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Exact Solutions for the Modified KdV and the Generalized KdV Equations via Exp-Function Method

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Abstract. An application of the Exp-function method (EFM) to search for exact solutions of nonlinear partial differential equations is analyzed. This method is used for the modified KdV equation and the generalized KdV equation. The EFM was used to construct periodic wave and solitary wave solutions of nonlinear evolution equations (NLEEs). This method is developed for searching exact travelling wave solutions of nonlinear partial differential equations. It is shown that the Exp-function method, with the help of symbolic computation, provides a straightforward and powerful mathematical tool for solving nonlinear evolution equations in mathematical physics and applied mathematics.

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1. Introduction

Nonlinear phenomena plays a fundamental role in applied mathematics and physics. Recently, the study of nonlinear partial differential equations in modelling physical phenomena, has become an important tool.

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The investigation of the traveling wave solutions plays an important role in nonlinear sciences. A variety of powerful methods have been presented, such as the inverse scattering transform ([3]), Hirota's bilinear method ([17, 23]), homotopy analysis method ([1, 7]), variational iteration method ([8, 9]), Adomian decomposition method ([10]), homotopy perturbation method ([11, 12, 19, 21, 22]), sine-cosine method ([24, 32]), tanh-function method ([4, 13, 25]), tanh-coth method ([26, 27]), Bäcklund transformation ([18, 20]) and so on. Here, we use an effective method for constructing a range of exact solutions for the following nonlinear partial differential equations which was first presented by J. H. He ([14]). A new method called the Exp-function method (EFM) is presented to look for traveling wave solutions of nonlinear evolution equations (NLEEs). The Exp-function method has successfully been applied to many situations. For example, He et al. ([15]) have solved the nonlinear wave equations by using the Exp-function method. Wu and He ([30]) have used the Exp-function method to give new periodic solutions for nonlinear evolution equations. He and Abdou ([16]) have examined the Exp-function method to find generalized solitary solutions and compacton-like solutions of the Jaulent-Miodek equations. Abdou ([2]) has solved generalized solitary and periodic solutions for nonlinear partial differential equations by the Exp-function method. Boz and Bekir ([5]) have applied the Exp-function method for (3+1)-dimensional nonlinear evolution equations. Chun ([6]) has obtained the solitons and periodic solutions for the fifth–order KdV equation by using the Exp–function method. The Exp-function method along with Hirota's and tanh-coth methods have been applied for computing solitary wave solutions of the generalized shallow water wave equation by Wazwaz ([28]). Wu and He ([31]) have applied the Exp-function method to nonlinear equations. The EFM has recently been generalized by Zhang ([33]) to high-dimensional nonlinear evolution equation. The KdV equation, that plays an important role in the solitary wave theory, is given by

$$\mathrm{u_t} + 6\mathrm{u}\mathrm{u_x} + \mathrm{u_{xxx}} = 0, \tag{1}$$

The balance between the weak nonlinear steepening of $\mathrm{uu_x}$ and the dispersion effect of $\mathrm{u_{xxx}}$ in Eq. (1) generates solitons. The KdV equation is a spatially one dimensional model. The KdV equation is completely integrable ([29]), admits multiple–soliton solutions and exhibits an infinite number of conservation laws of energy. In this article, we have used the Exp-function method to investigate the modified KdV equation and the generalized KdV equation given by [25]

$$u_t + 3au^2u_x + bu_{xxx} = 0, \tag{2}$$

and

$$u_t + a(n+1)u^n u_x + bu_{xxx} = 0. \tag{3}$$

By using the Exp-function method we obtained various solutions for the modified KdV equation and the generalized KdV equation and some new results are formally developed in this article. Our aim of this paper is to obtain analytical solutions of the modified KdV equation and the generalized KdV equation, and to determine the accuracy of the Exp–function method in solving these kinds of problems. The remainder of the paper is organized as follows: In Section 2, a brief discussions for the Exp–function method is presented and exact solutions of Eqs. (2) and (2) are obtained. In Section 3, we describe this method briefly and apply this technique to the modified KdV equation. In Section 4, exact results are considered for the generalized KdV equation by the aforementioned method. Section 5, ends this report with a brief conclusion.

2. Basic Idea of the Exp-Function Method

We first consider the nonlinear equation of the form:

$$\mathcal{N}(u, u_t, u_x, u_{xx}, u_{tt}, u_{tx}, ...) = 0, \tag{4}$$

and introduce a transformation

$$u(x,t) = u(\eta), \qquad \eta = kx + \omega t, \tag{5}$$

where k and ω are constant to be determined later. Therefore Eq. (4) is reduced to an ODE as follows

$$\mathcal{M}(u, \omega u', ku', k^2u'', ...) = 0. \tag{6}$$

The EFM is based on the assumption that travelling wave solutions as in ([15]) can be expressed in the form

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \tag{7}$$

where c, d, p and q are positive integers which could be freely chosen and a_n's and b_m's are unknown constants to be determined. To determine the values of c and p, we balance the linear term of highest order in Eq. (6) with the highest order nonlinear term. Also to determine the values of d and q, we balance the linear term of lowest order in Eq. (6) with the lowest order nonlinear term.

3. The Modified KdV Equation

In this section we employ the Exp-function method to the modified KdV equation

$$u_t + 3au^2u_x + bu_{xxx} = 0, \tag{8}$$

and by using the wave variable $\eta = \mu(x - ct)$ reduces it to an ODE [25],

$$-c\mu u' + 3a\mu u^2 u' + b\mu^3 u''' = 0. \tag{9}$$

Then by integrating Eq. (9) and neglecting the constant of integration we obtain

$$-cu + au^3 + b\mu^2 u'' = 0. \tag{10}$$

In order to determine the values of c and p, we balance the linear term of the highest order u'' with the highest order nonlinear term u^3 in Eq. (10) and get

$$u'' = \frac{c_1 \exp((c + 3p)\eta) + ...}{c_2 \exp(4p\eta) + ...} \tag{11}$$

$$u^3 = \frac{c_3 \exp(3c\eta) + ...}{c_4 \exp(3p\eta) + ...} = \frac{c_3 \exp((3c + p)\eta) + ...}{c_4 \exp(4p\eta) + ...} \tag{12}$$

respectively. Balancing highest order of the Exp–function in (11) and (12) and get

$$c + 3p = 3c + p, \tag{13}$$

which leads to the result $c = p$.

Similarly, to determine the values of d and q, for the terms u'' and u^3 in Eq. (10) by simple calculation, we obtain

$$u'' = \frac{... + d_1 \exp(-(d + 3q)\eta)}{... + d_2 \exp(-4q\eta)}, \tag{14}$$

$$u^3 = \frac{... + d_3 \exp(-3d\eta)}{... + d_4 \exp(-3q\eta)} = \frac{... + d_3 \exp(-(3d + q)\eta)}{... + d_4 \exp(-4q\eta)}, \tag{15}$$

respectively. Balancing lowest order of the Exp-function in (14) and (15) gives us

$$-(d + 3q) = -(3d + q), \tag{16}$$

which leads to the result $d = q$.
Now we consider the following cases:

Case I: $p = c = 1$ and $q = d = 1$.
For simplicity, we set $b_1 = 1$, $p = c = 1$ and $d = q = 1$. Then (7) reduces to

$$u(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{17}$$

Substituting (17) into Eq. (10), and using the well-known Maple software, we have

$$\frac{1}{A}\big[C_3 \exp(3\eta) + C_2 \exp(2\eta) + C_1 \exp(\eta) + C_0 + C_{-1} \exp(-\eta)+ \tag{18}$$

$$C_{-2} \exp(-2\eta) + C_{-3} \exp(-3\eta)\big] = 0,$$

where

$$A = \big[b_{-1} \exp(-\eta) + b_0 + \exp(\eta)\big]^3, \tag{19}$$

and C_n' are coefficients of $\exp(n\eta)$'. Equating the coefficients of $\exp(n\eta)$ to zero, we obtain the following set of algebraic equations for $a_1, a_0, a_{-1}, b_0, b_{-1}$ and c, as

$$\begin{cases} C_3 = 0,\ C_2 = 0,\ C_1 = 0, \\ C_0 = 0, \\ C_{-3} = 0,\ C_{-2} = 0,\ C_{-1} = 0. \end{cases} \tag{20}$$

Solving the system of algebraic equations with the help of Maple gives the following set of non-trivial solutions

(I) The first set is:

$$a_{-1} = 0 \quad b_0 = 0, \quad b_{-1} = \frac{1}{8}\frac{aa_0^2}{\mu^2 b}, \quad a_0 = a_0, \quad a_1 = 0, \quad c = \mu^2 b, \quad \mu = \mu, \tag{21}$$

which gives:

$$u_1(x,t) = \frac{a_0}{\frac{1}{8}\frac{aa_0^2}{\mu^2 b}\exp[-\mu(x-\mu^2 bt)] + \exp[(\mu(x-\mu^2 bt)]}. \tag{22}$$

If we choose $a_0 = 2\sqrt{\frac{2\mu^2 b}{a}}$ or $a_0 = 2\sqrt{-\frac{2\mu^2 b}{a}}$, then the Solution (22) becomes to

$$u_{1,1}(x,t) = \sqrt{\frac{2\mu^2 b}{a}} sech[\mu(x-\mu^2 bt)],$$
$$u_{1,2}(x,t) = \sqrt{-\frac{2\mu^2 b}{a}} csch\left[\mu(x-\mu^2 bt)\right].$$

(II) The second set is:

$$a_{-1} = -a_1 b_{-1},\ b_0 = 0,\ b_{-1} = b{-1},\ a_0 = 0,\ a_1 = a_1,\ c = aa_1^2,\ \mu = \sqrt{-\frac{a}{2b}}a_1, \tag{23}$$

which gives:

$$u_2(x,t) = \frac{-a_1 b_{-1}\exp\left[-\sqrt{-\frac{a}{2b}}a_1(x-aa_1^2 t)\right] + a_1\exp\left[\sqrt{-\frac{a}{2b}}a_1(x-aa_1^2 t)\right]}{b_{-1}\exp\left[-\sqrt{-\frac{a}{2b}}a_1(x-aa_1^2 t)\right] + \exp\left[\sqrt{-\frac{a}{2b}}a_1(x-aa_1^2 t)\right]} \tag{24}$$

If we choose $b_{-1} = 1$ or $b_{-1} = -1$, then the Solution (24) becomes to

$$u_{2,1}(x,t) = a_1\tanh\left[\sqrt{-\frac{a}{2b}}a_1(x-aa_1^2 t)\right],$$
$$u_{2,2}(x,t) = a_1\coth\left[\sqrt{-\frac{a}{2b}}a_1(x-aa_1^2 t)\right].$$

Respectively, where a_1 is an arbitrary constant. In the case μ is an imaginary number, each of the obtained solitonary solutions above can be converted into a periodic solution or compact-like solution. Here, we only discuss the solution given by (24). If $\mu = i\overline{\mu}$ in (24), then we obtain

$$\exp[i\overline{\mu}(x - 2b\overline{\mu}^2 t)] = \cos[\overline{\mu}(x - 2b\overline{\mu}^2 t)] + i\sin[\overline{\mu}(x - 2b\overline{\mu}^2 t)], \tag{25}$$

and

$$\exp[-i\overline{\mu}(x - 2b\overline{\mu}^2 t)] = \cos[\overline{\mu}(x - 2b\overline{\mu}^2 t)] - i\sin[\overline{\mu}(x - 2b\overline{\mu}^2 t)]. \tag{26}$$

and (24) becomes:

$$u_2(x,t) = a_1 \frac{(1 - b_{-1})\cos[\overline{\mu}(x - 2b\overline{\mu}^2 t)] + i(1 + b_{-1})\sin[\overline{\mu}(x - 2b\overline{\mu}^2 t)]}{(1 + b_{-1})\cos[\overline{\mu}(x - 2b\overline{\mu}^2 t)] + i(1 - b_{-1})\sin[\overline{\mu}(x - 2b\overline{\mu}^2 t)]}. \tag{27}$$

or

$$u_2(x,t) = a_1 \frac{(1 - b_{-1}) + i(1 + b_{-1})\tan[\overline{\mu}(x - 2b\overline{\mu}^2 t)]}{(1 + b_{-1}) + i(1 - b_{-1})\tan[\overline{\mu}(x - 2b\overline{\mu}^2 t)]}. \tag{28}$$

If we choose $b_{-1} = 1$ or $b_{-1} = -1$, then the complex Solution (28) respectively gives:

$$u_{2,3}(x,t) = a_1 i\tan[\overline{\mu}(x - 2b\overline{\mu}^2 t)],\ \ u_{2,4}(x,t) = a_1 i\cot[\overline{\mu}(x - 2b\overline{\mu}^2 t)],$$

where a_1 is an arbitrary constant.

Case II: $p = c = 2$ and $q = d = 1$.
For simplicity, we set $b_2 = 1$, $p = c = 2$ and $d = q = 1$. Then (7) reduces to

$$u(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(2\eta) + b_1\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{29}$$

Substituting (29) into Eq. (10), and using the well-known Maple software, we get

$$\frac{1}{A}\big[C_6\exp(6\eta) + C_5\exp(5\eta) + C_4\exp(4\eta) + C_3\exp(3\eta) + C_2\exp(2\eta) \tag{30}$$

$$\mathrm{C_1}\exp(\eta) + \mathrm{C_0} + \mathrm{C_{-1}}\exp(-\eta) + \mathrm{C_{-2}}\exp(-2\eta) + \mathrm{C_{-3}}\exp(-3\eta)] = 0,$$

where

$$\mathrm{A} = \left[\mathrm{b_{-1}}\exp(-\eta) + \mathrm{b_0} + \mathrm{b_1}\exp(\eta) + \exp(2\eta)\right]^3, \tag{31}$$

and $\mathrm{C_n}$' are coefficients of $\exp(\mathrm{n}\eta)$'. Equating the coefficients of $\exp(n\eta)$ to zero, we obtain the following set of algebraic equations for $\mathrm{a_1, a_0, a_{-1}, b_1, b_0, b_{-1}}$ and c, as

$$\begin{cases} \mathrm{C_6} = 0,\ \mathrm{C_5} = 0,\ \mathrm{C_4} = 0,\ \mathrm{C_3} = 0,\ \mathrm{C_2} = 0,\ \mathrm{C_1} = 0, \\ \mathrm{C_0} = 0, \\ \mathrm{C_{-3}} = 0,\ \mathrm{C_{-2}} = 0,\ \mathrm{C_{-1}} = 0. \end{cases} \tag{32}$$

Solving the system of algebraic equations gives the following sets of non-trivial solutions

(I) The first set is:

$$\mathrm{a_{-1}} = -\mathrm{a_2 b_{-1}},\ \ \mathrm{b_{-1}} = \mathrm{b_{-1}},\ \ \mathrm{a_0} = 0\ \ \mathrm{b_0} = 0,\ \ \mathrm{a_2} = \mathrm{a_2},\ \ \mathrm{c} = \mathrm{a a_2^2}, \tag{33}$$

$$\mathrm{a_1} = 0, \qquad \mathrm{b_1} = 0, \qquad \mu = \sqrt{-\frac{2\mathrm{a}}{9\mathrm{b}}}\mathrm{a_2},$$

which gives:

$$\mathrm{u_1(x,t)} = \frac{-\mathrm{a_2 b_{-1}}\exp\left[-\sqrt{-\frac{2\mathrm{a}}{9\mathrm{b}}}\mathrm{a_2(x - a a_2^2 t)}\right] + \mathrm{a_2}\exp\left[2\sqrt{-\frac{2\mathrm{a}}{9\mathrm{b}}}\mathrm{a_2(x - a a_2^2 t)}\right]}{\mathrm{b_{-1}}\exp\left[-\sqrt{-\frac{2\mathrm{a}}{9\mathrm{b}}}\mathrm{a_2(x - a a_2^2 t)}\right] + \exp\left[2\sqrt{-\frac{2\mathrm{a}}{9\mathrm{b}}}\mathrm{a_2(x - a a_2^2 t)}\right]}. \tag{34}$$

If we choose $\mathrm{b_{-1}} = 1$ or $\mathrm{b_{-1}} = -1$, then the solution (34) respectively gives

$$u_{1,1}(x,t) = a_2\tanh\left[\sqrt{-\tfrac{2a}{9b}}a_2(x - a a_2^2 t)\right],$$

$$u_{1,2}(x,t) = a_2\coth\left[\sqrt{-\tfrac{2a}{9b}}a_2(x - a a_2^2 t)\right].$$

where $\mathrm{a_2}$ is an arbitrary constant.

(II) The second set is:

$$\mathrm{a_{-1}} = \frac{1}{8}\frac{(\mathrm{a_2^2 b_1^2} - 4\mathrm{a_2^2 b_0} - \mathrm{a_1^2})(\mathrm{a_2 b_1} - \mathrm{a_1})}{\mathrm{a_2^2}},\ \ \mathrm{a_0} = -\frac{1}{2}\frac{\mathrm{a_2^2 b_1^2} - 2\mathrm{a_2^2 b_0} - \mathrm{a_1^2}}{\mathrm{a_2}}, \tag{35}$$

$$b_{-1} = -\frac{1}{8}\frac{(a_2^2b_1^2 - 4a_2^2b_0 - a_1^2)(a_2b_1 - a_1)}{a_2^3}, \quad b_0 = b_0, \quad a_2 = a_2, \quad c = aa_2^2,$$

$$a_1 = a_1, \qquad b_1 = b_1, \qquad \mu = \sqrt{-\frac{a}{2b}}a_2, \qquad \eta = \sqrt{-\frac{a}{2b}}a_2(x - aa_2^2t),$$

which gives:

$$u_2(x,t) = \frac{\frac{1}{8}\frac{(a_2^2b_1^2-4a_2^2b_0-a_1^2)(a_2b_1-a_1)}{a_2^2}e^{(-\eta)} - \frac{1}{2}\frac{a_2^2b_1^2-2a_2^2b_0-a_1^2}{a_2} + a_1e^{(\eta)} + a_2e^{(2\eta)}}{-\frac{1}{8}\frac{(a_2^2b_1^2-4a_2^2b_0-a_1^2)(a_2b_1-a_1)}{a_2^3}e^{(-\eta)} + b_0 + b_1e^{(\eta)} + e^{(2\eta)}}.$$

Case III: $p = c = 2$ and $q = d = 2$.
Since the values of c and d can be freely chosen, we set $p = c = 2$ and $d = q = 2$ and then the trial function (7), becomes

$$u(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)}{b_2\exp(2\eta) + b_1\exp(\eta) + b_0 + b_{-1}\exp(-\eta) + b_{-2}\exp(-2\eta)}. \quad (36)$$

There are some free parameters in (36). We set $b_2 = 1$ and $b_1 = b_{-1} = 0$ for simplicity, the trial function, Eq. (36) is simplified as follows

$$u(\eta) = \frac{a_2\exp(2\eta) + a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)}{\exp(2\eta) + b_0 + b_{-2}\exp(-2\eta)}. \quad (37)$$

Substituting (37), into Eq. (10), and using the well-known Maple software, we will have

$$\begin{aligned}&\frac{1}{A}\left[C_6\exp(6\eta) + C_5\exp(5\eta) + C_4\exp(4\eta) + C_3\exp(3\eta) + C_2\exp(2\eta)\right. \\ &+C_1\exp(\eta) + C_0 + C_{-1}\exp(-\eta) + C_{-2}\exp(-2\eta) + C_{-3}\exp(-3\eta) + \\ &\quad \left. C_{-4}\exp(-4\eta) + C_{-5}\exp(-5\eta) + C_{-6}\exp(-6\eta)\right] = 0,\end{aligned} \quad (38)$$

where

$$A = \left[b_{-1}\exp(-\eta) + b_0 + b_1\exp(\eta) + \exp(2\eta)\right]^3, \quad (39)$$

and C_n' are coefficients of $\exp(n\eta)$'. Equating the coefficients of $\exp(n\eta)$ to zero, we obtain the following set of algebraic equations for $a_2, a_{-2}, a_1, a_0, a_{-1}, b_0, b_{-2}$ and c, as

$$\begin{cases} C_6 = 0, \; C_5 = 0, \; C_4 = 0, \; C_3 = 0, \; C_2 = 0, \; C_1 = 0. \\ C_0 = 0, \\ C_{-6} = 0, \; C_{-5} = 0, \; C_{-4} = 0, \; C_{-3} = 0, \; C_{-2} = 0, \; C_{-1} = 0. \end{cases} \quad (40)$$

Solving the system of algebraic equations we get

(I) The first set is:

$$\mathrm{a}_1=\mathrm{a}_1,\ \mathrm{a}_{-2}=0,\ \mathrm{a}_{-1}=-\frac{1}{8}\frac{\mathrm{a}_1(\mathrm{a}\mathrm{a}_1^2-8\mathrm{b}\mu^2\mathrm{b}_0)}{\mathrm{b}\mu^2},\ \mathrm{b}_0=\mathrm{b}_0,\ \mathrm{a}_0=0, \quad (41)$$

$$\mathrm{c}=\mathrm{b}\mu^2,\qquad \mathrm{b}_{-2}=-\frac{1}{64}\frac{\mathrm{a}\mathrm{a}_1^2(\mathrm{a}\mathrm{a}_1^2-8\mathrm{b}\mu^2\mathrm{b}_0)}{\mathrm{b}^2\mu^4},\qquad \mathrm{a}_2=0,\quad \mu=\mu,$$

which gives:

$$\mathrm{u}_1(\mathrm{x},\mathrm{t})=\frac{\mathrm{a}_{-1}\mathrm{e}^{-\mathrm{A}}+\mathrm{a}_1\mathrm{e}^{\mathrm{A}}}{\mathrm{b}_{-2}\mathrm{e}^{-2\mathrm{A}}+\mathrm{b}_0+\mathrm{e}^{2\mathrm{A}}},$$

where $\mathrm{A}=\mu(\mathrm{x}-\mathrm{b}\mu^2\mathrm{t})$, a_0, b_0, a_1 and a_2 are arbitrary constants.

(II) The second set is:

$$\mathrm{a}_1=0,\ \ \mathrm{a}_{-2}=0,\ \ \mathrm{a}_{-1}=0,\ \ \mathrm{b}_0=0,\ \ \mathrm{a}_0=\mathrm{a}_0,\ \ \mathrm{a}_2=0, \quad (42)$$

$$\mathrm{c}=4\mathrm{b}\mu^2,\qquad \mathrm{b}_{-2}=\frac{1}{32}\frac{\mathrm{a}\mathrm{a}_0^2}{\mathrm{b}\mu^2},\quad \mu=\mu,$$

which gives:

$$\mathrm{u}_2(\mathrm{x},\mathrm{t})=\frac{\mathrm{a}_0}{\frac{1}{32}\frac{\mathrm{a}\mathrm{a}_0^2}{\mathrm{b}\mu^2}\exp[-2\mu(\mathrm{x}-4\mathrm{b}\mu^2\mathrm{t})]+\exp[2\mu(\mathrm{x}-4\mathrm{b}\mu^2\mathrm{t})]}. \quad (43)$$

If we choose $\mathrm{a}_0=4\sqrt{\frac{2\mu^2\mathrm{b}}{\mathrm{a}}}$ or $\mathrm{a}_0=4\sqrt{\frac{-2\mu^2\mathrm{b}}{\mathrm{a}}}$, then the solution (43) respectively gives

$$u_{2,1}(x,t) = \sqrt{\frac{8b\mu^2}{a}}sech\left[2\mu(x-4b\mu^2t)\right],$$
$$u_{2,2}(x,t) = \sqrt{\frac{-8b\mu^2}{a}}csch\left[2\mu(x-4b\mu^2t)\right].$$

Obtained results in above are the exact solutions of the modified KdV equation.

4. The Generalized KdV Equation

We next apply the Exp-function method to the generalized KdV equation

$$u_t + a(n+1)u^n u_x + bu_{xxx} = 0, \tag{44}$$

and by using the wave variable $\eta = \mu(x - ct)$ reduce it to an ODE [25].

$$-c\mu u' + a(n+1)\mu u^n u' + b\mu^3 u''' = 0, \tag{45}$$

where by integrating (45) and neglecting the constant of integration we obtain

$$-cu + au^{n+1} + b\mu^2 u'' = 0. \tag{46}$$

For various values of n we introduce the transformation

$$u = v^{\frac{1}{n}}, \tag{47}$$

that carries Eq. (46) to

$$-cn^2v^2 + an^2v^3 + bn\mu^2 vv'' + b(1-n)\mu^2(v')^2 = 0. \tag{48}$$

In order to determine the values of c and p, we balance v^3 with vv'' in Eq. (48), to get

$$vv'' = \frac{c_1 \exp((2c+3p)\eta) + ...}{c_2 \exp(5p\eta) + ...}, \tag{49}$$

$$v^3 = \frac{c_3 \exp(3c\eta) + ...}{c_4 \exp(3p\eta) + ...} = \frac{c_3 \exp((3c+2p)\eta) + ...}{c_4 \exp(5p\eta) + ...}, \tag{50}$$

respectively. Balancing highest order of the Exp–function in (49) and (50), we obtain

$$c = p. \tag{51}$$

Similarly, to determine the values of d and q, we balance v^3 with vv'' in Eq. (48), we obtain

$$vv'' = \frac{... + d_1 \exp(-(2d+3q)\eta)}{... + d_2 \exp(-5q\eta)}, \tag{52}$$

$$v^3 = \frac{... + d_3 \exp(-3d\eta)}{... + d_4 \exp(-3q\eta)} = \frac{... + d_3 \exp(-(3d + 2q)\eta)}{... + d_4 \exp(-5q\eta)}, \tag{53}$$

respectively. Balancing lowest order of the Exp–function in (52) and (53), we obtain

$$d = q. \tag{54}$$

Case I: $p = c = 1$ and $q = d = 1$.
For simplicity, we set $b_1 = 1$, $p = c = 1$ and $d = q = 1$. Then (7) reduces to

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{55}$$

Substituting (55) into Eq. (48) and using the well-known Maple software, we will have

$$\frac{1}{A}\left[C_4 \exp(4\eta) + C_3 \exp(3\eta) + C_2 \exp(2\eta) + C_1 \exp(\eta) + C_0 \right. \tag{56}$$
$$\left. + C_{-1} \exp(-\eta) + C_{-2} \exp(-2\eta) + C_{-3} \exp(-3\eta) + C_{-4} \exp(-4\eta)\right] = 0$$

where

$$A = [b_{-1} \exp(-\eta) + b_0 + \exp(\eta)]^4, \tag{57}$$

and C_n' are coefficients of $\exp(n\eta)$'. Equating the coefficients of $\exp(n\eta)$ to zero, we obtain the following set of algebraic equations for $a_1, a_0, a_{-1}, b_0, b_{-1}$ and c, as

$$\begin{cases} C_4 = 0,\ C_3 = 0,\ C_2 = 0,\ C_1 = 0, \\ C_0 = 0, \\ C_{-4} = 0,\ C_{-3} = 0,\ C_{-2} = 0,\ C_{-1} = 0. \end{cases} \tag{58}$$

Solving the system of algebraic equations we get

(I) The first set is:

$$a_{-1} = 0\ b_0 = b_0,\ b_{-1} = \frac{1}{4}b_0^2,\ a_0 = \frac{b\mu^2 b_0(n+2)}{an^2},\ a_1 = 0,\ c = \frac{\mu^2 b}{n^2},\ \mu = \mu, \tag{59}$$

which gives:

$$v_1(x,t) = \frac{\frac{b\mu^2 b_0(n+2)}{an^2}}{\frac{1}{4}b_0^2 \exp[-\mu(x - \frac{\mu^2 b}{n^2}t)] + b_0 + \exp[(\mu(x - \frac{\mu^2 b}{n^2}t)]},$$

$$u_1(x,t) = \left\{ \frac{\frac{b\mu^2 b_0(n+2)}{an^2}}{\frac{1}{4}b_0^2\exp[-\mu(x-\frac{\mu^2 b}{n^2}t)] + b_0 + \exp[(\mu(x-\frac{\mu^2 b}{n^2}t)]} \right\}^{\frac{1}{n}} \quad (60)$$

If we choose $b_0 = 2$ or $b_0 = -2$, then the solution (60) respectively gives (cf. Eqs. (38) and (37) in [25])

$$u_{1,1}(x,t) = \left\{ \frac{b\mu^2(n+2)}{2an^2} \operatorname{sech}^2\left[\frac{\mu}{2}\left(x - \frac{\mu^2 b}{n^2}t\right)\right]\right\}^{\frac{1}{n}},$$

$$u_{1,2}(x,t) = \left\{ -\frac{b\mu^2(n+2)}{2an^2} \operatorname{csch}^2\left[\frac{\mu}{2}\left(x - \frac{\mu^2 b}{n^2}t\right)\right]\right\}^{\frac{1}{n}}.$$

If $\mu = i\overline{\mu}$ in (60) then we obtain

$$\exp\left[i\overline{\mu}\left(x + \frac{\overline{\mu}^2 b}{n^2}t\right)\right] = \cos\left[\overline{\mu}\left(x + \frac{\overline{\mu}^2 b}{n^2}t\right)\right] + i\sin\left[\overline{\mu}\left(x + \frac{\overline{\mu}^2 b}{n^2}t\right)\right], \quad (61)$$

and

$$\exp\left[-i\overline{\mu}\left(x + \frac{\overline{\mu}^2 b}{n^2}t\right)\right] = \cos\left[\overline{\mu}\left(x + \frac{\overline{\mu}^2 b}{n^2}t\right)\right] - i\sin\left[\overline{\mu}\left(x + \frac{\overline{\mu}^2 b}{n^2}t\right)\right]. \quad (62)$$

and (60) becomes

$$u_1(x,t) = \left\{ -\frac{\frac{4b\overline{\mu}^2 b_0(n+2)}{an^2}}{(4+b_0^2)\cos\left[\overline{\mu}\left(x+\frac{\overline{\mu}^2 b}{n^2}t\right)\right] + 4b_0 + i(4-b_0^2)\sin\left[\overline{\mu}\left(x+\frac{\overline{\mu}^2 b}{n^2}t\right)\right]} \right\}^{\frac{1}{n}}. \quad (63)$$

It should be pointed out that the transformation $\mu = i\overline{\mu}$ was first used by He and Wu in [16] to find periodic solutions or compact-like solutions from the obtained solitary solutions. If we search for a periodic solution or compact-like solution, the imaginary part in the denominator of (63) must be zero, that requires that

$$b_0 = \pm 2. \quad (64)$$

by applying (64) then (63) reduces to a compact-like solution, which reads

$$u_1(x,t) = \left\{ \mp \frac{\frac{b\overline{\mu}^2(n+2)}{an^2}}{\cos\left[\overline{\mu}\left(x+\frac{\overline{\mu}^2 b}{n^2}t\right)\right] \pm 1} \right\}^{\frac{1}{n}}. \quad (65)$$

Also (65) is further simplified to obtain a periodic solution respectively (cf. Eqs. (40) and (39) in [25])

$$u_{1,3}(x,t) = \left\{ -\frac{b\overline{\mu}^2(n+2)}{an^2} \sec^2\left[\frac{\overline{\mu}}{2}\left(x + \frac{\overline{\mu}^2 b}{n^2}t\right)\right]\right\}^{\frac{1}{n}}, \tag{66}$$

$$u_{1,4}(x,t) = \left\{ -\frac{b\overline{\mu}^2(n+2)}{an^2} \csc^2\left[\frac{\overline{\mu}}{2}\left(x + \frac{\overline{\mu}^2 b}{n^2}t\right)\right]\right\}^{\frac{1}{n}}. \tag{67}$$

Case II: $p = c = 2$ and $q = d = 1$.
For simplicity, we set $b_2 = 1$, $p = c = 2$ and $d = q = 1$. Then (7) reduces to

$$v(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{68}$$

Substituting (68) into Eq. (48), we have

$$\begin{gathered}\frac{1}{A}\big[C_8 \exp(8\eta) + C_7 \exp(7\eta) + C_6 \exp(6\eta) + C_5 \exp(5\eta) + C_4 \exp(4\eta) + \\ C_3 \exp(3\eta) + C_2 \exp(2\eta) + C_1 \exp(\eta) + C_0 + C_{-1}\exp(-\eta) + \\ C_{-2}\exp(-2\eta) + C_{-3}\exp(-3\eta) + C_{-4}\exp(-4\eta)\big] = 0,\end{gathered} \tag{69}$$

where

$$A = \big[b_{-1}\exp(-\eta) + b_0 + b_1 \exp(\eta) + \exp(2\eta)\big]^4, \tag{70}$$

and C_n' are coefficients of $\exp(n\eta)$'. Equating the coefficients of $\exp(n\eta)$ to zero, we obtain the following set of algebraic equations for $a_1, a_0, a_{-1}, b_1, b_0, b_{-1}$ and c, as

$$\begin{cases} C_8 = 0, C_7 = 0, \mathbb{C}_6 = 0, \mathbb{C}_5 = 0, \mathbb{C}_4 = 0, \mathbb{C}_3 = 0, \mathbb{C}_2 = 0, \mathbb{C}_1 = 0, \\ C_0 = 0, \\ C_{-4} = 0, \mathbb{C}_{-3} = 0, \mathbb{C}_{-2} = 0, \mathbb{C}_{-1} = 0. \end{cases} \tag{71}$$

By the same manipulation as illustrated above, we get

(I) The first set is:

$$a_{-1} = 0, \quad b_{-1} = b_{-1}, \quad a_0 = a_0, \quad a_1 = \frac{4b^2\mu^4 b_{-1}(n+2)^2}{a^2 n^4 a_0}, \tag{72}$$

$$c = \frac{\mu^2 b}{n^2}, \quad \mu = \mu, \quad a_2 = 0,$$

$$b_0 = \frac{a_0^3 a^3 n^6 + 4b^3\mu^6 b_{-1}^2 n^3 + 24b^3\mu^6 b_{-1}^2 n^2 + 48b^3\mu^6 b_{-1}^2 n + 32b^3\mu^6 b_{-1}^2}{b\mu^2 a^2 n^4 (n+2) a_0^2},$$

$$b_1 = \frac{a_0^3 a^3 n^6 + 16b^3\mu^6 b_{-1}^2 n^3 + 96b^3\mu^6 b_{-1}^2 n^2 + 192b^3\mu^6 b_{-1}^2 n + 128b^3\mu^6 b_{-1}^2}{4b^2\mu^4 a a_0 n^2 (n+2)^2 b_{-1}},$$

which gives:

$$v_1(x,t) = \frac{a_0 + \frac{4b^2\mu^4 b_{-1}(n+2)^2}{a^2 n^4 a_0} e^{\left[\mu\left(x - \frac{\mu^2 b}{n^2} t\right)\right]}}{b_{-1} e^{\left[-\mu\left(x - \frac{\mu^2 b}{n^2} t\right)\right]} + b_0 + b_1 e^{\left[\mu\left(x - \frac{\mu^2 b}{n^2} t\right)\right]} + e^{\left[2\mu\left(x - \frac{\mu^2 b}{n^2} t\right)\right]}},$$

$$u_1(x,t) = \left\{ \frac{a_0 + \frac{4b^2\mu^4 b_{-1}(n+2)^2}{a^2 n^4 a_0} e^{\left[\mu\left(x - \frac{\mu^2 b}{n^2} t\right)\right]}}{b_{-1} e^{\left[-\mu\left(x - \frac{\mu^2 b}{n^2} t\right)\right]} + b_0 + b_1 e^{\left[\mu\left(x - \frac{\mu^2 b}{n^2} t\right)\right]} + e^{\left[2\mu\left(x - \frac{\mu^2 b}{n^2} t\right)\right]}} \right\}^{\frac{1}{n}}.$$

Case III: $p = c = 2$ and $q = d = 2$.

Since the values of c and d can be freely chosen, we set $p = c = 2$ and $d = q = 2$. Then the trial function, (7) becomes

$$v(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)}{b_2 \exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1}\exp(-\eta) + b_{-2}\exp(-2\eta)}. \tag{73}$$

There are some free parameters in (73), we set $b_2 = 1, b_1 = b_{-1} = 0$ for simplicity, the trial function, (73) is simplified as follows

$$u(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta) + a_{-2}\exp(-2\eta)}{\exp(2\eta) + b_0 + b_{-2}\exp(-2\eta)}. \tag{74}$$

Substituting (74) into Eq. (48), to get

$$\frac{1}{A}[C_8 \exp(8\eta) + C_7 \exp(7\eta) + C_6 \exp(6\eta) + C_5 \exp(5\eta) + C_4 \exp(4\eta) + \tag{75}$$

$$C_3\exp(3\eta)+C_2\exp(2\eta)+C_1\exp(\eta)+C_0+C_{-1}\exp(-\eta)+$$
$$C_{-2}\exp(-2\eta)+C_{-3}\exp(-3\eta)+C_{-4}\exp(-4\eta)+C_{-5}\exp(-5\eta)+$$
$$C_{-6}\exp(-6\eta)+C_{-7}\exp(-7\eta)+C_{-8}\exp(-8\eta)]=0,$$

where

$$A=\left[b_{-1}\exp(-\eta)+b_0+b_1\exp(\eta)+\exp(2\eta)\right]^4, \tag{76}$$

and C_n' are coefficients of $\exp(n\eta)$'. Equating the coefficients of $\exp(n\eta)$ to zero, we obtain the following set of algebraic equations for $a_2, a_{-2}, a_1, a_0, a_{-1}, b_0, b_{-2}$ and c, as

$$\begin{cases} C_8=0, \mathbb{C}_7=0, \mathbb{C}_6=0, \mathbb{C}_5=0, \mathbb{C}_4=0, \mathbb{C}_3=0, \mathbb{C}_2=0, \mathbb{C}_1=0. \\ C_0=0, \\ C_{-8}=0, \mathbb{C}_{-7}=0, \mathbb{C}_{-6}=0, \mathbb{C}_{-5}=0, \mathbb{C}_{-4}=0, \mathbb{C}_{-3}=0, \mathbb{C}_{-2}=0, \mathbb{C}_{-1}=0. \end{cases} \tag{77}$$

By the same manipulation as illustrated above, we get

(I) The first set is:

$$a_1=0, \quad a_{-2}=0, \quad a_{-1}=0, \quad b_0=b_0, \quad a_0=\frac{4b_0\mu^2 b(n+2)}{an^2},$$
$$c=\frac{4b\mu^2}{n^2}, \quad b_{-2}=\frac{1}{4}b_0^2, \quad a_2=0, \quad \mu=\mu, \tag{78}$$

which gives:

$$v_1(x,t)=\frac{\frac{4b_0\mu^2 b(n+2)}{an^2}}{\frac{1}{4}b_0^2\exp\left[-2\mu\left(x-\frac{4\mu^2 b}{n^2}t\right)\right]+b_0+\exp\left[2\mu\left(x-\frac{4\mu^2 b}{n^2}t\right)\right]},$$

$$u_1(x,t)=\left\{\frac{\frac{4b_0\mu^2 b(n+2)}{an^2}}{\frac{1}{4}b_0^2\exp\left[-2\mu\left(x-\frac{4\mu^2 b}{n^2}t\right)\right]+b_0+\exp\left[2\mu\left(x-\frac{4\mu^2 b}{n^2}t\right)\right]}\right\}^{\frac{1}{n}}. \tag{79}$$

If we choose $b_0=2$ or $b_0=-2$ and then the solution (79) respectively gives (cf. Eqs. (38) and (37) in [25])

$$u_{1,1}(x,t)=\left\{\frac{b\mu^2(n+2)}{2an^2}\operatorname{sech}^2\left[\mu\left(x-\frac{4\mu^2 b}{n^2}t\right)\right]\right\}^{\frac{1}{n}},$$

$$u_{1,2}(x,t)=\left\{-\frac{b\mu^2(n+2)}{2an^2}\ \mathrm{csch}^2\left[\mu\left(x-\frac{4\mu^2 b}{n^2}t\right)\right]\right\}^{\frac{1}{n}}.$$

As illustrated in the previous case, the obtained solitonary solutions can be converted into periodic solutions or compact-like solutions if μ is chosen as an imaginary number. Here, we only discuss the solution given by (79). If $\mu = i\overline{\mu}$, then it becomes

$$u_1(x,t)=\left\{-\frac{\frac{4b\overline{\mu}^2 b_0(n+2)}{an^2}}{(4+b_0^2)\cos(B)+4b_0+i(4-b_0^2)\sin(B)}\right\}^{\frac{1}{n}}. \quad (80)$$

where $B = 2\overline{\mu}\left(x+\frac{4\overline{\mu}^2 b}{n^2}t\right)$. Elimination of the imaginary part requires that

$$b_0 = \pm 2. \quad (81)$$

We therefore, by applying (81) obtain from (80) the periodic solutions

$$u_1(x,t)=\left\{\mp\frac{\frac{b\overline{\mu}^2(n+2)}{an^2}}{\cos\left[2\overline{\mu}\left(x+\frac{4\overline{\mu}^2 b}{n^2}t\right)\right]\pm 1}\right\}^{\frac{1}{n}}. \quad (82)$$

and (82) is further simplified to obtain a periodic solution respectively (cf. Eqs. (40) and (39) in [25])

$$u_{1,3}(x,t)=\left\{-\frac{b\overline{\mu}^2(n+2)}{an^2}\ \sec^2\left[\overline{\mu}\left(x+\frac{4\overline{\mu}^2 b}{n^2}t\right)\right]\right\}^{\frac{1}{n}}, \quad (83)$$

$$u_{1,4}(x,t)=\left\{-\frac{b\overline{\mu}^2(n+2)}{an^2}\ \csc^2\left[\overline{\mu}\left(x+\frac{4\overline{\mu}^2 b}{n^2}t\right)\right]\right\}^{\frac{1}{n}}. \quad (84)$$

Obtained results in above are the exact solutions of the generalized KdV equation.

5. Conclusion

Based on the Exp-function method, some nonlinear evolution equations are solved exactly. In this article we investigated the modified KdV equation and the generalized KdV equation. The Exp-function method is a useful method for finding travelling wave solutions of nonlinear evolution equations. This method has been successfully applied to obtain some new generalized solitonary solutions to the modified KdV equation and the generalized KdV equation. The Exp-function method is more powerful in searching for exact solutions of NLPDEs. Comparing our results and Wazwaz's results [25] it can be seen that the results are the same. Also, new results are formally developed in this article. It can be concluded that the this method is a very powerful and efficient technique in finding exact solutions for wide classes of problems.

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