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Some Properties of C^*-Graded Metric Spaces and Fixed Point Results

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Abstract. In this paper, by using of C^*-algebra we give a metric with range in positive unit ball of a C^*-algebra. Indeed this is a generalization of a fuzzy metric space. Some definitions of compatible mappings of types (I) and (II) are introduced and some fixed and common fixed point theorems are proved.

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1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [34] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. Deng [8], Ereeg [9], Fang [10], George [11], Kaleva and Seikkala [18], Kramosil and Michalek [19] have introduced the concept of fuzzy metric spaces in different ways.

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In fuzzy metric spaces given by Kramosil and Michalek [19], Grabiec [12] obtained the fuzzy version of Banach contraction principle, which has been improved and extended by some authors.
Sessa ([28]) defined a generalization of commutativity introduced by Jungck ([16]), which is called the weak commutativity. Further, Jungck ([17]) introduced more generalized commutativity, so called compatibility. Mishra et al. [21] obtained some common fixed point theorems for compatible mappings in fuzzy metric spaces. Pathak et al. [23] introduced the concept of compatible mappings of type (I) and (II) in metric spaces.
Many authors ([13,20,27,26,29]) have also proved some fixed point theorems in fuzzy (*probabilistic*) metric spaces (see [1-6,10,12,14,15,30]).
Now we give basic definitions and their properties as follows:
Recall that, a complex algebra is a vector space A over the field $\mathbb{C}$ with one multiplication from $A \times A$ into A that satisfies the following relations;

1. $x(yz) = (xy)z$,
2. $x(y+z) = xy + xz, \quad (x+y)z = xz + yz$,
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$,

for all x, y, z in A and α in $\mathbb{C}$.
If A is a Banach space with respect to $||.||$ that satisfies multiplicative inequality
$$||xy|| \leqslant ||x|| \, ||y|| \qquad (x \in A, y \in A)$$
and A contains unit element e such that
$$xe = ex = x \qquad (x \in A)$$
and $||e|| = 1$, then A is called a Banach algebra.
The map $x \to x^*$ from complex algebra A into itself is called an involution if for all x and λ in $\mathbb{C}$ we have,

1. $(x+y)^* = x^* + y^*$
2. $(\lambda x)^* = \overline{\lambda} x^*$

3. $(xy)^* = y^*x^*$

4. $x^{**} = x$.

Every Banach algebra with an involution $x \to x^*$ with the following relation is called a C^*-algebra,

$$||xx^*|| = ||x||^2 \qquad (x \in A).$$

From now on, A is a C^*-algebra. For every $x \in A$, we define the spectrum of x, is $\sigma(x)$, as follows,

$$\sigma(x) = \{\lambda \in \mathbb{C} : \lambda e - x \quad \text{is not invertible in A}\}.$$

In [28], it has been that for all $x \in A$, where A is a Banach algebra, $\sigma(x)$ is non-empty and compact.
We say that $x \in A$ is positive (in symbol $x \succeq 0$) if $x = x^*$ and $\sigma(x) \subseteq [0, \infty)$ and similarly x is strictly positive $(x \succ 0)$ if $x = x^*$ and $\sigma(x) \subseteq (0, \infty)$. Positive unit ball of A is denoted by B_A^+ i.e

$$B_A^+ := \{x \in A : x \succeq 0, ||x|| < 1\} \cup \{e\}.$$

Let A^+ be the set of all positive elements of A.
Now we introduce two binary relation on A^+ as, $a \succeq b$ and $a \succ b$, which means that, $a - b \succeq 0$ and $a - b \succ 0$,respectively. These two binary relations have some good properties that we are going to summarize some of it:

1. if a, b and $c \in A^+$ and $a \succeq b$ then $a + c \succeq b + c$,

2. if $a \succeq b$ then $ta \succeq tb$ for all non-negative real number,

3. $a \succeq b$ iff $-b \succeq -a$.

Theorem 1.1. *Let A be a C^*- algebra*

1. If $a \succeq 0$ and $b \succeq 0$ then $a + b \succeq 0$.

2. $A^+ = \{a^*a| \quad a \in A\}$.

3. *If* $a, b \in A^*$ *and* $c \in A$, *then* $b \succeq a$ *implies* $c^*bc \succeq c^*ac$.

4. *If* $b \succeq a \succeq 0$, *then* $||b|| \geqslant ||a||$.

5. *If* $a, b \in A^*$ *and* a, b *are invertible elements, then* $b \succeq a$ *implies* $a^{-1} \succeq b^{-1} \succeq 0$.

6. *If* $r \geqslant s$, *for all* $r, s \in \mathbb{R}^+$, *then* $re \succeq se$.

7. *If* $a, b \in A^+$, *then* $a \succeq b$ *and* $b \succeq a$, *implies* $a = b$.

Proof. See [21]. □

Definition 1.2. *A binary operation* $\bullet : B_A^+ \times B_A^+ \longrightarrow B_A^+$ *is a continuous* t *-norm if it satisfies the following conditions:*

1. $\bullet$ *is associative and commutative,*

2. $\bullet$ *is continuous,*

3. $a \bullet e = a$ *for all* $a \in B_A^+$,

4. $a \bullet b \preceq c \bullet d$ *whenever* $a \preceq c$ *and* $b \preceq d$ *for all* $a, b, c, d \in B_A^+$,

5. $||a \bullet b||e = ||a||e \bullet ||b||e$.

Example 1.3. $\bullet : B_A^+ \times B_A^+ \longrightarrow B_A^+$ with $\bullet(a, b) = ab$, where ab is the product of C^* -algebra A, is a continuous t- norm.

Definition 1.4. *The pair* (M, F) *with* $F : X \longrightarrow B_A^+$ *is called a* C^**-graded set on* X. *Every* $F(x)$ *in* B_A^+ *is celled graded membership of* x *in* B_A^+.

Definition 1.5. *A triple* $(X, M, \bullet)$ *is called a* C^**-graded metric space if* X *is a non-empty arbitrary set,* $\bullet$ *is a continuous* t*-norm and* M *is a* C^**-graded set on* $X^2 \times (0, \infty)$ *satisfying the following conditions for all* $x, y, z \in X$,$u \in B_A^+$ *and* $t, s \succ 0$.

1. $M(x, y, .) : (0, \infty) \longrightarrow B_A^+$ *is continuous,*

2. $M(x, y, t) \succ 0$,

3. $M(x, y, t) = e$ *if* $x = y$,

4. $M(x, y, t) = M(y, x, t)$,

5. $M(x, y, t) \bullet M(y, z, s) \preceq M(x, z, t + s)$.

Definition 1.6. *Let* $(X, M, \bullet)$ *be a* C^**-graded metric space. For any* $t > 0$ *and* $x \in X$ *we define the open ball* $B(x, r, t)$ *with center* x *and radius* $0 < r < 1$ *is defined by,*

$$B(x, r, t) = \{y \in X : M(x, y, t) \succ (1 - r)e\}.$$

Definition 1.7. *Let* $(X, M, \bullet)$ *be a* C^**-graded metric space and* $A \subset X$.

1. *A sequence* $\{x_n\}$ *in* X *converges to* x *if and only if* $M(x_n, x, t) \to e$ *as* $n \to \infty$ *for all* $t > 0$.

2. *Sequence* $\{x_n\}$ *in* X *is called a Cauchy sequence if for each* $0 < \epsilon < 1$ *and* $t > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that* $M(x_n, x_m, t) \succ (1 - \varepsilon)e$, *for any* $n, m \geqslant n_0$.

3. *The* C^**-graded metric space* $(X, M, *)$ *is said to be complete if every Cauchy sequence is convergent.*

4. *Let* τ *be the set of all* $A \subset X$ *with* $x \in A$ *if and only if there exist* $t > 0$ *and* $0 < r < 1$ *such that* $B(x, r, t) \subset A$. *Then* τ *is a topology on* X *(induced by the* C^**-graded metric* M*).*

Example 1.8. Let $a \bullet\ b = ab$ (ordinary product in A). For any $t \in (0, \infty)$, define

$$M(x, y, t) = (\frac{t}{t + ||x - y||})e \quad , \quad (x, y \in X)$$

then $(M, X, \bullet)$ is a C^*-graded metric space.

Lemma 1.9. *Let $(X, M, \bullet)$ be a C^*-graded metric space then $M(x, y, t)$ is non-decreasing with respect to t for all x, y in X.*

Proof. If $t_1 \leqslant t_2$, then $t_2 = t_1 + \varepsilon$, for some $\epsilon > 0$. By using (5) from Definition 1.5., if $y = x$, $s = t_1$ and $t = \varepsilon$ we have $M(x, x, \varepsilon) \bullet M(x, z, t_1) \preceq M(x, z, t_1 + \varepsilon)$ then from (3) of Definition 1.5., and (3) of Definition 1.2., we have

$$M(x, z, t_1) \preceq M(x, z, t_1 + \varepsilon) = M(x, z, t_2)$$

for all x, y in X. Note that $M(x, z, t_1) \preceq M(x, z, t_2)$ means that $M(x, z, t_2) - M(x, z, t_1)$ is a positive element with norm less than one of C^*-algebra A or

$$M(x, z, t_2) - M(x, z, t_1) \in B_A^+. \quad \square$$

Definition 1.10. *Let $(X, M, \bullet)$ be a C^*-graded metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if*

$$lim_n M(x_n, y_n, t_n) = M(x, y, t)$$

whenever a sequence $\{(x_n, y_n, t_n)\}_n$ in $X^2 \times (0, \infty)$, exists such that

$$lim_n M(x_n, x, t) = lim_n M(y_n, y, t) = e,$$

$$lim_n M(x, y, t_n) = M(x, y, t).$$

Remark 1.11. *Note that B_A^+ is a closed subset of A. Let $\{b_n\}_n$ be a sequence in B_A^+ such that converges to $b \in A$, we show that $b \in B_A^+$. Since $b_n \longrightarrow b$ and $||b_n|| \leqslant 1$ for every n, and norm is a continuous function then $||b|| \leqslant 1$. Moreover $\sigma(b_n) \subseteq [0, \infty)$, for all n. If we prove that $\sigma(b) \subseteq [0, \infty)$, proof will be completed. Note that $\sigma(b_n) = \widehat{b}_n(\triangle)$ where $\triangle$ is a compact and Hausdorff space, $||\widehat{b}_n||_\infty = |\widehat{b}_n(h)| \leqslant ||b_n||$ and $\widehat{b}_n - \widehat{b} = \widehat{b_n - b}$, so $||\widehat{b}_n - \widehat{b}||_\infty = ||\widehat{b_n - b}|| \leqslant ||b_n - b||$. Since $b_n \longrightarrow b$, the right hand side of the last inequality tends to zero, so $\widehat{b}_n \longrightarrow \widehat{b}$. Now since $\sigma(b_n) = \widehat{b}_n(\triangle) \subseteq [0, \infty)$, and $\widehat{b} \longrightarrow \widehat{b}$, then $\sigma(b_n) \longrightarrow \sigma(b)$. Therefore $\sigma(b) \subseteq [0, \infty)$.*

Lemma 1.12. *Let $(X, M, \bullet)$ be a C^*-graded metric space, then M is a continuous function on $X^2 \times (0, \infty)$.*

Proof. Let $M(x_{n'}, y_{n'}, t_{n'})$ be a sequence in B_A^+, then there is a subsequence $\{M(x_n, y_n, t_n)\}$ such that converges to u belongs to B_A^+. Since $t_n \longrightarrow t$, for $0 < \delta < \frac{t}{2}$, there exists N such that for all $n \geqslant N$, $|t_n - t| < \delta$. From axiom (5) of C^*-graded metric we have,

$$M(x_n, y_n, t_n) \succeq M(x_n, x, \frac{\delta}{2}) \bullet M(x, y, t - 2\delta) \bullet M(y, y_n, \frac{\delta}{2})$$

So,

$$u = lim_n M(x_n, y_n, t_n) \succeq e \bullet M(x, y, t - 2\delta) \bullet e.$$

Therefore,

$$u \succeq M(x, y, t - 2\delta) \qquad .$$

In a similar way, we have,

$$M(x, y, t + \frac{3\delta}{2}) \succeq M(x, x_n, \frac{\delta}{4}) \bullet M(x_n, y_n, t_n) \bullet M(y_n, y, \frac{\delta}{4}).$$

Therefore,

$$M(x, y, t + \frac{3\delta}{2}) \succeq lim_n M(x_n, y_n, t_n) = u.$$

Now from axiom (1) of C^*-graded metric space, since $\delta > 0$ is arbitrary we deduce that

$$u \succeq M(x, y, t) \quad and \quad M(x, y, t) \succeq u.$$

So $u = M(x, y, t)$ and therefore M is a continuous function on $X^2 \times (0, \infty)$. $\square$

Lemma 1.13. *Let $(X, M, \bullet)$ be a C^*-graded metric space. If we define $E_{\lambda,M} : X^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,M}(x, y) = inf\{t > 0 : ||M(x, y, t)||e \succ (1 - \lambda)e\}$$

for all $\lambda \in (0, 1)$ and $x, y \in X$ then the sequence $\{x_n\}$ is convergent in C^*-graded metric space $(X, M, \bullet)$ iff $E_{\lambda,M}(x_n, x) \longrightarrow 0$. Also,the *sequence* $\{x_n\}$ *is a Cauchy sequence iff it is a Cauchy sequence with* $E_{\lambda,M}$.((*i.e*) $E_{\lambda,M}(x_n, x_m)$ *converges to zero*).

Proof. Since M is continuous in it's third place and

$$E_{\lambda,M}(x,y) = inf\{t > 0 : ||M(x,y,t)||e \succ (1-\lambda)e\},$$

we have

$$||M(x_n, x, \eta)||e \succ (1-\lambda)e \quad iff \quad E_{\lambda,M}(x_n, x) \prec \eta$$

for all $\eta \succ 0$. □

Lemma 1.14. *Let* $(X, M, \bullet)$ *be a* C^**-graded metric space, and* $\{x_n\}_n$ *be a sequence in* X *such that*

$$||M(x_n, x_m, t)||e \succeq ||M(x_0, x_1, k^n t)||e$$

for $k \geqslant 1$ *and* $m > n$, *then the sequence* $\{x_n\}$ *is a Cauchy sequence.*

Proof. For all $\lambda \in (0,1)$ we have

$$\begin{aligned} E_{\lambda,M}(x_n, x_m) &= inf\ \{t > 0 : ||M(x_n, x_m, t)||e \succ (1-\lambda)e\} \\ &\leqslant inf\ \{t > 0 : ||M(x_o, x_1, k^n t)||e \succ (1-\lambda)e\} \\ &= \{\frac{t}{k^n} : ||M(x_0, x_1, t)||e \succ (1-\lambda)e\} \\ &= \frac{1}{k^n} E_{\lambda,M}(x_0, x_1). \end{aligned}$$

So from Lemma 1.13., $\{x_n\}$ is a Cauchy sequence. □

2. Compatible Mappings of Type (I) And (II)

Definition 2.1. *Let* F *and* S *be mappings from a* C^**-graded metric space* $(X, M, \bullet)$ *into itself, then the pair* (F, S) *is said to be compatible of type* (I) *if , for all* $t > 0$,

$$lim_n ||M(FSx_n, x, t)||e \preceq ||M(Sx, x, t)||e$$

whenever $\{x_n\}$ *is a sequence in* X *such that*

$$lim_n Fx_n = lim_n Sx_n = x$$

and similarly we say (F, S) is compatible of type (II) iff (S, F) be compatible of type (I).

Proposition 2.2. *Let F and S be mappings from a C^*-graded metric space $(X, M, \bullet)$ into itself. Suppose that the pair (F, S) is compatible of type (I),(respectively, (II)) and $Fz = Sz$ for some $z \in X$ then for all $t > 0$, $||M(Fz, SSz, t)||e \succeq ||M(Fz, FSz, t)||e$, (respectively $||M(Sz, FFz, t)||e \succeq ||M(Sz, SFz, t)||e$).*

Proof. Just take $x_n = z$ for all n in Definition 2.1. □

3. Main Results

Let Φ be the class of all continuous and increasing functions $\phi : (B_A^+)^5 \to B_A^+$ in any coordinate and

$$\phi(te, te, te, te, te) \succ te$$

for all $t \in [0, 1)$.

Example 3.1. The function $\phi : (B_A^+)^5 \to B_A^+$ defined as,

$$\phi(x_1, x_2, x_3, x_4, x_5) = (\min\{||x_i||\})^h e$$

for some $0 < h < 1$, belongs to Φ.

Example 3.2. $\phi(x_1, x_2, x_3, x_4, x_5) = ||x_1||^h e$, for some $0 < h < 1$, belongs to Φ.

Example 3.3. $\phi(x_1, x_2, x_3, x_4, x_5) = (\sum_{i=1}^{5} a_i(t)||x_i||)^h e$, such that $0 < h < 1$ and for all $t > 0$, $a_i : \mathbb{R}^+ \to (0, 1]$ are functions with, $\sum_{i=1}^{5} a_i(t) = 1$.

Theorem 3.4. *Let $(X, M, \bullet)$ be a complete C^*-graded metric space with $a \bullet a = a$ for all $a \in B_A^+$. let F, B, S and T be mappings from X into itself such that,*

1. *$F(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,*

2. *there exists a constant* $k \in (0, \frac{1}{2})$ *such that*

$$||M(Fx, By, kt)||e \succeq \phi \begin{pmatrix} ||M(Sx, Ty, t)||e, \\ ||M(Fx, Sx, t)||e, \\ ||M(By, Ty, t)||e, \\ ||M(Fx, Ty, \alpha t)||e, \\ ||M(By, Sx, (2-\alpha)t)||e \end{pmatrix},$$

for all $x, y \in X$, $\alpha \in (0, 2), t > 0$ *and* $\phi \in \Phi$. *If the mappings* F, B, S *and* T *satisfy any one of the following conditions:*

3. *the pairs* (F, S) *and* (B, T) *are compatible of type (II) and* F *or* B *is continuous,*
4. *the pairs* (F, S) *and* (B, T) *are compatible of type (I) and* S *or* T *is continuous,*

then F, B, S *and* T *have a unique common fixed point in* X.

Proof. Let $x \in X$ be an arbitrary point. Since $F(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, there exists $x_1, x_2 \in X$ such that $Fx_0 = Tx_1$, $Bx_1 = Sx_2$. Inductively, construct the sequences y_n and x_n in X such that

$$y_{2n} = Fx_{2n} = Tx_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

for $n = 0, 1, 2, \cdots$. Then, by $\alpha = 1 - q$ and $q \in (\frac{1}{2}, 1)$, if we set $d_m(t) = ||M(y_m, y_{m+1}, t)||e$ for all $t > 0$, then we prove that $d_m(t)$ is increasing with respect to m. Setting $m = 2n$, then we have

$$\begin{aligned} d_{2n}(kt) = ||M(y_{2n}, y_{2n+1}, kt)||e &= ||M(Ax_{2n}, Bx_{2n+1}, kt)||e \\ &\succeq \phi \begin{pmatrix} ||M(Sx_{2n}, Tx_{2n+1}, t)||e, \\ ||M(Fx_{2n}, Sx_{2n}, t)||e, \\ ||M(Bx_{2n+1}, Tx_{2n+1}, t)||e, \\ ||M(Fx_{2n}, Tx_{2n+1}, (1-q)t||e, \\ ||M(Bx_{2n+1}, Sx_{2n}, (1+q))t)||e \end{pmatrix} \end{aligned}$$

$$= \phi \begin{pmatrix} ||M(y_{2n-1}, y_{2n}, t)||e, \\ ||M(y_{2n}, y_{2n-1}, t)||e, \\ ||M(y_{2n+1}, y_{2n}, t)||e, \\ ||M(y_{2n}, y_{2n}, (1-q)t||e, \\ ||M(y_{2n+1}, y_{2n-1}, (1+q))t)||e \end{pmatrix}$$

$$= \phi(d_{2n-1}(t), d_{2n-1}(t), d_{2n}(t), e, ||M(y_{2n+1}, y_{2n-1}, (1+q)t)||e),$$

that is,

$$d_{2n}(kt) \succeq \phi(d_{2n-1}(qt), d_{2n-1}(qt), d_{2n}(qt), e, d_{2n-1}(t) \bullet d_{2n}(qt)). \quad (1)$$

We claim that, for all $n \in \mathbb{N}$, $d_{2n}(t) \succeq d_{2n-1}(t)$. In fact if $d_{2n}(t) \prec d_{2n-1}(t)$ then, since $d_{2n}(qt) \bullet d_{2n-1}(t) \succeq d_{2n}(qt) \bullet d_{2n}(qt) = d_{2n}(qt)$. In the inequality (1) we have

$$d_{2n}(kt) \succeq \phi(d_{2n}(qt), d_{2n}(qt), d_{2n}(qt), d_{2n}(qt), d_{2n}(qt)) \succ d_{2n}(qt)$$

that is, $d_{2n}(kt) \succ d_{2n}(qt)$, which is a contradiction. Hence $d_{2n}(t) \succeq d_{2n-1}(qt)$ for all $n \in \mathbb{N}$ and $t > 0$.
Similarly, for $m = 2n+1$, we have $d_{2n+1}(t) \succeq d_{2n}(t)$ and so $\{d_n(t)\}$ is an increasing sequence in B_A^+. By the inequality (1), we have

$$\begin{aligned} d_{2n}(kt) &\succeq \phi(d_{2n-1}(qt), d_{2n-1}(qt), d_{2n-1}(qt), d_{2n-1}(qt), d_{2n-1}(qt)) \\ &\succ d_{2n-1}(qt). \end{aligned}$$

Similarly for $m = 2n+1$ we have $d_{2n+1}(kt) \succeq d_{2n}(qt)$ and so $d_n(kt) \succeq d_{n-1}(qt)$ for all $n \in \mathbb{N}$. That is,

$$||M(y_n, y_{n+1}, t)||e \succeq ||M(y_{n-1}, y_n, \frac{q}{k}t)||e \succeq ... \succeq ||M(y_0, y_1, (\frac{q}{k})^n t)||e.$$

Hence, by Lemma 1.14., $\{y_n\}$ is a cauchy sequence and, by the completeness of X, $\{y_n\}$ converges to a point z in X. Let $lim_n y_n = z$. Hence we have

$$\begin{aligned} lim_n y_{2n} &= lim_n F x_{2n} = lim_n T x_{2n+1} \\ &= lim_n y_{2n+1} = lim_n B x_{2n+1} \\ &= lim_n S x_{2n+2} = z. \end{aligned}$$

Now, suppose that T is continuous and the pairs (F,S) and (B,T) are compatible of type (I). Hence we have

$$lim_n TTx_{2n+1} = Tz,$$

$$||M(Tz,z,t)||e \succeq lim_n ||M(BTx_{2n-1},z,t)||e.$$

Now, for $\alpha = 1$, put $x = x_{2n}$ and $y = Tx_{2n+1}$ in the inequality (1) then we obtain

$$||M(Fx_{2n},BTx_{2n+1},kt)||e \succeq \phi \begin{pmatrix} ||M(Sx_{2n},TTx_{2n+1},t)||e, \\ ||M(Fx_{2n},Sx_{2n},t)||e, \\ ||M(BTx_{2n+1},TTx_{2n+1},t)||e, \\ ||M(Fx_{2n},TTx_{2n},t)||e, \\ ||M(BTx_{2n+1},Sx_{2n},t)||e \end{pmatrix}.$$

Letting $n \to \infty$, then we have

$$\begin{aligned} ||M(z,lim_n BTx_{2n+1},kt)||e &\succeq \phi \begin{pmatrix} ||M(z,Tz,t)||e, \\ ||M(z,z,t)||e, \\ ||M(lim_n BTx_{2n+1},Tz,t)||e, \\ ||M(z,Tz,t)||e, \\ ||M(lim_n BTx_{2n+1},z,t)||e \end{pmatrix} \\ &\succeq \phi \begin{pmatrix} ||M(z,Tz,\frac{t}{2})||e, \\ ||M(z,z,\frac{t}{2})||e, \\ ||M(lim_n BTx_{2n+1},Tz,\frac{t}{2})||e, \\ ||M(z,Tz,\frac{t}{2})||e, \\ ||M(lim_n BTx_{2n+1},z,\frac{t}{2})||e \end{pmatrix}. \end{aligned}$$

Thus it follows that

$$\begin{aligned} lim_n ||M(BTx_{2n+1},Tz,t)||e \succeq & lim_n ||M(BTx_{2n+1},z,\frac{t}{2})||e \\ & \bullet lim_n ||M(z,Tz,\frac{t}{2})||e \end{aligned}$$

and so

$$lim_n ||M(BTx_{2n+1},Tz,t)||e \succeq lim_n ||M(BTx_{2n+1},z,\frac{t}{2})||e.$$

Since $\phi(t,t,t,t,t) \succ t$, by the above inequalities we have

$$||M(z, lim_{n\to\infty} BTx_{2n+1}, kt)||e \succ ||M(z, lim_{n\to\infty} BTx_{2n+1}, \frac{t}{2})||e,$$

which is a contradiction. It follows that $lim_n BTx_{2n+1} = z$.
Now, using the compatibility of type (I), we have

$$||M(Tz, z, t)||e \succeq lim_n ||M(z, BTx_{2n+1}, t)||e = e$$

and so $Tz = z$.
Again, replacing x by x_{2n} and y by z in (3.1). For $\alpha = 1$, we have

$$||M(Fx_{2n}, Bz, kt)||e \succeq \phi \begin{pmatrix} ||M(Sx_{2n}, Tz, t)||e, \\ ||M(Fx_{2n}, Sx_{2n}, t)||e, \\ ||M(Bz, Tz, t)||e, \\ ||M(Fx_{2n}, Tz, t)||e, \\ ||M(Bz, Sx_{2n}, t)||e \end{pmatrix}$$

Letting $n \to \infty$, we have

$$||M(Bz, z, kt)||e \succ ||M(Bz, z, t)||e.$$

Which implies that $Bz = z$. Since $B(X) \subseteq S(X)$, there exist $u \in X$ such that $Su = z = Bz$. So, for $\alpha = 1$, we have

$$||M(Fu, Bz, kt)||e \succeq \phi \begin{pmatrix} ||M(Su, Tz, t)||e, \\ ||M(Fu, Su, t)||e, \\ ||M(Bz, Tz, t)||e, \\ ||M(Fu, Tz, t)||e, \\ ||M(Bz, Su, t)||e \end{pmatrix}$$

and so,

$$||M(Fu, z, kt)||e \succ ||M(z, Fu, t)||e,$$

which implies that $Fu = z$. Since the pair (F, S) is compatible of type (I) and $Fu = z$, by proposition (2.2.), we have

$$||M(Fu, SSu, t)||e \succeq ||M(Fz, FSu, t)||e$$

and so,

$$||M(z, Sz, t)||e \succeq ||M(z, Fz, t)||e.$$

Again, for $\alpha = 1$,we have

$$||M(Fz, Bz, kt)||e \succeq \phi \begin{pmatrix} ||M(Sz, Tz, t)||e, \\ ||M(Fz, Sz, t)||e, \\ ||M(Bz, Tz, t)||e, \\ ||M(Fz, Tz, t)||e, \\ ||M(Bz, Sz, t)||e \end{pmatrix}.$$

It follows that

$$\begin{aligned} M(Fz, Sz, t) &\succeq M(Fz, z, \frac{t}{2}) \bullet M(z, Sz, \frac{t}{2}) \\ &\succeq M(z, Fz, \frac{t}{2}) \bullet M(z, Fz, \frac{t}{2}) \\ &= M(z, Fz, \frac{t}{2}). \end{aligned}$$

Hence we have

$$||M(Fz, z, kt)||e \succeq \phi \begin{pmatrix} ||M(Sz, z, \frac{t}{2})||e, \\ ||M(Fz, z, \frac{t}{2})||e, \\ ||M(Fz, z, \frac{t}{2})||e, \\ ||M(Fz, z, \frac{t}{2})||e, \\ ||M(z, Fz, \frac{t}{2})||e \end{pmatrix} \succ ||M(z, Fz, \frac{t}{2})||e$$

and so $Fz = z$. Therefore, $Fz = Bz = Sz = Tz = z$ and z is a common fixed point for the self- mappings F, B, S and T.

The uniqueness of a common fixed point of the mappings F, B, S, T is easily verified by using (1). In fact, if $\acute{z}$ is another fixed point for F, B, S

and T, then for $\alpha = 1$, we have

$$
\begin{aligned}
||M(z,\acute{z},t)||e &= ||M(Fz,B\acute{z},kt)||e \\
&\succeq \phi\begin{pmatrix} ||M(Sz,T\acute{z},t)||e, \\ ||M(Fz,Sz,t)||e, \\ ||M(B\acute{z},T\acute{z},t)||e, \\ ||M(Fz,T\acute{z},t)||e, \\ ||M(B\acute{z},Sz,t)||e \end{pmatrix} \\
&\succ ||M(z,\acute{z},t)||e
\end{aligned}
$$

and so $z = \acute{z}$. □

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References

[1] R. Badard, Fixed point theorems for fuzzy numbers, *Fuzzy Set Syst.*, 13 (1984), 291-302.

[2] B. K. Bose and D. Sabani, Fuzzy mappings and fixed point theorems, *Fuzzy Set Syst.*, 21 (1987), 53-58.

[3] D. Butnarin, Fixed point for fuzzy mappings, *Fuzzy Set Syst.*, 7 (1982), 191-207.

[4] S. S. Chang, Fixed point theorems for fuzzy mappings, *Fuzzy Set Syst.*, 7 (1985), 181-187.

[5] S. S. Chang, Y. J. Cho, B. S. Lee, J. S. Jung, and S. M. Kang, Coincidence point and minimization theorems in fuzzy metric spaces, *Fuzzy Set Syst.*, 88 (1997), 119-128.

[6] S. S. Chang, Y. J. Cho, B. E. Lee, and G. M. Lee, Fixed degree and fixed point theorems for fuzzy mappings, *Fuzzy Set Syst.*, 87 (1997), 325-334.

[7] Y. J. Cho, S. Sedghi and Nabi shobe, Generalized fixed point theorems for compatible mappings with some types in fuzzy metric spaces, Chaos, *Solitons and Fractals*, (2009), 2233-2244.

[8] Z. K. Deng, Fuzzy psendo-metric spaces, *J. Math. Anal. Appl.*, 86 (1982), 74-95.

[9] M. A. Ereeg, Metric spaces in fuzzy set theory, *J. Math. Anal. Appl.*, 69 (1979), 338-353.

[10] J. X. Fang, On fixed point theorems in fuzzy metric spaces, *Fuzzy Set Syst.*, 46 (1992), 107-113.

[11] A. George and P. Veeramani, On some result in fuzzy metric space, *Fuzzy Set Syst.*, 64 (1994), 395-399.

[12] M. Grabiec, Fixed points in fuzzy metric spaces, *Fuzzy Set Syst.*, 27 (1988), 385-389.

[13] V. Gregori and A. Sapena, On fixed-point theorem in fuzzy metric spaces, *Fuzzy Set Syst.*, 125 (2002), 245-252.

[14] O. Hadzic, Fixed point theorems for multi-valued mappings in some classes of fuzzy metric spaces, *Fuzzy Set Syst.*, 29 (1989), 115-125.

[15] J. S. Jung, Y. J. Cho, S. S. Chang, and S. M. Kang, Coincidence theorems for set-valued mappings and Eklands variational principle in fuzzy metric spaces, *Fuzzy Set Syst.*, 79 (1996), 239-250.

[16] G. Jungck, Commuting maps and fixed points, *Amer. Math.*, Monthly, 83 (1976), 261-263.

[17] G. Jungck, Compatible mappings and common fixed points, *Int. J. Math. Math Sci.*, 9 (1986), 771-779.

[18] O. Kaleva and S. Seikkala, On fuzzy metric spaces, *Fuzzy Set Syst.*, 12 (1984), 215-229.

[19] I. Kramosil and J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetika*, 11 (1975), 326-334.

[20] D. Mihet, A Banach contraction theorem in fuzzy metric spaces, *Fuzzy Set Syst.*, 144 (2004), 431-439.

[21] S. N. Mishra, S. N. Sharma, and S. L. Singh, Common fixed points of maps in fuzzy metric spaces, *Int. J. Math. Math Sci.*, 17 (1994), 253-258.

[22] G. Y. Murphy, *C^*-algebras and aperator theory*, Academic press.Inc., 1990.

[23] H. K. Pathak, N. Mishra, and A. K. Kalinde, Common fixed point theorems with applications to nonlinear integral equation, *Demonstratio Math.*, 32 (1999), 517-564.

[24] J. Rodriguez Lopez and S. Ramaguera, The Hausdorff fuzzy metric on compact sets, *Fuzzy Set Syst.*, 47 (2004), 273-283.

[25] W. Rudin, *Functionel Analysis*, McGraw-Hill, Inc., 1973.

[26] R. Saadati and S. Sedghi, A common fixed point theorem for R-weakly commuting maps in fuzzy metric spaces, *6th Iranian conference on fuzzy systems*, (2006), 387-391.

[27] B. Schweizer, H. Sherwood, and R. M. Tardiff, Contractions on PM-space examples and counterexamples, *Stochastica*, (1988), 5-17.

[28] S. Sessa, On some weak commutativity condition of mappings in fixed point considerations, *Publ. Inst. Math.*, (Beograd), 32 (1982), 149-153.

[29] S. Sedghi, D. Turkoglu, and N. Shobe, Generalization common fixed point theorem in complete fuzzy metric spaces, *J.Comput. Anal. Applictat*, 93 (2007), 337-348.

[30] S. Sharma and B. Desphaude, Common fixed points of compatible maps of type (β) on fuzzy metric spaces, *Demonsratio Math.*, (2002), 165-174.

[31] B. Singh and S. Jain, A fixed point theorem in Menger space through weak compatibility, *J. Math. Anal. Appl.*, 301 (2005), 439-448.

[32] Y. Tanaka , Y. Mizno, and T. Kado, Chaotic dynamics in Friedmann equation, *Chaos, Solitons and Fractals*, 24 (2005), 407-422.

[33] D. Turkoglu, I. Altun, and Y. J. Cho, *Compatible maps and compatible maps of types (α) and (β) in intuitionistic fuzzy metric spaces*, in press.

[34] L. A. Zadeh, Fuzzy sets, *Inform Control*, 8 (1965), 338-353.

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