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# Module Amenability and Tensor Product of Semigroup Algebras

**A. Bodaghi**
Islamic Azad University-Garmsar Branch

**Abstract.** Let $S$ be an inverse semigroup with an upward directed set of idempotents $E$. In this paper we prove that if $S$ is amenable, then $\ell^1(S)\widehat{\bigotimes}\ell^1(S)$ is module amenable as an $\ell^1(E)$-module. Also we show that $\ell^1(S)\widehat{\bigotimes}\ell^1(S)$ is module super-amenable if an appropriate group homomorphic image of $S$ is finite.

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## 1. Introduction

The notion of amenability of Banach algebras was introduced by Barry Johnson in [9]. A Banach algebra $\mathcal{A}$ is *amenable* if every bounded derivation from $\mathcal{A}$ into any dual Banach $\mathcal{A}$-module is inner, equivalently if $H^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$-module $X$, where $H^1(\mathcal{A}, X^*)$ is the *first Hochschild cohomology group* of $\mathcal{A}$ with coefficients in $X^*$. He proved in [9, Proposition 5.4] that if $\mathcal{A}$ and $\mathcal{B}$ are amenable Banach algebra, then so is $\mathcal{A}\widehat{\bigotimes}\mathcal{B}$ (see also [6, Corollary 2.9.62]). Also $\mathcal{A}$ is called *super-amenable* (*contractible*) if $H^1(\mathcal{A}, X) = \{0\}$ for every Banach $\mathcal{A}$-bimodule $X$ (see [6,12]). It is known $\mathcal{A}\widehat{\bigotimes}\mathcal{B}$ is super-amenable if $\mathcal{A}$ and $\mathcal{B}$ are super-amenable [12, Exercise 4.1.4].

For a discrete semigroup $S$, $\ell^\infty(S)$ is the Banach algebra of bounded complex-valued functions on $S$ with the supremum norm and pointwise

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multiplication. For each $t \in S$ and $f \in \ell^\infty(S)$, let $L_t f$ and $R_t f$ denote the left and the right translations of $f$ by $t$, that is $\langle L_t f, s\rangle = \langle f, ts\rangle$ and $\langle R_t f, s\rangle = \langle f, st\rangle$, for each $s \in S$. Then a linear functional $m \in (\ell^\infty(S))^*$ is called a *mean* if $\|m\| = \langle m, 1\rangle = 1$; $m$ is called a *left* (*right*) *invariant mean* if $\langle m, L_t f\rangle = \langle m, f\rangle$ ($\langle m, R_t f\rangle = \langle m, f\rangle$, respectively) for all $s \in S$ and $f \in \ell^\infty(S)$. A discrete semigroup $S$ is called *amenable* if there exists a mean $m$ on $\ell^\infty(S)$ which is both left and right invariant (see [7]). An *inverse semigroup* is a discrete semigroup $S$ such that for each $s \in S$, there is a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. Elements of the form $ss^*$ are called *idempotents* of $S$. For an inverse semigroup $S$, a left invariant mean on $\ell^\infty(S)$ is right invariant and vise versa.

M. Amini in [1] introduced the concept of module amenability for a Banach algebra. He showed that for an inverse semigroup $S$ with set of idempotents $E$, the semigroup algebra $\ell^1(S)$ is $\ell^1(E)$-module amenable if and only if $S$ is amenable.

This extends the Johnson's theorem [9, Theorem 2.5] in the discrete case) which asserts that for a discrete group $G$, $\ell^1(G)$ is amenable if and only if $G$ is amenable. The author and Amini in [4] introduced the concept of module super-amenability and showed that for an inverse semigroup $S$, the semigroup algebra $\ell^1(S)$ is module super-amenable if and only if the group homomorphic image $S/\approx$ of $S$ is finite, where $\approx$ is an equivalence relation on $S$.

In part two of this paper, we show that when $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left then under some mild conditions, module amenability of $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ implies amenability of $\mathcal{A}/J\widehat{\bigotimes}\mathcal{A}/J$ and vise versa, where $J$ is the closed ideal of $\mathcal{A}$ generated by $\alpha\cdot(ab) - (ab)\cdot\alpha$ for all $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. There is a similar result for super amenability.

Finally, we prove that if $S$ is an amenable inverse semigroup with an upward directed set of idempotents $E$, then $\ell^1(S)\widehat{\bigotimes}\ell^1(S)$ is module amenable as an $\ell^1(E)$-module. Also we show that $\ell^1(S)\widehat{\bigotimes}\ell^1(S)$ is module super-amenable when the appropriate group homomorphic image $S/\approx$ is finite.

# 2. Module Amenability of the Tensor Product of Banach Algebras

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions, as follows

$$\alpha \cdot (ab) = (\alpha \cdot a)b, \quad (ab) \cdot \alpha = a(b \cdot \alpha) \quad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Let $X$ be a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule with compatible actions, that is

$$\alpha{\cdot}(a{\cdot}x) = (\alpha{\cdot}a){\cdot}x, \; a{\cdot}(\alpha{\cdot}x) = (a{\cdot}\alpha){\cdot}x, \; (\alpha{\cdot}x){\cdot}a = \alpha{\cdot}(x{\cdot}a) \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X)$$

and the same for the right or two-sided actions. Then we say that $X$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module. If moreover $\alpha \cdot x = x \cdot \alpha$ for all $\alpha \in \mathfrak{A}, x \in X$, then $X$ is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module. If $X$ is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is $X^*$, the first dual space of $X$, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on $X^*$ are defined as follows

$$\langle \alpha \cdot f, x \rangle = \langle f, x \cdot \alpha \rangle, \; \langle a \cdot f, x \rangle = \langle f, x \cdot a \rangle \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X, f \in X^*)$$

and the same for the right actions.
Note that, in general, $\mathcal{A}$ is not an $\mathcal{A}$-$\mathfrak{A}$-module because $\mathcal{A}$ does not satisfy in the compatibility condition $a \cdot (\alpha \cdot b) = (a \cdot \alpha) \cdot b$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$ [2]. But when $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module.
It is well known that $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$, the projective tensor product of $\mathcal{A}$ and $\mathcal{A}$ is a Banach algebra with respect to the canonical multiplication defined by $(a \otimes b)(c \otimes d) = (ac \otimes bd)$. Also it is a Banach $\mathcal{A}$-bimodule and a Banach $\mathfrak{A}$-bimodule by the following usual actions:

$$\alpha \cdot (a \otimes b) = (\alpha \cdot a) \otimes b, \quad c \cdot (a \otimes b) = (ca) \otimes b \quad (\alpha \in \mathfrak{A}, a, b, c \in \mathcal{A}),$$

Similarly, for the right actions consider the module projective tensor product $\mathcal{A}\widehat{\bigotimes}_{\mathfrak{A}}\mathcal{A}$ which is isomorphic to the quotient space $(\mathcal{A}\widehat{\bigotimes}\mathcal{A})/I$, where $I$ is the closed ideal of the projective tensor product $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ generated by elements of the form $\alpha \cdot a \otimes b - a \otimes b \cdot \alpha$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$ [11]. Also we consider $J$, the closed ideal of $\mathcal{A}$ generated by elements

of the form $(\alpha \cdot a)b - a(b \cdot \alpha)$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$. Then $\mathcal{A}/J$ is Banach $\mathcal{A}$-$\mathfrak{A}$-module when $\mathcal{A}$ acts on $\mathcal{A}/J$ canonically.
Let $\mathcal{A}$ and $\mathfrak{A}$ be as in the above and $X$ be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A bounded map $D : \mathcal{A} \longrightarrow X$ is called a *module derivation* if

$$D(a \pm b) = D(a) \pm D(b), \;\; D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in \mathcal{A}),$$

and

$$D(\alpha \cdot a) = \alpha \cdot D(a), \quad D(a \cdot \alpha) = D(a) \cdot \alpha \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Although $D$ is not necessary linear, but still its boundedness implies its norm continuity (since it preserves subtraction). When $X$ is commutative $\mathcal{A}$-$\mathfrak{A}$-module, each $x \in X$ defines a module derivation

$$D_x(a) = a \cdot x - x \cdot a \quad (a \in \mathcal{A}).$$

These are called *inner* module derivations. The Banach algebra $\mathcal{A}$ is called *module amenable* (as an $\mathfrak{A}$-module) if for any commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module $X$, each module derivation $D : \mathcal{A} \longrightarrow X^*$ is inner [1]. Similarly, $\mathcal{A}$ is called *module super-amenable* if each module derivation $D : \mathcal{A} \longrightarrow X$ is inner [4].
We say the Banach algebra $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left (right) if for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, $\alpha \cdot a = f(\alpha)a$ $(a \cdot \alpha = f(\alpha)a)$, where $f$ is a continuous linear functional on $\mathfrak{A}$. The following lemma is proved in [3].

**Lemma 2.1.** *Let $\mathcal{A}$ be a Banach algebra and Banach $\mathfrak{A}$-module with compatible actions, and $J_0$ be a closed ideal of $\mathcal{A}$ such that $J \subseteq J_0$. If $\mathcal{A}/J_0$ has a left or right identity $e + J_0$, then for each $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$ we have $a \cdot \alpha - \alpha \cdot a \in J_0$, i.e., $\mathcal{A}/J_0$ is commutative Banach $\mathfrak{A}$-module.*

Recall that $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$ if there is a bounded net $\{\gamma_i\}$ in $\mathfrak{A}$ such that for each $a \in \mathcal{A}$, $\|\gamma_i \cdot a - a\| \to 0$ and $\|a \cdot \gamma_i - a\| \to 0$, as $i \longrightarrow \infty$.

**Theorem 2.2.** *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module with trivial left action and $\mathcal{A}/J$ has an identity. If $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ is module amenable (module super-amenable), then $\mathcal{A}/J\widehat{\bigotimes}\mathcal{A}/J$ is amenable (module super-amenable). The converse is true if $\mathfrak{A}$ has a bounded approximate identity for $\mathcal{A}$.*

**Proof.** We prove the result for the module amenability. Let $X$ be a unital $\mathcal{A}/J\widehat{\bigotimes}\mathcal{A}/J$-bimodule and $D:\mathcal{A}/J\widehat{\bigotimes}\mathcal{A}/J\longrightarrow X^*$ be a bounded derivation (see [5, Lemma 43.6]). Then $X$ is an $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$-bimodule with module actions given by

$$(a\otimes b)\cdot x:=((a+J)\otimes(b+J))\cdot x,\ \ x\cdot(a\otimes b):=x\cdot((a+J)\otimes(b+J))\ \ (x\in X, a\in\mathcal{A}),$$

and $X$ is $\mathfrak{A}$-bimodule with trivial actions, that is $\alpha\cdot x=x\cdot\alpha=f(\alpha)x$, for each $x\in X$ and $\alpha\in\mathfrak{A}$ which $f$ is a continuous linear functional on $\mathfrak{A}$. Since $f(\alpha)a-a\cdot\alpha\in J$ (see Lemma 2.1.), we have $f(\alpha)a+J=a\cdot\alpha+J$, for each $\alpha\in\mathfrak{A}$, and the actions of $\mathfrak{A}$ and $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$ on $X$ are compatible. Therefore $X$ is commutative Banach $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$-$\mathfrak{A}$-module. Consider $\Phi:(\mathcal{A}\widehat{\bigotimes}\mathcal{A})/I\longrightarrow\mathcal{A}/J\widehat{\bigotimes}\mathcal{A}/J$ defined by

$$\Phi((a\otimes b)+I)=(a+J)\otimes(b+J).$$

For each $a,b\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$ we have

$$\begin{aligned}(\alpha\cdot a+J)\otimes(b+J)-(a+J)\otimes(b\cdot\alpha+J)&=(f(\alpha)a+J)\otimes(b+J)\\&-(a+J)\otimes(f(\alpha)b+J)\\&=f(\alpha)(a+J)\otimes(b+J)\\&-f(\alpha)(a+J)\otimes(b+J)=0.\end{aligned}$$

We have used Lemma 2.1., in the first equality, hence $\Phi$ is well defined. Obviously $\Phi$ is $\mathfrak{A}$-bimodule morphism. We show that the map $\overline{D}=D\circ\Phi\circ\pi:\mathcal{A}\widehat{\bigotimes}\mathcal{A}\longrightarrow X^*$ is module derivation where $\pi:\mathcal{A}\widehat{\bigotimes}\mathcal{A}\longrightarrow(\mathcal{A}\widehat{\bigotimes}\mathcal{A})/I$ is the projection map. For each $a,b,c,d\in\mathcal{A}$ and $\alpha\in\mathfrak{A}$, we have

$$\begin{aligned}\overline{D}((a\otimes b)(c\otimes d))&=D(((a+J)\otimes(b+J))((c+J)\otimes(d+J)))\\&=D((a+J)\otimes(b+J))\cdot((c+J)\otimes(d+J))\\&+((a+J)\otimes(b+J))\cdot D((c+J)\otimes(d+J))\\&=\overline{D}(a\otimes b)\cdot(c\otimes d)+(a\otimes b)\cdot\overline{D}(c\otimes d).\end{aligned}$$

For each $a,b\in\mathcal{A}$ we have $\overline{D}((a\otimes b)\pm(c\otimes d))=\overline{D}(a\otimes b)\pm\overline{D}(c\otimes d)$.

Also $\mathcal{A}/J\widehat{\bigotimes}\mathcal{A}/J$ is an $\mathfrak{A}$-bimodule, hence for $\alpha \in \mathfrak{A}$, we have

$$\begin{aligned}\overline{D}((a\otimes b)\cdot\alpha) &= D((a+J)\otimes(b\cdot\alpha+J))\\ &= D((a+J)\otimes(f(\alpha)b+J))\\ &= f(\alpha)D((a+J)\otimes(b+J))\\ &= \overline{D}(a\otimes b)\cdot\alpha.\end{aligned}$$

On the other hand, since the left $\mathfrak{A}$-module actions on $\mathcal{A}$ and $X$ are trivial, $\overline{D}(\alpha\cdot(a\otimes b)) = \overline{D}(f(\alpha)(a\otimes b)) = \alpha\cdot\overline{D}(a\otimes b)$. Therefore there exists $x^* \in X^*$ such that $\overline{D}(a\otimes b) = (a\otimes b)\cdot x^* - x^*\cdot(a\otimes b)$ , hence $D((a+J)\otimes(b+J)) = ((a+J)\otimes(b+J))\cdot x^* - x^*\cdot((a+J)\otimes(b+J))$, and so $D$ is inner.
For the converse, we note that for every derivation $D : \mathcal{A} \longrightarrow X$ on unital Banach algebra $\mathcal{A}$ with identity $e$, we have $D(e) = 0$ and without loss of generality we can assume that $e\cdot D(a) = D(a)\cdot e = D(a)$ for all $a \in \mathcal{A}$. We use this fact in the rest of the proof. Now, suppose that $X$ is a commutative Banach $\mathcal{A}\widehat{\bigotimes}\mathcal{A}$-$\mathfrak{A}$-module. We consider the following module actions $\mathcal{A}/J\widehat{\bigotimes}\mathcal{A}/J$ on $X$,

$$((a+J)\otimes(b+J))\cdot x := (a\otimes b)\cdot x,\quad x\cdot((a+J)\otimes(b+J)) := x\cdot(a\otimes b)\quad (x\in X, a\in\mathcal{A}).$$

For each $a,b,c,d \in \mathcal{A}$, $x \in X$, and $\alpha,\beta \in \mathfrak{A}$, we have

$$\begin{aligned}((\alpha\cdot ab - ab\cdot\alpha)\otimes(\beta\cdot cd - cd\cdot\beta))\cdot x &= (\alpha\cdot ab\otimes\beta\cdot cd - \alpha\cdot ab\otimes cd\cdot\beta\\ &\quad - ab\cdot\alpha\otimes\beta\cdot cd\\ &\quad + ab\cdot\alpha\otimes cd\cdot\beta)\cdot x\\ &= \beta\cdot((f(\alpha)ab\otimes cd)\cdot x)\\ &\quad - ((f(\alpha)ab\otimes cd)\cdot x)\cdot\beta\\ &\quad - \beta\cdot((ab\cdot\alpha\otimes cd)\cdot x)\\ &\quad + ((ab\cdot\alpha\otimes cd)\cdot x)\cdot\beta = 0.\end{aligned}$$

Similarly if $a \in J$ or $b \in J$, we can show that $(a\otimes b)\cdot x = 0$ and $x\cdot(a\otimes b) = 0$. Therefore $X$ is a Banach $\mathcal{A}/J\widehat{\bigotimes}\mathcal{A}/J$-bimodule. Suppose that $D : \mathcal{A}\widehat{\bigotimes}\mathcal{A} \longrightarrow X^*$ is a module derivation, and consider $\tilde{D} : \mathcal{A}/J\widehat{\bigotimes}\mathcal{A}/J \longrightarrow X^*$ defined by $\tilde{D}((a+J)\otimes(b+J)) := D(a\otimes b)$, for all $a,b \in \mathcal{A}$. Suppose that $e+J$ is identity for $\mathcal{A}/J$, we have

$$\begin{aligned} D(a\otimes(\alpha\cdot cd-cd\cdot\alpha)) &= \alpha\cdot D(a\otimes cd)-D(a\otimes cd)\cdot\alpha\\ &= \alpha\cdot D(ae\otimes cd)-D(ae\otimes cd)\cdot\alpha\\ &= \alpha\cdot D(a\otimes c)\cdot(e\otimes d)+\alpha\cdot(a\otimes c)\cdot D(e\otimes d)\\ &- D(a\otimes c)\cdot(e\otimes d)\cdot\alpha-(a\otimes c)\cdot D(e\otimes d)\cdot\alpha=0. \end{aligned}$$

Although $ae$ is not equal with $a$, but we have

$$D(a\otimes cd)=\tilde{D}((a+J)\otimes(cd+J))=\tilde{D}((ae+J)\otimes(cd+J))=D(ae\otimes cd).$$

By the above observation, $\tilde{D}$ is also well-defined. Suppose that $\mathfrak{A}$ has a bounded approximate identity $(\gamma_i)$ for $\mathcal{A}$. Since $f$ is bounded, $\{|f(\gamma_i)|\}$ is a bounded sequence in $\mathbb{C}$. Without loss of generality, we may assume that $f(\gamma_i)\longrightarrow 1$, as $i\longrightarrow\infty$. Then for each $\lambda\in\mathbb{C}$ we have

$$e\cdot(\lambda\gamma_i)-f(\gamma_i)e=(\lambda e)\cdot\gamma_i-f(\gamma_i)e\longrightarrow\lambda e-e$$

in norm. Since $J$ is a closed ideal of $\mathcal{A}$, $\lambda e-e\in J$. Next, for each $\lambda\in\mathbb{C}$, and $a,b\in\mathcal{A}$, we have

$$\begin{aligned} \tilde{D}((\lambda a+J)\otimes(b+J)) &= \tilde{D}((a+J)\otimes(b+J))(e+J)\otimes(\lambda e+J))\\ &= \tilde{D}((a+J)\otimes(b+J))\cdot((e+J)\otimes(\lambda e+J))\\ &+((a+J)\otimes(b+J))\cdot\tilde{D}((e+J)\otimes(\lambda e+J))\\ &= \lambda\tilde{D}((a+J)\otimes(b+J))\cdot((e+J)\otimes(e+J))\\ &+((a+J)\otimes(b+J))\cdot\tilde{D}((e+J)\otimes(e+J))\\ &= \lambda\tilde{D}((a+J)\otimes(b+J)). \end{aligned}$$

Thus $\tilde{D}$ is $\mathbb{C}$-linear, and so it is inner. Therefore $D$ is an inner module derivation. $\square$

In this part we find conditions on a (discrete) inverse semigroup $S$ such that the tensor product $\ell^1(S)\widehat{\bigotimes}\ell^1(S)$ is $\ell^1(E)$-module amenable and super-amenable, where $E$ is the set of idempotents of $S$, acting on $S$ trivially from left and by multiplication from right. Let $S$ be an inverse semigroup with set idempotent $E$, where the order of $E$ is defined by

$$e\leqslant d\Longleftrightarrow ed=e\quad(e,d\in E).$$

It is easy to show that $E$ is a (commutative) subsemigroup of $S$ [8, Theorem V.1.2]. In particular $\ell^1(E)$ could be regard as a subalgebra of $\ell^1(S)$, and thereby $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E)$-module with compatible actions ([1]). Here we let $\ell^1(E)$ act on $\ell^1(S)$ by multiplication from right and trivially from left, that is

$$\delta_e \cdot \delta_s = \delta_s, \ \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \quad (s \in S, e \in E).$$

In this case, the ideal $J$ is the closed linear span of $\{\delta_{set} - \delta_{st} : \ s, t \in S, e \in E\}$. We consider an equivalence relation on $S$ as follows

$$s \approx t \Longleftrightarrow \delta_s - \delta_t \in J \ \ (s, t \in S).$$

Recall that $E$ is called *upward directed* if for every $e, f \in E$ there exists $g \in E$ such that $eg = e$ and $fg = f$. This is precisely the assertion that $S$ satisfies the $D_1$ condition of Duncan and Namioka [7]. It is shown in [10, Theorem 3.2.], that if $E$ is upward directed, then the quotient $S/\approx$ is a discrete group. As in [10, Theorem 3.3], we may observe that $\ell^1(S)/J \cong \ell^1(S/\approx)$. With the above notations, $\ell^1(S)/J \cong \ell^1(S/\approx)$ is a commutative $\ell^1(E)$-bimodule with the following actions

$$\delta_e \cdot (\delta_s + J) = \delta_s + J, \ (\delta_s + J) \cdot \delta_e = \delta_{se} + J \quad (s \in S, e \in E).$$

**Theorem 2.3.** *Let $S$ be an inverse semigroup with an upward directed set of idempotents $E$ and $\ell^1(S)$ be a Banach $\ell^1(E)$-module with trivial left action and canonical right action. Then the following statements hold:*

*(i) If $S$ is amenable, then $\ell^1(S)\widehat{\bigotimes}\ell^1(S)$ is module amenable.*

*(ii) If $S/\approx$ is finite, then $\ell^1(S)\widehat{\bigotimes}\ell^1(S)$ is module super-amenable.*

**Proof.** $(i)$ The semigroup algebra $S$ is amenable if and only if $\ell^1(S)$ is module amenable [1. Theorem 3.1]. Thus $\ell^1(S/\approx)$ is unital amenable Banach algebra by [3, Proposition 3.2], and so the tensor product $\ell^1(S/\approx)\widehat{\bigotimes}\ell^1(S/\approx)$ is amenable [6. Corollary 2.9.62]. Now the proof is completed by using Theorem 2.2.

$(ii)$ Since $S/\approx$ is a finite (discrete)group, $\ell^1(S)$ is module super-amenable as $\ell^1(E)$-module, hence $\ell^1(S/\approx)$ is super-amenable by [4, Lemma 2.7]. By [12, Exercise 4.1.4], $\ell^1(S/\approx)\widehat{\bigotimes}\ell^1(S/\approx)$ is super-amenable. Now the result follows from Theorem 2.2 with $\mathcal{A} = \ell^1(S)$ and $\mathfrak{A} = \ell^1(E)$. $\square$

**Example 2.4.** (i) Let $\mathcal{C}$ be the bicyclic inverse semigroup generated by $a$ and $b$, that is

$$\mathcal{C} = \{a^m b^n : m, n \geqslant 0\}, \;\; (a^m b^n)^* = a^n b^m.$$

The set of idempotents of $\mathcal{C}$ is $E_{\mathcal{C}} = \{a^n b^n : n = 0, 1, ...\}$ which is totally ordered (and so is upward directed) with the following order

$$a^n b^n \leqslant a^m b^m \Longleftrightarrow m \leqslant n.$$

It is shown in [3] that $\mathcal{C}/\approx$ is isomorphic to the group of integers $\mathbb{Z}$, hence $\mathcal{C}$ is amenable. Therefore the tensor product $\ell^1(\mathcal{C})\widehat{\bigotimes}\ell^1(\mathcal{C})$ is module amenable by Theorem 2.3.

(ii) Let $(\mathbb{N}, \vee)$ be the commutative semigroup of positive integers with maximum operation $m \vee n = max(m, n)$, then each element of $\mathbb{N}$ is an idempotent, that is $E_{\mathbb{N}} = \mathbb{N}$. Hence $\mathbb{N}/\approx$ is the trivial group with one element. Therefore by Theorem 2.2., the tensor product $\ell^1(\mathbb{N})\widehat{\bigotimes}\ell^1(\mathbb{N})$ is module super-amenable, as an $\ell^1(\mathbb{N})$-module.

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**Abasalt Bodaghi**
Department of Mathematics
Assistant Professor of Mathematics
Islamic Azad University, Garmsar-Branch
Garmsar, Iran.
E-mail: abasalt.bodaghi@gmail.com