Some Properties of C*-Graded Metric Spaces and Fixed Point Results

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Abstract. In this paper, by using of C^* -algebra we give a metric with range in positive unit ball of a C^* -algebra. Indeed this is a generalization of a fuzzy metric space. Some definitions of compatible mappings of types (I) and (II) are introduced and some fixed and common fixed point theorems are proved.

AMS Subject Classification: 54E40; 54E35; 54H25. **Keywords and Phrases:** C^* -algebra, common fixed point theorem, compatible mappings of type (I) and (II).

1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [34] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. Deng [8], Ereeg [9], Fang [10], George [11], Kaleva and Seikkala [18], Kramosil and Michalek [19] have introduced the concept of fuzzy metric spaces in different ways.

Received November 2009; Final Revised March 2010

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In fuzzy metric spaces given by Kramosil and Michalek [19], Grabiec [12] obtained the fuzzy version of Banach contraction principle, which has been improved and extended by some authors.

Sessa ([28]) defined a generalization of commutativity introduced by Jungck ([16]), which is called the weak commutativity. Further, Jungck ([17]) introduced more generalized commutativity, so called compatibility. Mishra et al. [21] obtained some common fixed point theorems for compatible mappings in fuzzy metric spaces. Pathak et al. [23] introduced the concept of compatible mappings of type (I) and (II) in metric spaces.

Many authors ([13,20,27,26,29]) have also proved some fixed point theorems in fuzzy (*probabilistic*) metric spaces (see [1-6,10,12,14,15,30]). Now we give basic definitions and their properties as follows:

Recall that, a complex algebra is a vector space A over the field \mathbb{C} with one multiplication from $A \times A$ into A that satisfies the following relations;

1.
$$x(yz) = (xy)z$$
,

2.
$$x(y+z) = xy + xz$$
, $(x+y)z = xz + yz$,

3.
$$\alpha(xy) = (\alpha x)y = x(\alpha y),$$

for all x, y, z in A and α in \mathbb{C} .

If A is a Banach space with respect to ||.|| that satisfies multiplicative inequality

$$||xy|| \le ||x|| ||y|| \qquad (x \in A, y \in A)$$

and A contains unit element e such that

$$xe = ex = x$$
 $(x \in A)$

and ||e|| = 1, then A is called a Banach algebra.

The map $x \to x^*$ from complex algebra A into itself is called an involution if for all x and λ in \mathbb{C} we have,

1.
$$(x+y)^* = x^* + y^*$$

2.
$$(\lambda x)^* = \overline{\lambda} x^*$$

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3.
$$(xy)^* = y^*x$$

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4. $x^{**} = x$.

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Every Banach algebra with an involution $x \to x^*$ with the following relation is called a C^* -algebra,

$$||xx^*|| = ||x||^2 \qquad (x \in A)$$

From now on, A is a C^* -algebra. For every $x \in A$, we define the spectrum of x, is $\sigma(x)$, as follows,

$$\sigma(x) = \{ \lambda \in \mathbb{C} : \lambda e - x \quad \text{is not invertible in A} \}.$$

In [28], it has been that for all $x \in A$, where A is a Banach algebra, $\sigma(x)$ is non-empty and compact.

We say that $x \in A$ is positive (in symbol $x \succeq 0$) if $x = x^*$ and $\sigma(x) \subseteq [0, \infty)$ and similarly x is strictly positive $(x \succ 0)$ if $x = x^*$ and $\sigma(x) \subseteq (0, \infty)$. Positive unit ball of A is denoted by B_A^+ i.e

$$B_A^+ := \{ x \in A : x \succeq 0, ||x|| < 1 \} \cup \{ e \}.$$

Let A^+ be the set of all positive elements of A.

Now we introduce two binary relation on A^+ as, $a \succeq b$ and $a \succ b$, which means that, $a - b \succeq 0$ and $a - b \succ 0$, respectively. These two binary relations have some good properties that we are going to summarize some of it:

- 1. if a, b and $c \in A^+$ and $a \succeq b$ then $a + c \succeq b + c$,
- 2. if $a \succeq b$ then $ta \succeq tb$ for all non-negative real number,
- 3. $a \succeq b$ iff $-b \succeq -a$.

Theorem 1.1. Let A be a C^* - algebra

- 1. If $a \succeq 0$ and $b \succeq 0$ then $a + b \succeq 0$.
- 2. $A^+ = \{a^*a | a \in A\}.$

- Archive of SID 3. If $a, b \in A^*$ and $c \in A$, then $b \succeq a$ implies $c^*bc \succeq c^*ac$.
 - 4. If $b \succeq a \succeq 0$, then $||b|| \ge ||a||$.
 - 5. If $a, b \in A^*$ and a, b are invertible elements, then $b \succeq a$ implies $a^{-1} \succeq b^{-1} \succeq 0$.
 - 6. If $r \ge s$, for all $r, s \in \mathbb{R}^+$, then $re \succeq se$.
 - 7. If $a, b \in A^+$, then $a \succeq b$ and $b \succeq a$, implies a = b.

Proof. See [21]. \Box

Definition 1.2. A binary operation $\bullet : B_A^+ \times B_A^+ \longrightarrow B_A^+$ is a continuous t -norm if it satisfies the following conditions:

- 1. is associative and commutative,
- 2. is continuous,
- 3. $a \bullet e = a$ for all $a \in B_A^+$,
- 4. $a \bullet b \preceq c \bullet d$ whenever $a \preceq c$ and $b \preceq d$ for all $a, b, c, d \in B_A^+$,
- 5. $||a \bullet b||e = ||a||e \bullet ||b||e$.

Example 1.3. $\bullet: B_A^+ \times B_A^+ \longrightarrow B_A^+$ with $\bullet(a, b) = ab$, where ab is the product of C^* -algebra A, is a continuous t- norm.

Definition 1.4. The pair (M, F) with $F : X \longrightarrow B_A^+$ is called a C^* -graded set on X. Every F(x) in B_A^+ is celled graded membership of x in B_A^+ .

Definition 1.5. A triple (X, M, \bullet) is called a C^* -graded metric space if X is a non-empty arbitrary set, \bullet is a continuous t-norm and M is a C^* -graded set on $X^2 \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X, u \in B^+_A$ and $t, s \succ 0$.

1. $M(x, y, .) : (0, \infty) \longrightarrow B^+_A$ is continuous,

- Archive of SID 2. $M(x, y, t) \succ 0$, 3. M(x, y, t) = e if x = y, 4. M(x, y, t) = M(y, x, t),
 - 5. $M(x, y, t) \bullet M(y, z, s) \preceq M(x, z, t+s).$

Definition 1.6. Let (X, M, \bullet) be a C^{*}-graded metric space. For any t > 0 and $x \in X$ we define the open ball B(x, r, t) with center x and radius 0 < r < 1 is defined by,

$$B(x, r, t) = \{ y \in X : M(x, y, t) \succ (1 - r)e \}.$$

Definition 1.7. Let (X, M, \bullet) be a C^* -graded metric space and $A \subset X$.

- 1. A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to e$ as $n \to \infty$ for all t > 0.
- 2. Sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $0 < \epsilon < 1$ and t > 0, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) \succ (1 \varepsilon)e$, for any $n, m \ge n_0$.
- 3. The C^* -graded metric space (X, M, *) is said to be complete if every Cauchy sequence is convergent.
- 4. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist t > 0 and 0 < r < 1 such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the C^{*}-graded metric M).

Example 1.8. Let $a \bullet b = ab$ (ordinary product in A). For any $t \in (0, \infty)$, define

$$M(x, y, t) = (\frac{t}{t + ||x - y||})e \quad , \quad (x, y \in X)$$

then (M, X, \bullet) is a C^{*}-graded metric space.

Lemma 1.9. Let (X, M, \bullet) be a C^* -graded metric space then M(x, y, t) is non-decreasing with respect to t for all x, y in X.

Proof. If $t_1 \leq t_2$, then $t_2 = t_1 + \varepsilon$, for some $\epsilon > 0$. By using (5) from Definition 1.5., if y = x, $s = t_1$ and $t = \varepsilon$ we have $M(x, x, \varepsilon) \bullet M(x, z, t_1) \preceq M(x, z, t_1 + \varepsilon)$ then from (3) of Definition 1.5., and (3) of Definition 1.2., we have

$$M(x, z, t_1) \preceq M(x, z, t_1 + \varepsilon) = M(x, z, t_2)$$

for all x, y in X. Note that $M(x, z, t_1) \preceq M(x, z, t_2)$ means that $M(x, z, t_2) - M(x, z, t_1)$ is a positive element with norm less than one of C^* -algebra A or

$$M(x, z, t_2) - M(x, z, t_1) \in B_A^+.$$

Definition 1.10. Let (X, M, \bullet) be a C^* -graded metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$lim_n M(x_n, y_n, t_n) = M(x, y, t)$$

whenever a sequence $\{(x_n, y_n, t_n)\}_n$ in $X^2 \times (0, \infty)$, exists such that

$$lim_n M(x_n, x, t) = lim_n M(y_n, y, t) = e,$$
$$lim_n M(x, y, t_n) = M(x, y, t).$$

Remark 1.11. Note that B_A^+ is a closed subset of A. Let $\{b_n\}_n$ be a sequence in B_A^+ such that converges to $b \in A$, we show that $b \in B_A^+$. Since $b_n \longrightarrow b$ and $||b_n|| \leq 1$ for every n, and norm is a continuous function then $||b|| \leq 1$. Moreover $\sigma(b_n) \subseteq [0, \infty)$, for all n. If we prove that $\sigma(b) \subseteq [0, \infty)$, proof will be completed. Note that $\sigma(b_n) = \hat{b}_n(\Delta)$ where Δ is a compact and Hausdorff space, $||\hat{b}_n||_{\infty} = |\hat{b}_n(h)| \leq ||b_n||$ and $\hat{b}_n - \hat{b} = \hat{b}_n - \hat{b}$, so $||\hat{b}_n - \hat{b}||_{\infty} = ||\hat{b}_n - \hat{b}|| \leq ||b_n - b||$. Since $b_n \longrightarrow \hat{b}$, the right hand side of the last inequality tends to zero, so $\hat{b}_n \longrightarrow \hat{b}$. Now since $\sigma(b_n) = \hat{b}_n(\Delta) \subseteq [0, \infty)$, and $\hat{b} \longrightarrow \hat{b}$, then $\sigma(b_n) \longrightarrow \sigma(b)$. Therefore $\sigma(b) \subseteq [0, \infty)$.

Lemma 1.12. Let (X, M, \bullet) be a C^* -graded metric space, then M is a continuous function on $X^2 \times (0, \infty)$.

Proof. Let $M(x_{n'}, y_{n'}, t_{n'})$ be a sequence in B_A^+ , then there is a subsequence $\{M(x_n, y_n, t_n)\}$ such that converges to u belongs to B_A^+ . Since $t_n \longrightarrow t$, for $0 < \delta < \frac{t}{2}$, there exists N such that for all $n \ge N$, $|t_n - t| < \delta$. From axiom (5) of C^* -graded metric we have,

$$M(x_n, y_n, t_n) \succeq M(x_n, x, \frac{\delta}{2}) \bullet M(x, y, t - 2\delta) \bullet M(y, y_n, \frac{\delta}{2})$$

So,

$$u = lim_n M(x_n, y_n, t_n) \succeq e \bullet M(x, y, t - 2\delta) \bullet e$$

Therefore,

$$u \succeq M(x, y, t - 2\delta)$$

In a similar way, we have,

$$M(x, y, t + \frac{3\delta}{2}) \succeq M(x, x_n, \frac{\delta}{4}) \bullet M(x_n, y_n, t_n) \bullet M(y_n, y, \frac{\delta}{4}).$$

Therefore,

$$M(x, y, t + \frac{3\delta}{2}) \succeq \lim_{n \to \infty} M(x_n, y_n, t_n) = u.$$

Now from axiom (1) of C*-graded metric space, since $\delta > 0$ is arbitrary we deduce that

$$u \succeq M(x, y, t)$$
 and $M(x, y, t) \succeq u$.

So u = M(x, y, t) and therefore M is a continuous function on $X^2 \times (0, \infty)$. \Box

Lemma 1.13. Let (X, M, \bullet) be a C^{*}-graded metric space. If we define $E_{\lambda,M}: X^2 \longrightarrow \mathbb{R}^+ \cup \{0\}$ by

$$E_{\lambda,M}(x,y) = \inf\{t > 0 : ||M(x,y,t)||e \succ (1-\lambda)e\}$$

for all $\lambda \in (0,1)$ and $x, y \in X$ then the sequence $\{x_n\}$ is convergent in C^* -graded metric space (X, M, \bullet) iff $E_{\lambda,M}(x_n, x) \longrightarrow 0$. Also, the

sequence $\{x_n\}$ is a Cauchy sequence iff it is a Cauchy sequence with $E_{\lambda,M}.((i.e) \ E_{\lambda,M}(x_n, x_m) \text{ converges to zero}).$

Proof. Since M is continuous in it's third place and

$$E_{\lambda,M}(x,y) = \inf\{t > 0 : ||M(x,y,t)|| e \succ (1-\lambda)e\},\$$

we have

$$||M(x_n, x, \eta)|| e \succ (1 - \lambda)e \quad iff \quad E_{\lambda, M}(x_n, x) \prec \eta$$

for all $\eta \succ 0$. \Box

Lemma 1.14. Let (X, M, \bullet) be a C^{*}-graded metric space, and $\{x_n\}_n$ be a sequence in X such that

$$||M(x_n, x_m, t)|| e \geq ||M(x_0, x_1, k^n t)|| e$$

for $k \ge 1$ and m > n, then the sequence $\{x_n\}$ is a Cauchy sequence.

Proof. For all $\lambda \in (0, 1)$ we have

$$\begin{split} E_{\lambda,M}(x_n, x_m) &= \inf \{ t > 0 : ||M(x_n, x_m, t)||e \succ (1 - \lambda)e \} \\ &\leq \inf \{ t > 0 : ||M(x_o, x_1, k^n t)||e \succ (1 - \lambda)e \} \\ &= \{ \frac{t}{k^n} : ||M(x_0, x_1, t)||e \succ (1 - \lambda)e \} \\ &= \frac{1}{k^n} E_{\lambda,M}(x_0, x_1). \end{split}$$

So from Lemma 1.13., $\{x_n\}$ is a Cauchy sequence. \Box

2. Compatible Mappings of Type (I) And (II)

Definition 2.1. Let F and S be mappings from a C^* -graded metric space (X, M, \bullet) into itself, then the pair (F, S) is said to be compatible of type (I) if, for all t > 0,

$$\lim_{n} ||M(FSx_n, x, t)|| e \leq ||M(Sx, x, t)|| e$$

whenever $\{x_n\}$ is a sequence in X such that

 $lim_n Fx_n = lim_n Sx_n = x$

and similarly we say (F, S) is compatible of type (II) iff (S, F) be compatible of type (I).

Proposition 2.2. Let F and S be mappings from a C^* -graded metric space (X, M, \bullet) into itself. Suppose that the pair (F, S) is compatible of type (I),(respectively, (II)) and Fz = Sz for some $z \in X$ then for all t > 0, $||M(Fz, SSz, t)||e \succeq ||M(Fz, FSz, t)||e$, (respectively $||M(Sz, FFz, t)||e \succeq ||M(Sz, SFz, t)||e$).

Proof. Just take $x_n = z$ for all n in Definition 2.1. \Box

3. Main Results

Let Φ be the class of all continuous and increasing functions $\phi: (B_A^+)^5 \to B_A^+$ in any coordinate and

$$\phi(te, te, te, te, te) \succ te$$

for all $t \in [0, 1)$.

Example 3.1. The function $\phi: (B_A^+)^5 \to B_A^+$ defined as,

$$\phi(x_1, x_2, x_3, x_4, x_5) = (\min\{||x_i||\})^h e^{-\frac{1}{2}}$$

for some 0 < h < 1, belongs to Φ .

Example 3.2. $\phi(x_1, x_2, x_3, x_4, x_5) = ||x_1||^h e$, for some 0 < h < 1, belongs to Φ .

Example 3.3. $\phi(x_1, x_2, x_3, x_4, x_5) = (\sum_{i=1}^5 a_i(t) ||x_i||)^h e$, such that 0 < h < 1 and for all t > 0, $a_i : \mathbb{R}^+ \to (0, 1]$ are functions with, $\sum_{i=1}^5 a_i(t) = 1$.

Theorem 3.4. Let (X, M, \bullet) be a complete C^* -graded metric space with $a \bullet a = a$ for all $a \in B_A^+$. let F, B, S and T be mappings from X into itself such that,

1. $F(X) \subseteq T(X), B(X) \subseteq S(X),$

2. there exists a constant $k \in (0, \frac{1}{2})$ such that

$$||M(Fx, By, kt)||e \succeq \phi \begin{pmatrix} ||M(Sx, Ty, t)||e, \\ ||M(Fx, Sx, t)||e, \\ ||M(By, Ty, t)||e, \\ ||M(Fx, Ty, \alpha t)||e, \\ ||M(By, Sx, (2 - \alpha)t)||e \end{pmatrix}$$

for all $x, y \in X$, $\alpha \in (0,2), t > 0$ and $\phi \in \Phi$. If the mappings F, B, S and T satisfy any one of the following conditions:

,

- 3. the pairs (F, S) and (B, T) are compatible of type (II) and F or B is continuous,
- 4. the pairs (F,S) and (B,T) are compatible of type (I) and S or T is continuous,

then F, B, S and T have a unique common fixed point in X.

Proof. Let $x \in X$ be an arbitrary point. Since $F(X) \subseteq T(X)$, $B(X) \subseteq S(X)$, there exists $x_1, x_2 \in X$ such that $Fx_0 = Tx_1$, $Bx_1 = Sx_2$. Inductively, construct the sequences y_n and x_n in X such that

$$y_{2n} = Fx_{2n} = Tx_{2n+1}, \quad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$$

for $n = 0, 1, 2, \cdots$. Then, by $\alpha = 1 - q$ and $q \in (\frac{1}{2}, 1)$, if we set $d_m(t) = ||M(y_m, y_{m+1}, t)||e$ for all t > 0, then we prove that $d_m(t)$ is increasing with respect to m. Setting m = 2n, then we have

$$d_{2n}(kt) = ||M(y_{2n}, y_{2n+1}, kt)||e = ||M(Ax_{2n}, Bx_{2n+1}, kt)||e$$
$$\geq \phi \begin{pmatrix} ||M(Sx_{2n}, Tx_{2n+1}, t)||e, \\ ||M(Fx_{2n}, Sx_{2n}, t)||e, \\ ||M(Bx_{2n+1}, Tx_{2n+1}, t)||e, \\ ||M(Fx_{2n}, Tx_{2n+1}, (1-q)t||e, \\ ||M(Bx_{2n+1}, Sx_{2n}, (1+q))t)||e \end{pmatrix}$$

$$= \phi \begin{pmatrix} ||M(y_{2n-1}, y_{2n}, t)||e, \\ ||M(y_{2n}, y_{2n-1}, t)||e, \\ ||M(y_{2n+1}, y_{2n}, t)||e, \\ ||M(y_{2n+1}, y_{2n}, (1-q)t||e, \\ ||M(y_{2n+1}, y_{2n-1}, (1+q))t)||e \end{pmatrix}$$
$$= \phi(d_{2n-1}(t), d_{2n-1}(t), e, ||M(y_{2n+1}, y_{2n-1}, (1+q)t)||e),$$

that is,

$$d_{2n}(kt) \succeq \phi(d_{2n-1}(qt), d_{2n-1}(qt), d_{2n}(qt), e, d_{2n-1}(t) \bullet d_{2n}(qt)).$$
(1)

We claim that, for all $n \in \mathbb{N}$, $d_{2n}(t) \succeq d_{2n-1}(t)$. In fact if $d_{2n}(t) \prec d_{2n-1}(t)$ then, since $d_{2n}(qt) \bullet d_{2n-1}(t) \succeq d_{2n}(qt) \bullet d_{2n}(qt) = d_{2n}(qt)$. In the inequality (1) we have

$$d_{2n}(kt) \succeq \phi(d_{2n}(qt), d_{2n}(qt), d_{2n}(qt), d_{2n}(qt), d_{2n}(qt)) \succ d_{2n}(qt)$$

that is, $d_{2n}(kt) \succ d_{2n}(qt)$, which is a contradiction. Hence $d_{2n}(t) \succeq d_{2n-1}(qt)$ for all $n \in \mathbb{N}$ and t > 0.

Similarly, for m = 2n + 1, we have $d_{2n+1}(t) \succeq d_{2n}(t)$ and so $\{d_n(t)\}$ is an increasing sequence in B_A^+ . By the inequality (1), we have

$$d_{2n}(kt) \succeq \phi(d_{2n-1}(qt), d_{2n-1}(qt), d_{2n-1}(qt), d_{2n-1}(qt), d_{2n-1}(qt)) \\ \succ d_{2n-1}(qt).$$

Similarly for m = 2n + 1 we have $d_{2n+1}(kt) \succeq d_{2n}(qt)$ and so $d_n(kt) \succeq d_{n-1}(qt)$ for all $n \in \mathbb{N}$. That is,

$$||M(y_n, y_{n+1}, t)||e \succeq ||M(y_{n-1}, y_n, \frac{q}{k}t)||e \succeq \dots \succeq ||M(y_0, y_1, (\frac{q}{k})^n t)||e.$$

Hence, by Lemma 1.14., $\{y_n\}$ is a cauchy sequence and, by the completeness of X, $\{y_n\}$ converges to a point z in X. Let $\lim_n y_n = z$. Hence we have

$$lim_{n}y_{2n} = lim_{n}Fx_{2n} = lim_{n}Tx_{2n+1}$$
$$= lim_{n}y_{2n+1} = lim_{n}Bx_{2n+1}$$
$$= lim_{n}Sx_{2n+2} = z.$$

Now, suppose that T is continuous and the pairs (F, S) and (B, T) are compatible of type (I). Hence we have

$$lim_n TTx_{2n+1} = Tz,$$

$$||M(Tz, z, t)|| e \succeq lim_n ||M(BTx_{2n-1}, z, t)|| e.$$

Now, for $\alpha = 1$, put $x = x_{2n}$ and $y = Tx_{2n+1}$ in the inequality (1) then we obtain

$$||M(Fx_{2n}, BTx_{2n+1}, kt)||e \succeq \phi \begin{pmatrix} ||M(Sx_{2n}, TTx_{2n+1}, t)||e, \\ ||M(Fx_{2n}, Sx_{2n}, t)||e, \\ ||M(BTx_{2n+1}, TTx_{2n+1}, t)||e, \\ ||M(Fx_{2n}, TTx_{2n}, t)||e, \\ ||M(BTx_{2n+1}, Sx_{2n}, t)||e \end{pmatrix}$$

Letting $n \to \infty$, then we have

$$\begin{split} ||M(z, lim_n BTx_{2n+1}, kt)||e \succeq \phi \begin{pmatrix} ||M(z, Tz, t)||e, \\ ||M(z, z, t)||e, \\ ||M(lim_n BTx_{2n+1}, Tz, t)||e, \\ ||M(lim_n BTx_{2n+1}, z, t)||e \end{pmatrix} \\ \succeq \phi \begin{pmatrix} ||M(z, Tz, \frac{t}{2})||e, \\ ||M(z, Tz, \frac{t}{2})||e, \\ ||M(lim_n BTx_{2n+1}, Tz, \frac{t}{2})||e, \\ ||M(lim_n BTx_{2n+1}, Tz, \frac{t}{2})||e, \\ ||M(lim_n BTx_{2n+1}, z, \frac{t}{2})||e, \\ ||M(lim_n BTx_{2n+1}, z, \frac{t}{2})||e \end{pmatrix} \end{split}$$

Thus it follows that

$$lim_{n}||M(BTx_{2n+1}, Tz, t)||e \succeq lim_{n}||M(BTx_{2n+1}, z, \frac{t}{2})||e$$

• $lim_{n}||M(z, Tz, \frac{t}{2})||e$

and so

$$\lim_{n \to \infty} ||M(BTx_{2n+1}, Tz, t)||e \ge \lim_{n \to \infty} ||M(BTx_{2n+1}, z, \frac{t}{2})||e.$$

Since $\phi(t, t, t, t, t) \succ t$, by the above inequalities we have

$$||M(z, lim_{n \to \infty} BTx_{2n+1}, kt)||e \succ ||M(z, lim_{n \to \infty} BTx_{2n+1}, \frac{t}{2})||e,$$

which is a contradiction. It follows that $lim_n BTx_{2n+1} = z$. Now, using the compatibility of type (I), we have

$$||M(Tz, z, t)||e \succeq lim_n||M(z, BTx_{2n+1}, t)||e = e$$

and so Tz = z.

Again, replacing x by x_{2n} and y by z in (3.1). For $\alpha = 1$, we have

$$||M(Fx_{2n}, Bz, kt)||e \succeq \phi \begin{pmatrix} ||M(Sx_{2n}, Tz, t)||e, \\ ||M(Fx_{2n}, Sx_{2n}, t)||e, \\ ||M(Bz, Tz, t)||e, \\ ||M(Fx_{2n}, Tz, t)||e, \\ ||M(Bz, Sx_{2n}, t)||e \end{pmatrix}$$

Letting $n \to \infty$, we have

$$||M(Bz, z, kt)||e \succ ||M(Bz, z, t)||e$$

Which implies that Bz = z. Since $B(X) \subseteq S(X)$, there exist $u \in X$ such that Su = z = Bz. So, for $\alpha = 1$, we have

$$||M(Fu, Bz, kt)||e \succeq \phi \begin{pmatrix} ||M(Su, Tz, t)||e, \\ ||M(Fu, Su, t)||e, \\ ||M(Bz, Tz, t)||e, \\ ||M(Fu, Tz, t)||e, \\ ||M(Bz, Su, t)||e \end{pmatrix}$$

and so,

$$||M(Fu, z, kt)||e \succ ||M(z, Fu, t)||e,$$

which implies that Fu = z. Since the pair (F, S) is compatible of type (I) and Fu = z, by proposition (2.2.), we have

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$$||M(Fu,SSu,t)||e \succeq ||M(Fz,FSu,t)||e$$

and so,

$$||M(z, Sz, t)||e \succeq ||M(z, Fz, t)||e|$$

Again, for $\alpha = 1$, we have

$$||M(Fz, Bz, kt)||e \succeq \phi \begin{pmatrix} ||M(Sz, Tz, t)||e, \\ ||M(Fz, Sz, t)||e, \\ ||M(Bz, Tz, t)||e, \\ ||M(Fz, Tz, t)||e, \\ ||M(Bz, Sz, t)||e \end{pmatrix}$$

It follows that

$$M(Fz, Sz, t) \succeq M(Fz, z, \frac{t}{2}) \bullet M(z, Sz, \frac{t}{2})$$
$$\succeq M(z, Fz, \frac{t}{2}) \bullet M(z, Fz, \frac{t}{2})$$
$$= M(z, Fz, \frac{t}{2}).$$

Hence we have

$$||M(Fz, z, kt)||e \succeq \phi \begin{pmatrix} ||M(Sz, z, \frac{t}{2})||e, \\ ||M(Fz, z, \frac{t}{2})||e, \\ ||M(Fz, z, \frac{t}{2})||e, \\ ||M(Fz, z, \frac{t}{2})||e, \\ ||M(z, Fz, \frac{t}{2})||e \end{pmatrix} \succ ||M(z, Fz, \frac{t}{2})||e$$

and so Fz = z. Therefore, Fz = Bz = Sz = Tz = z and z is a common fixed point for the self- mappings F, B, S and T.

The uniqueness of a common fixed point of the mappings F, B, S, T is easily verified by using (1). In fact, if \dot{z} is another fixed point for F, B, S

and T, then for $\alpha = 1$, we have

$$\begin{split} ||M(z, \dot{z}, t)||e &= ||M(Fz, B\dot{z}, kt)||e \\ &\succeq \phi \begin{pmatrix} ||M(Sz, T\dot{z}, t)||e, \\ ||M(Fz, Sz, t)||e, \\ ||M(B\dot{z}, T\dot{z}, t)||e, \\ ||M(Fz, T\dot{z}, t)||e, \\ ||M(B\dot{z}, Sz, t)||e \end{pmatrix} \\ &\succ ||M(z, \dot{z}, t)||e \end{split}$$

and so $z = \dot{z}$. \Box

Acknowledgments

The authors would like to thank referee for giving useful comments and suggestions for the improvement of this paper.

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