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On the Weakly Hypercyclic Composition Operators on Hardy Spaces

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Abstract. An operator T on a Banach space X is said to be weakly hypercyclic if there exists a vector $x \in X$ whose orbit under T is weakly dense in X. We show that every weakly hypercyclic composition operator on classic Hardy space H^2 is norm hypercyclic.

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1. Introduction

Let X be a separable, infinite-dimensional Banach space. If T is a bounded linear operator on X, the orbit of a vector $x \in X$ under T is defined by $orb(T, x) = \{T^n x : n \in \mathbb{N}\}$. The operator T is said to be hypercyclic if there exists a vector $x \in X$ whose orbit, is norm dense in X. The study of such operators on Banach spaces was initiated by Rolewicz ([6]), who showed that if B is the unilateral backward shift on $\ell^p(\mathbb{N})$, $1 \leqslant p < +\infty$, then the operator λB is hypercyclic for any scalar λ with $|\lambda| > 1$.

A generalization of hypercyclicity was proposed recently. It replaces the norm topology of X with the weak topology: T is said to be weakly hypercyclic if there is a vector $x \in X$ its orbit is dense in the weak topology of X. Since the norm topology is strictly stronger than the weak topology, every hypercyclic operator is weakly hypercyclic but it is

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shown in [3] that weakly hypercyclic operators may fail to be hypercyclic. A weakly hypercyclic operator has many of the same properties as a hypercyclic operator. For example, its adjoint, has no eigenvalue ([3]) and every component of whose spectrum must intersect the unite circle ([4]). In this paper, by searching the weak hypercyclicity in function spaces, we show for the class of composition operators on Hardy spaces, the weak hypercyclicity and hypercyclicity are equivalent. This answers quesion 5.8 in [3] affirmatively.

2. Main result

The Hardy space H^2 is the collection of functions $f \in H(\mathbb{U})$ with $\sum_{n=0}^{+\infty} |\hat{f}(n)|^2 < +\infty$ We equip H^2 with the norm that is naturally associated with its definition:

$$\|f\| = \left(\sum_{n=0}^{+\infty} |\hat{f}(n)|^2 \right)^{\frac{1}{2}}$$

and note that this norm arises from the natural inner product

$$< f, g >= \sum_{n=0}^{+\infty} \hat{f}(n)\overline{\hat{g}(n)} (f, g \in H^2).$$

Let $\{e_n\}$ be the canonical basis for ℓ^2. Then the operator $T : \ell^2 \to \ell^2$ defined by $Te_n = w_n e_{n-1}$ for $n > 1$ and $Te_0 = 0$ is a unilateral weighted backward shift where the weight sequence $\{w_n : n \geqslant 1\}$ is a bounded subset of positive real numbers. For a unilateral weighted backward shift on ℓ^2, it has been shown that norm hypercyclicity and weak hypercyclicity are equivalent ([3]). Thus it is natural to ask (Quastion 5.8., in [3]):

Question 2.1. *Are there other classes of operators for which norm hypercyclicity and weak hypercyclicity are equivalent?*
We search the answer of question in the class of composition operators. Every holomorphic self map φ of $\mathbb{U}$ induces a linear composition operator $C_\varphi : H(\mathbb{U}) \to H(\mathbb{U})$ by $C_\varphi(f)(z) = f(\varphi(z))$ for every $f \in H(\mathbb{U})$ and $z \in$

$\mathbb{U}$. Now suppose φ is a linear fractional self map of $\mathbb{U}$ whit no fixed point in $\mathbb{U}$. Then, we say φ is parabolic when φ has only one fixed point which must lie on the unit circle. Parabolic maps are conjugate to translations of the right half-plane into itself. Also, we say φ is hyperbolic when it has two fixed points, one of them must lies on the unit circle and another one is out of $\bar{\mathbb{U}}$, while in the automorphism case, both fixed points must lie on $\partial\mathbb{U}$ ([2]). Bourdon and Shapiro [2] characterize the hypercyclic composition operator on Hardy space, H^2.

Theorem 2.1. *For a linear fractional self map φ of $\mathbb{U}$, C_φ is hypercyclic on H^2 unless φ is a parabolic non-automorphism .*
For the proof see Theorem 1.47., in [1].

Theorem 2.2. *Let φ be a linear fractional self map of $\mathbb{U}$. Then C_φ is hypercyclic on H^2 if and only if C_φ is weakly hypercyclic.*

Proof. We need only consider the case that φ is parabolic non-automorphism, because for the other cases Theorem 2.1. says that hypercyclicity and weak hypercyclicity for C_φ are equivalent. Let φ be parabolic non-automorphism, so it has only one fixed point which lies on $\partial\mathbb{U}$. Without loss of generality we may take this fixed point to be $+1$. Let

$$\sigma(z) = \frac{1+z}{1-z} \text{ and } \Psi = \sigma \circ \varphi \circ \sigma^{-1}.$$

Then σ is a linear-fractional mapping of $\mathbb{U}$ onto the open right half plane $\mathbb{P}$, and one easily checks that $\Psi(w) = w + a$ where $Rea > 0$ and $w \in \mathbb{P}$. An easy computation shows that for each $z \in \mathbb{U}$,

$$1 - |\varphi_n(z)|^2 = \frac{4Re(\sigma(z) + na)}{|1 + \sigma(z) + na|^2}.$$

(φ_n means the nth iterate of φ) and

$$\varphi_n(z) - \varphi_n(0) = \frac{2(\sigma(z) - \sigma(0))}{(\sigma(0) + na + 1)(\sigma(z) + na + 1)}.$$

Also, for each pair of points $z; w \in \mathbb{U}$, and $f \in H^2$ the following estimate holds,

$$|f(z) - f(w)| \leqslant 2\|f\| \frac{|z-w|}{(\min\{1-|w|, 1-|z|\})^{\frac{3}{2}}},$$

which by substituting $\varphi_n(z), \varphi_n(0)$ instead of z, w respectively and by using the last estimate we get

$$|f(\varphi_n(z)) - f(\varphi_n(0))| \leqslant \frac{M}{\sqrt{n}},$$

where the constant M depends on f, z and φ. For the details see [2]. Now suppose that $\text{orb}(C_\varphi, f)$ is weakly dense for some vector $f \in H^2$. The linear functional $\Lambda_p : H^2 \longrightarrow C$ by $\Lambda_p(g) = g(p)$ is bounded, for every $p \in \mathbb{U}$. Let g be an arbitrary vector in H^2, $z \in \mathbb{U}$, and $\varepsilon > 0$. There exists an arbitrary large n such that $|f(\varphi_n(z)) - f(\varphi_n(0))| < \frac{\varepsilon}{4}$ and

$$|\Lambda_z(f \circ \varphi_n - g)| < \frac{\varepsilon}{4} \quad ; \quad |\Lambda_0(f \circ \varphi_n - g)| < \frac{\varepsilon}{4}.$$

Consequently

$$|f(\varphi_n(z)) - g(z)| < \frac{\varepsilon}{4} \quad ; \quad |f(\varphi_n(0)) - g(0)| < \frac{\varepsilon}{4},$$

so by the triangle inequality we can deduce $|g(z) - g(0)| < \varepsilon$. Hence, $g \equiv g(0)$. Thus only constant functions can be weak cluster points of the C_φ-orbit of an H^2 function and hence C_φ is not weakly hypercyclic. This ends the proof. $\square$

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