differential equations(ODEs) and those for SDEs.
Let $X = \{X(t); t \in [t_0, T]\}$ satisfying the scaler SDE (2) on $t_0 \leqslant t \leqslant T$, with the initial value $X(t_0) = x_0$ and for a given partition $t_0 = \tau_0 \leqslant \tau_1 \leqslant ... \leqslant \tau_n \leqslant \cdots \leqslant \tau_N = T$, of the time interval $[t_0, T]$.
Euler approximation is a continuous time stochastic process

$$Y = \{Y(t); t_0 \leqslant t \leqslant T\}$$

satisfying the iterative scheme equation ([3]),

$$Y_{n+1} = Y_n + f(\tau_n, Y_n)(\tau_{n+1} - \tau_n) + g(\tau_n, Y_n)(W(\tau_{n+1}) - W(\tau_n)), \tag{11}$$

for $n = 0, 1, \cdots, N-1$ with initial value $Y_0 = x_0$, where we have $Y_n = Y(\tau_n)$, $\Delta_n = \frac{T-t_0}{n}$, $\tau_n = t_0 + n\Delta$, $\Delta = max_n \Delta_n$, $\Delta W_n = W(\tau_{n+1}) - W(\tau_n) \sim N(0, \Delta_n)$, and from this we know that

$$E(\Delta W_n) = 0,$$

$$Var(\Delta W_n) = E(\Delta W_n^2) - \mu^2 = E(\Delta W_n^2) = 1.$$

The sequence $Y = \{Y_n; n = 0, 1, \cdots, N-1\}$ is the value of the EM approximation at the instants τ_n $(n = 0, 1, \cdots, N-1)$. The EM method has strong order of accuracy o.5, is numerically stable and converges to the Ito solution of system (2) ([12]). Beginning with $W_1 = 0$, we can run a simulation of a pollen track by using the random number generation capabilities of any programming language. In Matlab, for example, the *randn* routine returns a normally distributed random number with mean 0 and variance 1. Over 256 time steps, the particle track of a pollen grain is shown in Figure 1. Assume $R = 10\Omega$, $C = 0.1F$, $V(t) = v = 20V$ and $X(0) = 0$. The results of the deterministic and stochastic solutions of the RC circuit with both stochastic resistance and stochastic potential source $(\alpha = 1, \beta = 1)$ is shown in Table 1.

Table 1: Approximate values for the deterministic and stochastic models

Case	t	deterministic case	stochastic case
1	0	1	0.9856
2	0.0625	1.4647	1.4406
3	o.125	1.7134	1.7588
4	0.25	1.9179	1.3566
5	0.5	1.9932	1.8619
6	1	1.9999	1.6253

Figure 2, illustrates the average of the approximate solution with 256 independent trials using of the EM method.

5. Autoregressive Model

In statistics, an autoregressive (AR) model is a type of random process which is often used to model and predict various types of natural phenomena. The notation AR(p) refers to the autoregressive model of order p. The AR(p) model is defined as

$$X_t = \mu + \sum_{i=1}^{p} \varphi_i X_{t-i} + \xi_t,$$

where $\varphi_1, \varphi_2, \cdots, \varphi_p$ are the parameters of the model, and ξ_t, is white noise process with mean zero and Variance σ^2. Then, AR(1) process is given by:

$$X_t = \mu + \varphi X_{t-1} + \xi_t. \tag{12}$$

The model AR(1) is wide sense stationary if $|\varphi| < 1$ ([1]). In the AR(1) model:

$$E(X_t) = \mu,$$

$$Var(X_t) = E(X_t^2) - E^2(X_t) = \frac{\sigma^2}{1-\varphi^2}.$$

The autocovariance function is given by:

$$\gamma_n = E(X_t X_{t+n}) - E(X_t)E(X_{t+n}) = \frac{\sigma^2}{1-\varphi^2}\ \varphi^{|n|}.$$

In AR(1) model, $\rho_k = \varphi^k$ for $k = 0, 1, 2, \cdots$ and $\phi_{11} = 1$ and $\phi_{kk} = 0$ for $k \geqslant 2$ (ϕ_{kk} is partial autocorrelation function ([1])).

Now, we will explore the relationship between the continue-time SDE and AR(1) model(discreet-time SDE) for stochastic RC circuit model[7]. We convert continuous-time differential $dQ(t)$ in (10) to discreet-time difference $Q(t_{k+1}) - Q(t_k)$,

$$Q(t_{k+1}) - Q(t_k) = -\frac{Q(t)}{C(R+\alpha w(t))}dt + \frac{V(t)}{(R+\alpha W(t))}dt + \frac{\beta dN_2(t)}{(R+\alpha W(t))},$$

Then, we replacing $Q(t)$ with the arithmetic average $\frac{Q(t_{k+1})+Q(t_k)}{2}$.

$$Q(t_{k+1}) = \frac{Q(t_{k+1})+Q(t_k)}{2} - \frac{\frac{Q(t_{k+1})+Q(t_k)}{2}}{C(R+\alpha W(t))}dt + \frac{V(t)}{(R+\alpha W(t))}dt + \frac{\beta dN_2(t)}{(R+\alpha W(t))},$$

Let us consider discreet-time sampling times $t_0, t_1, ..., t_k, \cdots$, where $t_{k+1} - t_k = \Delta t$. With replacing the Wiener process $dN_2(t)$ with a complex discrete Gaussian process $\sqrt{\Delta t}\ \widetilde{N}(k)$, where $\widetilde{N}(k)$ is a standard complex Gaussian process with zero mean and unit variance [7].

$$Q(t_{k+1}) = \frac{Q(t_{k+1})+Q(t_k)}{2} - \frac{\frac{Q(t_{k+1})+Q(t_k)}{2}}{C(R+\alpha W(t))}\Delta t + \frac{V(t)}{(R+\alpha W(t))}\Delta t + \frac{\beta\sqrt{\Delta t}}{(R+\alpha W(t))}\widetilde{N}(k).$$

Then

$$Q(t_{k+1}) = \frac{2C\ V(t)\Delta t}{Ch(t)+\Delta t} + \frac{Ch(t)-\Delta t}{Ch(t)+\Delta t}Q(t_k) + \frac{2\beta C\sqrt{\Delta t}}{Ch(t)+\Delta t}\widetilde{N}(k), \tag{13}$$

where $h(t) = R + \alpha W(t)$. Equation (13) is a first-order AR process for the RC circuit. The AR coefficients are function of constant values C, $V(t) = V$, α, β and R, the sampling time interval Δt, and with $\mu = \frac{2CV\Delta t}{Ch(t)+\Delta t}$. Now, we define a difference process as [7].

$$\Delta_\gamma Q(t_k) = Q(t_{k+1}) - \gamma Q(t_k), \tag{14}$$

Then

$$\Delta_\gamma Q(t_k) = \mu + \frac{2\beta C\sqrt{\Delta t}}{Ch(t) + \Delta t}\widetilde{N}(k) \tag{15}$$

where

$$\gamma = \frac{Ch(t) - \Delta t}{Ch(t) + \Delta t}.$$

Model (15), is a complex Gaussian process with μ mean and variance $(\frac{2\beta C\sqrt{\Delta t}}{Ch(t)+\Delta t})^2$. Table 2, gives numerical results equation (10) using AR(1) model. The data is obtained for different some time. In this simulation, we suppose $Q(0) = Q(t_0) = 0$, $t_0 = 0$, $t_1 = 0.0625$, $t_2 = 0.125$, $t_3 = 0.25$, $t_4 = 0.5$ and $t_5 = 1$. However, supposed $\Delta t_k = t_k - t_{k-1}$, $(1 \leqslant k \leqslant 5)$. $W(t_k)$ is obtained of Fig 1.

Table 2: Numerical results of discrete-time approximation

Case	$\Delta(t_k)$	$W(t_k)$	$h(t_k)$	$\widetilde{N}(k)$	Exact value	$Q(t_{k+1})$
1	0.0625	-0.0794	0.9206	0.32	1.4647	1.7209
2	0.0625	-0.4656	0.5344	0.3187	1.7134	2.1592
3	o.125	-0.6911	0.3089	0.3187	1.9179	2.0484
4	0.25	-0.5621	0.4379	0.3150	1.9932	2.0732
5	0.5	-0.5828	0.4172	0.3006	1.9999	2.0165

6. Conclusion

We obtained stochastic model by adding noise in both the potential source and the resistance in the form deterministic model related to the Kirchhoff circuit second law. This paper shows an application of the Ito stochastic calculus in the RC modeling, including both analytical and numerical solutions.

We have presented a first-order stochastic AR model for a RC circuit, which is based on SDE modeling. Moreover, we explored the close relationship between the SDE and AR(1) model for electrical circuits. We have the intention to extend numerical solution the SDE with white noise. Euler-Maruyama method and AR(1) model are used for comparison in numerically solution of the stochastic model (10). By comparing tables 1 and 2, we conclude that discreet-time numerically approximation (AR(1) method) is more accurate than continuous-time approximation (EM method).

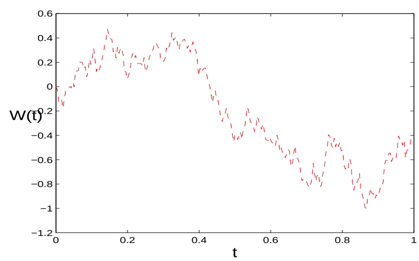

Figure 1: approximation solution

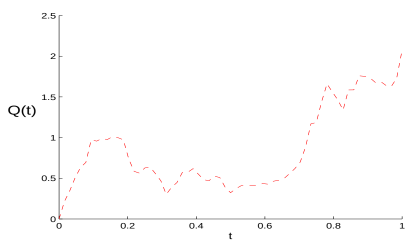

Figure 2: Brownian path simulation

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